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XIV.—On Substitutional Equations. By D. E. Rutherford, M.A., B.Sc., D.Math.,  
United College, University of St Andrews.

(MS. received June 12, 1944. Revised MS. received July 24, 1944. Read July 3, 1944)

1. It is well known that the  $n!$  permutations  $\epsilon, \sigma_2, \dots, \sigma_{n!}$  of  $n$  letters form a group which is usually denoted by  $S_n$ . The permutation  $\epsilon$  denotes the identity permutation and is the unit of the group. A linear combination of these permutations with numerical coefficients is called a *substitutional expression*, and any such expression will have the form

$$L = l_1 \epsilon + l_2 \sigma_2 + \dots + l_{n!} \sigma_{n!}.$$

Since the sum and product of two substitutional expressions are themselves substitutional expressions, these substitutional expressions are the elements of an algebra called by Weyl (p. 79) the *enveloping algebra* of the group  $S_n$ . It need hardly be pointed out that the basic elements  $\sigma_i$  of this algebra are linearly independent. Since the elements of this algebra may be regarded as operators acting on functions of the  $n$  permuted letters, we adopt the notation of Young and read products from right to left.

The simplest type of substitutional equation is

$$LX = 0,$$

where  $L$  is a given substitutional expression and  $X$  is an unknown one. Such equations were first studied by Young with a view to solving certain problems in the theory of invariants. We recapitulate some of his results in § 2.

It has been shown by Young (1927, p. 265) and others that corresponding to each partition  $\alpha$  of  $n$  a certain number  $(f^\alpha)^2$  of substitutional expressions

$$e_{rs}^\alpha, \quad (r, s = 1, \dots, f^\alpha)$$

can be constructed with the properties

$$\begin{aligned} e_{rs}^\alpha e_{pq}^\beta &= \delta^{\alpha\beta} \delta_{sp} e_{rq}^\alpha, \\ \sum_{\alpha} e_{rr}^\alpha &= \epsilon, \end{aligned} \tag{1}$$

where  $\delta$  is the Kronecker function. There are in all  $n!$  of these expressions and they form a basis of the enveloping algebra. Let

$$A = \sum_{\alpha\beta\gamma} a_{rs}^\alpha e_{rs}^\beta, \quad B = \sum_{\alpha\beta\gamma} b_{rs}^\alpha e_{rs}^\beta, \quad C = \sum_{\alpha\beta\gamma} c_{rs}^\alpha e_{rs}^\beta,$$

where  $a_{rs}^\alpha, b_{rs}^\alpha, c_{rs}^\alpha$  are numerical coefficients. If  $AB = C$ , then it can be deduced from (1) that  $U_A^\alpha U_B^\alpha = U_C^\alpha$ , where  $U_A^\alpha, U_B^\alpha, U_C^\alpha$  are respectively the matrices

$$[a_{rs}^\alpha], \quad [b_{rs}^\alpha], \quad [c_{rs}^\alpha].$$

Conversely, if  $U_A^\alpha U_B^\alpha = U_C^\alpha$  for every partition  $\alpha$  of  $n$ , then it may be shown that  $AB = C$ . Further, for every partition  $\alpha$  the matrix  $U_C^\alpha$  is a unit matrix of order  $f^\alpha$ .

2. Our first problem is that of solving the equation

$$LX = 0, \tag{2}$$

where  $L$  is known and  $X$  is unknown. If we write

$$L = \sum_i l_{\sigma_i} \sigma_i, \quad X = \sum_i x_{\sigma_i} \sigma_i,$$

where  $l_{\sigma_i}$  and  $x_{\sigma_i}$  are numerical, then

$$LX = \sum_{ij} l_{\sigma_i} x_{\sigma_j} \sigma_i \sigma_j = \sum_{kj} l_{\sigma_k \sigma_j^{-1}} x_{\sigma_j} \sigma_k.$$

Since the permutations  $\sigma_k$  are linearly independent, we conclude that the equations

$$\sum_j l_{\sigma_k \sigma_j^{-1}} x_{\sigma_j} = 0, \quad (k = 1, \dots, n!) \quad (3)$$

must hold if  $LX = 0$ . Evidently the number of linearly independent solutions of these equations depends upon the rank of the matrix  $\Lambda$  which has  $l_{\sigma_k \sigma_j^{-1}}$  for its  $k, j$ th element. If its rank is  $n!$ , the only solution of (3) is  $x_{\sigma_j} = 0$  for all  $j$ , and the only solution of (2) is  $X = 0$ . If, however, the rank  $\lambda$  is less than  $n!$ , there are  $n! - \lambda$  linearly independent non-zero solutions of (3), and the most general solution of (2) is a linear combination of  $n! - \lambda$  independent solutions (Young, 1900, p. 104).

A more fruitful method of attack is to express  $L$  and  $X$  in terms of the  $e_{rs}^\alpha$ . Let

$$L = \sum_{\alpha rs} l_{rs}^\alpha e_{rs}^\alpha, \quad X = \sum_{\alpha rs} x_{rs}^\alpha e_{rs}^\alpha.$$

Then as a result of (2) we have

$$U_L^\alpha U_X^\alpha = 0$$

for every partition  $\alpha$  of  $n$ . It is clear that the equations for the elements of any column of  $U_X^\alpha$  are the same as those for any other column. Thus, if  $u_X^\alpha$  denote a typical column, the equations for its elements are comprised in the single matrix equation

$$U_L^\alpha u_X^\alpha = 0.$$

If the rank of  $U_L^\alpha$  is  $\lambda^\alpha$ , then there are  $f^\alpha - \lambda^\alpha$  independent parameters in the most general solution for  $u_X^\alpha$ . In fact any solution  $u_X^\alpha$  may be expressed in the form

$$u_X^\alpha = \sum_i \xi_i^\alpha u_{X_i}^\alpha,$$

in which the  $u_{X_i}^\alpha$  are the  $f^\alpha - \lambda^\alpha$  independent solutions and the  $\xi_i^\alpha$  are a like number of arbitrary numerical coefficients. The coefficients  $\xi_i^\alpha$  will in general take different values for the different column vectors  $u_X^\alpha$ , but the  $u_{X_i}^\alpha$  may be taken to be the same for all columns of  $U_X^\alpha$ . There are therefore in all  $f^\alpha(f^\alpha - \lambda^\alpha)$  arbitrary parameters in the most general solution of the matrix equation. It follows that the number of linearly independent solutions of the equation  $LX = 0$  is (Young, 1927, p. 267)

$$\sum_\alpha f^\alpha(f^\alpha - \lambda^\alpha). \quad (4)$$

3. Before proceeding with the general case we consider the particular case where  $L$  is idempotent; that is, where  $L^2 = L$ . Since  $L$  is idempotent, so is the matrix  $U_L^\alpha$ ; but it is known that the rank of an idempotent matrix is equal to its trace. Further, if  $L = \sum_i l_i \sigma_i$ , then

$$\text{tr. } U_L^\alpha = \sum_i l_i (\text{tr. } U_{\sigma_i}^\alpha) = \sum_i l_i \chi_{\sigma_i}^\alpha,$$

since the trace of  $U_{\sigma_i}^\alpha$  is simply the character component  $\chi_{\sigma_i}^\alpha$ . It follows that if  $L$  is idempotent the number of linearly independent solutions of  $LX = 0$  is

$$\sum_{\alpha, i} f^\alpha(f^\alpha - l_i \chi_{\sigma_i}^\alpha).$$

Now it is well known that  $\sum_\alpha (f^\alpha)^2 = n!$  and that  $\sum_\alpha f^\alpha \chi_{\sigma_i}^\alpha$  is equal to  $n!$  or 0 according as  $\sigma_i$  is or is not the unit permutation  $\epsilon$  (Littlewood, p. 46). This leads to the following result.

THEOREM 1. If  $L$  be an idempotent substitutional expression, the number of linearly independent solutions of  $LX=0$  is  $n!(1-l_1)$ , where  $l_1$  is the coefficient of  $\epsilon$  in  $L$  when  $L$  is expressed in terms of the permutations.

From this point onwards we shall accommodate the printer by dropping the upper suffix  $\alpha$  wherever it occurs. This omission should present no difficulty to the reader.

Since  $U_L$  is idempotent, it follows that

$$U_X = U_{\epsilon} - U_L = U_{\epsilon-L}$$

is one solution of the equation

$$U_L U_X = 0. \quad (5)$$

Now if  $\lambda$  be the rank of  $U_L$ , the rank of  $U_{\epsilon-L}$  is just  $f-\lambda$ . There are therefore  $f-\lambda$  linearly independent columns in the matrix  $U_{\epsilon-L}$ , and this is exactly the number of linearly independent solutions of the equation

$$U_L U_X = 0.$$

The most general solution of this last equation is therefore

$$u_X = U_{\epsilon-L} u_Y,$$

where  $u_Y$  is an arbitrary column vector. Similarly, the most general solution of (5) is

$$U_X = U_{\epsilon-L} U_Y,$$

where  $U_Y$  is an arbitrary matrix. Since this is true for every  $\alpha$  we have the following result.

THEOREM 2. If  $L$  is idempotent, the most general solution of the equation  $LX=0$  is  $X=(\epsilon-L)Y$ , where  $Y$  is an arbitrary substitutional expression.

Although the solution exhibited in Theorem 2 involves  $n!$  arbitrary parameters, there are nevertheless only  $n!(1-l_1)$  linearly independent solutions. In the general case it is difficult to pick out this number of linearly independent ones from the  $n!$  solutions given by Theorem 2. In certain special cases, however, which are of some importance  $n!(1-l_1)$  independent solutions can be displayed, and these cases we now consider.

4. If the permutations  $\epsilon, \tau_2, \dots, \tau_h$  form a subgroup of  $S_n$  of order  $h$ , then  $\tau_i(\epsilon + \tau_2 + \dots + \tau_h) = (\epsilon + \tau_2 + \dots + \tau_h)$ . It follows immediately from this that

$$H \equiv \frac{1}{h}(\epsilon + \tau_2 + \dots + \tau_h)$$

is idempotent. The most general solution of the equation  $HX=0$  is therefore  $X=(\epsilon-H)Y$ , and so any such solution is a linear combination of the solutions

$$(\epsilon-H)\sigma_i, \quad (i=1, \dots, n!).$$

Theorem 1 informs us that of these only  $n!(h-1)/h$  are linearly independent, and it is our object to pick out this number from the  $n!$  solutions just quoted. By resolving  $S_n$  into associate complexes with respect to the subgroup, we can find  $n!/h (\equiv k)$  permutations  $\rho_i$  such that

$$\epsilon + \sigma_2 + \dots + \sigma_{n!} = (\epsilon + \tau_2 + \dots + \tau_h)(\epsilon + \rho_2 + \dots + \rho_k),$$

and it will be demonstrated that the  $n!(h-1)/h$  solutions

$$(\epsilon-H)\tau_i \rho_j \quad (i=2, \dots, h; j=1, \dots, k)$$

are linearly independent. There can be no relation between solutions involving different  $\rho_j$ , since distinct complexes have no elements in common. It is sufficient to show that no relation

$$\sum_{i=2}^h \eta_i (\epsilon-H)\tau_i = 0$$

exists unless each numerical coefficient  $\eta_i$  vanishes. Such a relation gives

$$\sum_{i=2}^h \eta_i \tau_i = \sum_{i=2}^h \eta_i H.$$

Comparing the coefficients of  $\epsilon$  on each side, we find that  $\sum_{i=2}^h \eta_i = 0$ , and so

$$\sum_{i=2}^h \eta_i \tau_i = 0.$$

Since the permutations  $\tau_i$  are linearly independent, we conclude that each  $\eta_i = 0$ .

Alternative independent solutions of the equation  $HX = 0$  are

$$(\epsilon - \tau_i) \rho_j, \quad (i = 2, \dots, h; j = 1, \dots, k).$$

They are linearly independent since the permutations of  $S_n$  are independent. Each obviously satisfies  $HX = 0$ .

Again, since  $H$  is idempotent, so is  $\epsilon - H$ . It follows that  $X = HY$  is the most general solution of

$$(\epsilon - H)X = 0.$$

By Theorem 1 there are  $k$  independent solutions. These may be chosen to be

$$H \rho_j, \quad (j = 1, \dots, k),$$

which are clearly independent as each solution involves elements belonging exclusively to one complex, and because different solutions arise from different complexes.

As a corollary to the foregoing, it is easily proved that if we write

$$H' \equiv \frac{1}{h}(\epsilon + \delta_2 \tau_2 + \dots + \delta_h \tau_h),$$

where  $\delta_i$  is  $+1$  or  $-1$  according as  $\tau_i$  is an even or an odd permutation, then  $H'$  is idempotent and the independent solutions of  $H'X = 0$  may be taken to be

$$(\epsilon - H') \delta_i \tau_i \rho_j, \quad (i = 2, \dots, h; j = 1, \dots, k),$$

or, alternatively,

$$(\epsilon - \delta_i \tau_i) \rho_j, \quad (i = 2, \dots, h; j = 1, \dots, k),$$

while those of  $(\epsilon - H')X = 0$  may be taken to be

$$H' \rho_j, \quad (j = 1, \dots, k).$$

5. We return now to the general equation

$$LX = 0,$$

where  $L$  is no longer assumed to be idempotent. The expressions  $\epsilon, L, L^2, \dots$  cannot all be independent since they are each linear functions of the  $n!$  permutations of  $S_n$ . Let

$$\phi_L(x) = 0$$

be the equation of least degree which is satisfied by  $L$ . It is called the *minimum equation* of  $L$  and  $\phi_L(x)$  is the *minimum function* of  $L$ . It follows from  $LX = 0$  that  $L^2X = 0, L^3X = 0, \dots$ , and hence  $\phi_L(L)X = \phi_L(0)X$ , where  $\phi_L(0)$  is numerical. Since, however,  $\phi_L(L)X = 0$ , it follows that the only solution of  $LX = 0$  is  $X = 0$  unless  $\phi_L(0) = 0$ , that is, unless  $\phi_L(x)$  has the factor  $x$ . It will be proved in this section that if  $\phi_L(x)$  has a factor  $x^i, i > 1$ , we can first of all replace the equation  $LX = 0$  by another equation  $MX = 0$  which has the same solutions,  $M$  being such that  $\phi_M(x)$  has the factor  $x$  unrepated. It will appear that the substitutional expression  $M$  is in fact equal to  $AL$ , where  $A$  is a properly chosen substitutional expression which possesses an inverse  $A^{-1}$ . This being so,  $LX = 0$  implies  $ALX = 0$ , or  $MX = 0$ , while  $MX = 0$  implies  $A^{-1}MX = 0$ , or  $LX = 0$ . It is thus ensured that  $LX = 0$  and  $MX = 0$  have the same solutions.

The minimum equation  $\phi_L(x) = 0$  is satisfied by the matrix  $U_L$  for each partition  $\alpha$ , and therefore  $\phi_L(x)$  contains the reduced characteristic function of each of these matrices as a factor.  $\phi_L(x)$  is in fact the L.C.M. of these reduced characteristic functions, as is easily

proved. For, if  $\theta(x)$  is the L.C.M. in question,  $\theta(x)=0$  is satisfied by each matrix  $U_L$  and hence by the expression  $L$ . Thus  $\theta(x)$  contains  $\phi_L(x)$  as a factor. On the other hand,  $\phi_L(x)$  has  $\theta(x)$  as a factor, since the reduced characteristic function of each  $U_L$  is a factor of  $\phi_L(x)$ . It follows that  $\phi_L(x)=\theta(x)$  apart from an irrelevant numerical factor.

Suppose now that

$$U_L = H[P + Q_1 + \dots + Q_s]H^{-1},$$

where the middle factor is the canonical form of  $U_L$ , where  $P$  is the non-singular submatrix corresponding to all the non-zero latent roots of  $U_L$ , and where  $Q_i$  is a submatrix of order  $q_i$  of the form

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

If we now write

$$U_A \equiv H[I + R_1 + \dots + R_s]H^{-1},$$

where  $I$  is a unit matrix of the same order as  $P$  and where  $R_i$  is a submatrix of order  $q_i$  of the form

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},$$

then, by actual multiplication,

$$U_A U_L = H[P + S_1 + \dots + S_s]H^{-1},$$

where  $S_i$  is a submatrix of order  $q_i$  of the form

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Thus  $P + S_1 + \dots + S_s$  is the canonical form of  $U_A U_L$ , and each canonical submatrix associated with a zero latent root is of order unity. This means that any elementary divisor of the matrix  $U_A U_L$  corresponding to a zero latent root is linear, and hence the reduced characteristic function of  $U_A U_L$  involves the factor  $x$  to the first power only. Thus, if  $A$  is the substitutional expression represented by the matrices  $U_A$ , the minimum function of  $AL$ , being the L.C.M. of the reduced characteristic functions of the matrices  $U_A U_L$ , contains the factor  $x$  to the first power only. We are led to the following conclusion.

**THEOREM 3.** *For any equation  $LX=0$ , where  $L$  does not possess an inverse, it is possible to find another equation  $MX=0$  with the same solutions and no others and such that  $\phi_M(x)$  has the factor  $x$  to the first degree only.*

6. Let the minimum equation of  $M$  be

$$\phi_M(x) \equiv x\psi(x) = 0, \quad (6)$$

where  $\psi(x)$  is prime to  $x$ . We may suppose that the constant term in  $\psi(x)$  is  $+1$  without lack of generality. This being so, it is clear that  $\epsilon - \psi(M)$  has a factor  $M$ . Since  $M$  satisfies equation (6), it follows that

$$\{\psi(M)\}^2 = \psi(M);$$

that is,  $\psi(M)$  and therefore  $\epsilon - \psi(M)$  are idempotents. Also, since  $\epsilon - \psi(M)$  has a factor  $M$ , the equation  $MX=0$  implies  $\{\epsilon - \psi(M)\}X=0$ . On the other hand,  $\{\epsilon - \psi(M)\}X=0$  implies  $M\{\epsilon - \psi(M)\}X=0$ ; that is,  $MX=0$ , since  $M$  satisfies (6). The equations  $MX=0$  and

$\{\epsilon - \psi(M)\}X$  therefore have the same solutions. Moreover, since  $\epsilon - \psi(M)$  is idempotent, the most general solution of these equations is  $X = \psi(M)Y$ , where  $Y$  is an arbitrary expression. The following theorem has now been proved.

**THEOREM 4.** *If the minimum equation of  $M$  is  $x\psi(x) = 0$ , where  $\psi(x)$  is prime to  $x$ , then  $X = \psi(M)Y$ , where  $Y$  is arbitrary, is the most general solution of the equation  $MX = 0$ .*

It will be seen that a combination of Theorems 3 and 4 enable us to solve any equation  $LX = 0$ . We first construct the expression  $M$  according to the method of § 5. When we have found the minimum equation of  $M$ , Theorem 4 gives the required solution. Neither  $A$  nor  $M$  is by any means unique, and indeed a suitable  $A$  can often be found by inspection. Theorem 3 gives a method for constructing an  $A$  when none can be found by inspection. We shall call  $MX = 0$  the *prepared* form of the equation  $LX = 0$ .

7. Successive applications of the methods described in the preceding sections enable us to solve a system of simultaneous equations

$$M_1X = 0, \quad L_2X = 0, \quad \dots, \quad L_cX = 0, \quad (7)$$

where we have indicated by the notation that the first equation has already been put into a prepared form. We assume that  $\phi_{M_1}(x)$ ,  $\phi_{L_2}(x)$ ,  $\dots$ ,  $\phi_{L_c}(x)$  each have a factor  $x$ , as otherwise  $X = 0$  is the only solution. The most general solution of  $M_1X = 0$  is

$$X = \psi_1(M_1)Y_1,$$

where  $Y_1$  is arbitrary.  $Y_1$  is not arbitrary, however, if  $X$  also satisfies  $L_2X = 0$ . In such case  $Y_1$  must satisfy the equation

$$L_2\psi_1(M_1)Y_1 = 0.$$

Suppose that the prepared form of this equation is  $M_2Y_1 = 0$ . Then

$$Y_1 = \psi_2(M_2)Y_2,$$

where  $Y_2$  is arbitrary unless  $X$  satisfies a further equation  $L_3X = 0$ . Proceeding in this manner we eventually obtain the most general solution of the equations (6). It is

$$X = \psi_1(M_1)\psi_2(M_2) \dots \psi_c(M_c)Y_c,$$

where  $Y_c$  is an arbitrary expression.

8. We have seen that if an equation  $MX = 0$  is in a prepared form, then the most general solution can be expressed as a function of  $M$  post-multiplied by an arbitrary expression  $Y$ . It is natural to ask whether this is always true even in the case of an unprepared equation  $LX = 0$ . We shall show presently that this is not so.

Consider first the case where the minimum equation of  $L$  is

$$\phi_L(x) \equiv x^i = 0, \quad (i < r).$$

It is at once obvious that  $X = L^{i-1}Y$  where  $Y$  is arbitrary is a solution of  $LX = 0$ , and indeed it is the most general solution which can be expressed as a function of  $L$  post-multiplied by an arbitrary expression. It is not, however, necessarily the most general solution of the equation  $LX = 0$ . Since each  $U_L$  satisfies the equation  $x^i = 0$ , all the latent roots of each  $U_L$  are zero, and each elementary divisor of each  $U_L$  must be of the form  $x^j$ , where  $j < i$ . Suppose those of  $U_L$  are  $x^{j_1}, \dots, x^{j_s}$ ; then  $U_L$  is similar to a direct sum of submatrices of the type  $Q_i$  used in § 5 (MacDuffee, p. 73; Turnbull and Aitken, p. 69), and the orders  $j_k$  of these submatrices are each less than or equal to  $i$ . It follows that the rank of  $U_L$  is  $\sum_k (j_k - 1)$  or  $f - s$ . If

$X = L^{i-1}Y$  is to be the most general solution of  $LX = 0$ , then the rank of  $(U_L)^{i-1}$  must be  $s$ . This can only be so if  $j_1 = \dots = j_s = i$  for each partition  $\alpha$  (Turnbull and Aitken, p. 62). It is thus only in exceptional cases that  $X = L^{i-1}Y$  is the most general solution of  $LX = 0$ .

Suppose next that the minimum equation of  $L$  is

$$\phi_L(x) \equiv x^i\psi(x) = 0,$$

where  $\psi(x)$  is prime to  $x$ . Functions  $\eta(x)$  and  $\zeta(x)$  can be found such that

$$x^i\eta(x) + \psi(x)\zeta(x) = 1,$$

where  $\zeta(x)$  is of degree  $i-1$  at most and does not contain the factor  $x$ . Now

$$F(L) \equiv L^i \eta(L) = \epsilon - \psi(L) \zeta(L)$$

is idempotent, for

$$\begin{aligned} \{F(L)\}^2 &= L^i \eta(L) \{\epsilon - \psi(L) \zeta(L)\} \\ &= L^i \eta(L) - \phi_L(L) \eta(L) \zeta(L) \\ &= F(L). \end{aligned}$$

Since  $F(L)$  has a factor  $L$ , every solution of  $LX=0$  satisfies  $F(L)X=0$ , and is therefore of the form

$$X = \{\epsilon - F(L)\}Y = \psi(L)\zeta(L)Y.$$

Not every solution of this form, however, satisfies  $LX=0$ . It will only do so if  $Y$  satisfies the equation

$$L\psi(L)\zeta(L)Y=0,$$

which we now write

$$KY=0,$$

where

$$K \equiv L\psi(L)\zeta(L) = L\{\epsilon - F(L)\}.$$

Now since  $\epsilon - F(L)$  is idempotent,

$$\begin{aligned} K^2 &= L^2\{\epsilon - F(L)\}, \\ K^3 &= L^3\{\epsilon - F(L)\}, \\ &\dots \end{aligned}$$

If  $\phi_K(x)$  be the minimum function of  $K$ , then

$$\phi_K(K) = \phi_K(L)\{\epsilon - F(L)\} = 0,$$

and since  $\epsilon - F(L)$  does not have a factor  $L$ ,  $\phi_K(L)$  must contain a factor  $L^i$ ; that is,  $\phi_K(x)$  contains a factor  $x^i$ . On the other hand,

$$K^i = L^i\{\epsilon - F(L)\} = L^i\psi(L)\zeta(L) = 0,$$

and so

$$\phi_K(x) = x^i,$$

which is the case already discussed. It follows that the most general solution of  $KY=0$  which can be expressed as a function of  $K$  post-multiplied by an arbitrary expression  $Z$  is

$$Y = K^{i-1}Z = L^{i-1}\{\epsilon - F(L)\}Z.$$

Substituting this result in the value for  $X$  we find that

$$\begin{aligned} X &= \{\epsilon - F(L)\}L^{i-1}\{\epsilon - F(L)\}Z \\ &= L^{i-1}\{\epsilon - F(L)\}Z \\ &= L^{i-1}\psi(L)\zeta(L)Z. \end{aligned}$$

Since  $L^i\psi(L)=0$ , we can write this solution in the form

$$X = L^{i-1}\psi(L)\zeta(0)Z,$$

or, including the numerical factor  $\zeta(0)$  in the arbitrary  $Z$ , this solution can be written

$$X = L^{i-1}\psi(L)Z.$$

The fact that this is always a solution of  $LX=0$  could of course have been deduced immediately from the fact that  $\phi_L(x) = x^i\psi(x)$ , but the foregoing argument tells us more than this. It gives a criterion by which we can say whether or not it is the most general solution. It is the most general solution if and only if each elementary divisor of each  $U_K$  is  $x^i$ . In any case it is the most general solution which can be expressed as a function of  $L$  post-multiplied by an arbitrary expression. This is clear from the fact that if we put  $Z=\epsilon$  and substitute in  $LX=0$  we get  $\phi_L(L)=0$ .



9. The methods which we have developed may also be used to solve substitutional equations of the form

$$LX = R,$$

where  $L$  and  $R$  are given substitutional expressions.

If  $L$  possesses an inverse  $L^{-1}$ , the above equation has a unique solution

$$X = L^{-1}R.$$

Suppose then that  $L$  does not possess an inverse. There must be at least one of the matrices  $U_L$  which is singular and where minimal function has a factor  $x$ . It follows that  $\phi_L(x)$  has a factor  $x$ . Using the same notation as in previous sections, we first construct the expression  $A$  such that  $A$  has an inverse and such that the minimum function of  $M(=AL)$  has a factor  $x$  unrepeated. Writing  $S=AR$ , our equation now takes the form

$$MX = S.$$

Since  $M\psi(M)=0$ , it follows that this equation has no solutions unless  $\psi(M)S=0$ . On the other hand, we can show that if  $\psi(M)S=0$ , then solutions of the equation  $MX=S$  certainly exist. Since the constant term in  $\psi(x)$  is  $+1$ , we can write

$$\psi(M) \equiv \epsilon - M\theta(M).$$

Now

$$S - M\theta(M)S = \psi(M)S = 0,$$

from which it follows that  $X=\theta(M)S$  satisfies the equation  $MX=S$ . The following criterion has now been established.

**THEOREM 5.** *The necessary and sufficient condition that a prepared equation  $MX=S$  should have a solution is that  $\psi(M)S=0$ .*

Let us now assume that the preceding condition is satisfied. Since  $\theta(M)S$  is one solution of the equation  $MX=S$ , any other solution  $X$  of this equation must satisfy the equation

$$M(X - \theta(M)S) = 0.$$

It follows from Theorem 4 that the most general solution of this equation is

$$X - \theta(M)S = \psi(M)Y,$$

where  $Y$  is arbitrary. That is, any solution  $X$  of the equation  $MX=S$  can be expressed in the form

$$X = \theta(M)S + \psi(M)Y$$

as the sum of a particular solution  $\theta(M)S$  and a complementary function  $\psi(M)Y$ . The fact that  $X$  must consist of the sum of a particular solution and a complementary function was pointed out by Young (1900, p. 106), although he did not obtain the explicit solution derived above. The following result has now been achieved.

**THEOREM 6.** *If the minimum equation of  $M$  is  $x\psi(x) \equiv x(1-x\theta(x))=0$  and if the condition of Theorem 5 is satisfied, then the most general solution of the equation  $MX=S$  is  $X=\theta(M)S+\psi(M)Y$ , where  $Y$  is arbitrary.*

We can also deal with the simultaneous equations

$$L_1X = R_1, \quad L_2X = R_2, \quad \dots, \quad L_0X = R_0.$$

If any one of these equations has no solution, the set cannot have a solution and so we need not proceed further. If any  $L_i$  has an inverse, then  $X=L_i^{-1}R_i$  if a solution exists, but if  $L_iL_i^{-1}R_i \neq R_j$  for some  $j$ , the equations are inconsistent and have no solution. We may suppose then that each equation has solutions and that no  $L_i$  has an inverse. Let the prepared form of  $L_1X=R_1$  be  $M_1X=S_1$ . Then

$$X = \theta_1(M_1)S_1 + \psi_1(M_1)Y_1.$$

If this solution also satisfies  $L_2X=R_2$ , we find that  $Y_1$  must satisfy the equation

$$L_2\psi_1(M_1)Y_1 = R_2 - L_2\theta_1(M_1)S_1.$$

This equation may have no solution, or, if it has a solution,  $L_2\psi_1(M_1)$  may have an inverse, in which case we proceed as before. If neither of these possibilities is true, let

$$M_2Y_1=S_2$$

be the prepared form of this equation. Then

$$Y_1=\theta_2(M_2)S_2+\psi_2(M_2)Y_2$$

and

$$X=\theta_1(M_1)S_1+\psi_1(M_1)\{\theta_2(M_2)S_2+\psi_2(M_2)Y_2\}.$$

Further equations  $L_3X=R_3, \dots$  imply further conditions which must be satisfied by  $Y_2, \dots$ , but it is now clear that this process comes to an end after  $c$  steps and that the most general solution of the simultaneous equations is eventually achieved.

10. We conclude by giving two simple illustrations of the foregoing theory in the case  $n=3$ . We shall solve the equations

$$LX=0 \quad \text{and} \quad LX=R,$$

where

$$L=(23)-(31)-(321)+(123),$$

$$R=\epsilon+(12)-(23)-(123).$$

It is easily verified that  $L^2=0$ , so  $\phi_L(x)=x^2$ , and the equations are not in a prepared form. Take  $A=(23)$ . Then

$$M \equiv AL = \epsilon - (12) + (31) - (123),$$

$$S \equiv AR = -\epsilon + (23) - (31) + (321).$$

Since we can verify that  $M^2=3M$ , it follows that

$$\phi_M = x - \frac{1}{3}x^2, \quad \psi(x) = 1 - \frac{1}{3}x, \quad \theta = \frac{1}{3}.$$

Now

$$\psi(M) = \epsilon - \frac{1}{3}M = \frac{1}{3}\{2\epsilon + (12) - (31) + (123)\},$$

$$\theta(M) = \frac{1}{3}\epsilon.$$

The most general solution of  $LX=0$  is therefore

$$X = \frac{1}{3}\{2\epsilon + (12) - (31) + (123)\}Y,$$

and that of  $LX=R$  is

$$X = \frac{1}{3}\{-\epsilon + (23) - (31) + (321)\} + \frac{1}{3}\{2\epsilon + (12) - (31) + (123)\}Y,$$

where in each case  $Y$  is an arbitrary expression.

Since the coefficient of  $\epsilon$  in  $\psi(M)$  is  $\frac{1}{3}$ , the equations under consideration have  $3! \cdot \frac{1}{3}$ , i.e. 4 linearly independent solutions. These can be taken to be given by

$$Y = \epsilon, \quad (12), \quad (23), \quad (123)$$

in turn.

## SUMMARY

Substitutional equations of the type considered by the late Alfred Young are shown to be intimately related with the theory of idempotents. Any equation  $LX=0$  possessing solutions other than  $X=0$  is shown to have the same solutions as another equation  $MX=0$ , where  $M$  is obtained from  $L$  by premultiplying the latter by a suitably chosen expression  $A$  and where the minimum equation of  $M$  is  $x\psi(x)=0$ ,  $\psi(x)$  being prime to  $x$ . The expression  $\psi(M)$  is then idempotent, and it is shown that the most general solution of  $LX=0$  is  $X=\psi(M)Y$ , where  $Y$  is an arbitrary expression. The number of linearly independent solutions of  $LX=0$  is  $kn!$ , where  $k$  is the coefficient of the unit permutation in  $\psi(M)$  when that expression is expressed in terms of the permutations of the symmetric group  $S_n$ .

Corresponding results are obtained for the equation  $LX=R$ , and methods are given for solving sets of simultaneous equations of both types.

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XV.—Quantum Mechanics of Fields. III. Electromagnetic Field and Electron Field in Interaction. By Professor Max Born, F.R.S., and H. W. Peng, Ph.D., Carnegie Research Fellow, University of Edinburgh.

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# INTRODUCTION

STUDYING the interaction of different pure fields, we have been led to some essential modifications of the ideas on which our quantum mechanics of fields is based. We shall explain these here for the example of the interaction of the Maxwell and the Dirac field.

In Part I \* we showed that a pure field in a given volume  $\Omega$  can be described by considering the potentials and field components as matrices, not attached to single points in  $\Omega$  (as the theory of Heisenberg and Pauli), but to the whole volume. Further, we assumed the total energy and momentum to be the product of  $\Omega$  and the corresponding densities. In Part II † we showed that this conception has to be modified; the eigenvalues of the energy and momentum as defined in Part I represent neither the states of single particles nor of a system of particles, but of something intermediate which corresponds to the simple oscillators of Heisenberg-Pauli and which we have called *apeirons*. The total energy and momentum of the system is a sum over the contributions of an assembly of apeirons. Mathematically the differences of the quantum mechanics of a field from that of a set of mass points (as treated in ordinary quantum mechanics) is the fact that the matrices representing a field are reducible (while those representing co-ordinates of mass points are irreducible); each irreducible submatrix corresponds to an apeiron.

The considerations of Part II make it obvious that the correct theory of quantised fields must be a much closer union of mechanics and statistics than we had anticipated in Part I.

A second indication of the need for more precise definitions and modifications is now obtained from the consideration of the nature of the interaction terms in the Lagrangian. We have assumed in Part I that the Lagrangian characterising a system is given by the usual function of the field quantities taken from the classical theory. So long as one has to do with pure fields this leads to no ambiguity. In the classical theories of the photon, meson, or electron field the Lagrangian is of the second degree in the field quantities. If one introduces a Fourier transformation (transition to the momentum space) the new expressions are still of the second degree and essentially the same as the original ones. It is true, we have also considered the case of arbitrary functions (non-linear equations); but we cannot ascertain whether the general function assumed to be the Lagrangian has as arguments the field quantities in the ordinary space or in the Fourier space. Hence the general theory is formally correct but has very little concrete content.

If we now consider the coupling of two pure fields we cannot avoid a decision about the function by which we represent the interaction. In the case (Maxwell plus Dirac)-field this function is of the 3rd degree in the space representation; its Fourier transformed, also of the 3rd degree, is a quite different and much more complicated function. Which of them is the correct one? If this question has been decided, the function chosen will then be considered as a matrix function and remains as such unchanged for every canonical (matrix) transformation. Hence if we have to choose the Fourier transformed function we would maintain that this rather complicated function is also the correct expression for the quantum theory of the fields in the space representation; the simpler classical expressions would then be only approximately true, in the sense of the correspondence principle.

Now we have a guide for this decision in the fact that the Heisenberg-Pauli theory, apart from the well-known awkward divergent terms, satisfactorily represents the facts of absorption.

\* These *Proceedings*, LXII, 1944, 40.

† *Ibid.*, 92.

emission, scattering, etc. This theory is based on the Fourier representation (momentum space). Hence we have to consider this representation as fundamental, and we have to choose the Lagrangian correspondingly, replacing Fourier coefficients by matrices in the same way as in the first approach to quantum mechanics (Heisenberg-Born-Jordan).

The whole argument which leads to a new and satisfactory theory for interacting fields is based on the correspondence postulate, and in order to make this clear it seemed to us advisable to go back to first principles. Therefore we begin with the classical form of the theory and introduce then the quantisation method of Heisenberg and Pauli. It is now an easy step to replace this semi-classical procedure by our new method, which is a generalised quantum mechanics using only non-commuting quantities.

A main feature of this theory is the definition of the total energy and momentum by the traces of the matrices representing the densities; this is an obvious generalisation of the apeiron sums used in Part II. It is further necessary to generalise the commutation laws for the field quantities given in Part I in such a way that they include the commutation of any quantity with that which is produced from it by permuting the apeirons; in this manner the full correspondence with the Heisenberg-Pauli commutation laws is obtained.

That the new theory is not less satisfactory than that of Heisenberg and Pauli is obvious from its derivation. The difference can be expressed by saying that the new theory admits an arbitrary apeiron distribution, while that of Heisenberg and Pauli assumes a uniform apeiron distribution (in momentum space). Hence all results not involving this distribution will be the same, while the divergent integrals produced by the uniform distribution become now convergent and may lead to new results.

We think, however, that the new theory may have more far-reaching consequences concerning the connection of the ultimate particles. These difficult problems, which have to be considered in relation to the most general principles of quantum theory, will be postponed for another article.

# 1. THE INTRODUCTION OF THE INTERACTION BETWEEN THE ELECTRON FIELD AND THE ELECTROMAGNETIC FIELD IN GENERAL

Without interaction, let the electromagnetic field be described in general by a Lagrangian  $L'(u_{gh})$  of the field strengths

$$u_{gh} = \frac{\partial u_h}{\partial x_g} - \frac{\partial u_g}{\partial x_h}, \quad (1.1)$$

and the electron field be described by a Lagrangian  $L''$  of the spinors  $v, v^*$  and their derivatives  $v_g, v_g^*$ . The combined field, together with interaction, is then † described by the Lagrangian

$$L = L'(u_{gh}) + L''(v, v^*; v_g, v_g^*), \quad (1.2)$$

where only the definition of  $v_g$  and  $v_g^*$  is now altered so as to take account of the interaction:

$$v_g = D_g v, \quad v_g^* = D_g^* v^*; \quad D_g = \frac{\partial}{\partial x_g} + \frac{ie}{\hbar c} u_g, \quad D_g^* = \frac{\partial}{\partial x_g} - \frac{ie}{\hbar c} u_g. \quad (1.3)$$

For abbreviation let

$$U_{gh} = \frac{\partial L}{\partial u_{gh}}; \quad V^* = \frac{\partial L}{\partial v}, \quad V_g^* = \frac{\partial L}{\partial v_g} \quad (1.4)$$

and similarly for the conjugates. The variational equations with respect to  $u_h, v^*$  and  $v$  are

$$-\frac{\partial U_{gh}}{\partial x_g} + \frac{ie}{\hbar c} \{ (v V_h^*) - (V_h v^*) \} = 0. \quad (1.5)$$

$$-D_g V_g + V = 0, \quad -D_g^* V_g^* + V^* = 0. \quad (1.6)$$

† This section represents a generalisation of the introduction of the interaction with Maxwell's field given by Pauli, *Rev. Mod. Phys.*, **xiii**, 1941, 207.

In order that (1.5) may be integrable the charge-current vector

$$s_h = -\frac{ie}{\hbar c} \{ (v V_h^*) - (V_h v^*) \} \quad (1.7)$$

must satisfy the equation of continuity, which is by (1.3) and (1.6)

$$\frac{\partial s_h}{\partial x_h} = -\frac{ie}{\hbar c} \{ (v_h V_h^*) + (v V^*) - (V_h v_h^*) - (V v^*) \} = 0. \quad (1.8)$$

This is the case if the Lagrangian is invariant under the general gauge transformation

$$v \rightarrow v e^{i\gamma}, \quad u_g \rightarrow u_g - \frac{\hbar c}{e} \frac{\partial \gamma}{\partial x_g} \quad \text{and hence} \quad v_g \rightarrow v_g e^{i\gamma}, \quad (1.9)$$

where  $\gamma$  is an arbitrary real function of the co-ordinates. Then (1.8) is verified by varying  $v$ ,  $v_g$  and their conjugates according to (1.9) and demanding the total variation of  $L$  with respect to  $\gamma$  to vanish.

The energy-momentum tensor  $T_{gh}$  for the combined field including the interaction can also be partitioned into two parts in a gauge-invariant way:

$$T_{gh} = T'_{gh} + T''_{gh}, \quad (1.10)$$

$$T'_{gh} = u_{gj} U_{hj} - L' \delta_{gh}, \quad (1.11)$$

$$T''_{gh} = (v_g V_h^*) + (V_h v_g^*) - L'' \delta_{gh}. \quad (1.12)$$

The divergence of the part  $T'_{gh}$  which may be attributed to the electromagnetic field is, by using the cyclic divergence equations resulting from (1.1) and in virtue of the source term of (1.5),

$$\frac{\partial T'_{gh}}{\partial x_h} = u_{gj} \frac{\partial U_{hj}}{\partial x_h} = -u_{gj} s_j. \quad (1.13)$$

The divergence of the part  $T''_{gh}$  which may be attributed to the electrons in the presence of the electromagnetic field is, by using the identity

$$\frac{\partial}{\partial x_j} v_g V_h^* = D_j v_g \cdot V_h^* + v_g \cdot D_j^* V_h^* \quad (1.14)$$

and (1.6) and (1.4),

$$\begin{aligned} \frac{\partial T''_{gh}}{\partial x_h} &= (D_h v_g \cdot V_h^*) + (V_h \cdot D_h^* v_g) + (v_g V^*) + (V v_g^*) \\ &\quad - \left( \frac{\partial v}{\partial x_g} V^* \right) - \left( \frac{\partial v_h}{\partial x_g} V_h^* \right) - \left( V \frac{\partial v^*}{\partial x_g} \right) - \left( V_h \frac{\partial v_h^*}{\partial x_g} \right). \end{aligned} \quad (1.15)$$

By using (1.8) the  $\partial/\partial x_g$  here can be replaced by  $D_g$  where it operates on  $v$  and  $v_h$ , and by  $D_g^*$  where it operates on  $v^*$  and  $v_h^*$ . By using (1.13) and cancelling terms (1.15) becomes

$$\frac{\partial T''_{gh}}{\partial x_h} = (D_h v_g - D_g v_h, V_h^*) + (V_h, D_h^* v_g^* - D_g^* v_h^*). \quad (1.16)$$

By using (1.3) again and noting that, by (1.1),

$$D_h D_g - D_g D_h = \frac{ie}{\hbar c} u_{hg}, \quad D_h^* D_g^* - D_g^* D_h^* = -\frac{ie}{\hbar c} u_{hg}, \quad (1.17)$$

(1.16) becomes, by using also (1.7),

$$\frac{\partial T''_{gh}}{\partial x_h} = \frac{ie}{\hbar c} u_{hg} \{ (v V_h^*) - (V_h v^*) \} = -u_{hg} s_h = u_{gh} s_h. \quad (1.18)$$

This verifies the Lorentz law of force in general. The sum of (1.13) and (1.18) yields the conservation laws for the energy and momentum

$$\frac{\partial T_{gh}}{\partial x_h} = 0. \quad (1.19)$$

## 2. CHANGE OF NOTATION FOR THE PASSAGE FROM THE LAGRANGIAN FORMALISM TO THE HAMILTONIAN FORMALISM. MAXWELL'S FIELD AND DIRAC'S FIELD

In order to prepare the classical theory of § 1 for quantisation by the method of Heisenberg and Pauli or by the new method, it is necessary to pass from the Lagrangian formalism to the Hamiltonian formalism by treating the time differently from the spatial co-ordinates. It is then appropriate to use the space-vector notation.

We specialise to Maxwell's field and Dirac's field in interaction described by the Lagrangian

$$L = \frac{1}{2} u_{gh} u_{gh} + \frac{\hbar c}{2i} \{ (v_g \alpha_g v^*) - (v \alpha_g v_g^*) \} + mc^2 (v \beta v^*), \quad (2.1)$$

where  $\alpha_1, \alpha_2, \alpha_3$  and  $\beta$  are Dirac's matrices, but  $\alpha_4 = i = \sqrt{-1}$  in consequence of our use of pseudo-Euclidian metric. In space-vector notation let

$$\left. \begin{aligned} (x_1, x_2, x_3) &= \mathbf{r}, & x_4 &= ict; \\ (u_1, u_2, u_3) &= \mathbf{A}, & u_4 &= i\phi; \\ (u_{23}, u_{31}, u_{12}) &= \mathbf{H}, & (u_{41}, u_{42}, u_{43}) &= i\mathbf{E}; \\ (\alpha_1, \alpha_2, \alpha_3) &= \boldsymbol{\alpha}, & \alpha_4 &= i; \\ (s_1, s_2, s_3) &= \mathbf{j}, & s_4 &= i\rho; \\ (v_1, v_2, v_3) &= \mathbf{f} + \frac{ie}{\hbar c} \mathbf{A}v, & v_4 &= i \left( -\frac{1}{c} \frac{\partial v}{\partial t} + \frac{ie}{\hbar c} \phi v \right). \end{aligned} \right\} \quad (2.2)$$

Among the components of the energy-momentum tensor we shall need only  $T_{04}$ , which by integration over a volume  $\Omega$  give the total momentum  $\mathbf{p}$  and energy  $E$  contained in  $\Omega$ , as follows:—

$$\int_{\Omega} (T_{14}, T_{24}, T_{34}) d\tau = i c \mathbf{p}, \quad \int_{\Omega} T_{44} d\tau = -E. \quad (2.3)$$

The equations of § 1 will now be specialised for the Lagrangian (2.1) and written in the notation (2.2) and (2.3), as far as they will be needed in the following sections. They can be ordered in four groups:

(a) Equations which are merely definitions of the densities of current and charge, and the total momentum and energy:

$$\mathbf{j} = -e(v\boldsymbol{\alpha}v^*), \quad \rho = -e(vv^*), \quad (2.4), (2.5)$$

$$\mathbf{p} = \int_{\Omega} \left( \frac{1}{c} \mathbf{E} \wedge \mathbf{H} - \frac{1}{c} \mathbf{A} \rho + \frac{\hbar}{2i} \{ (\mathbf{f}v^*) - (v\mathbf{f}^*) \} \right) d\tau, \quad (2.6)$$

$$E = \int_{\Omega} \left( \frac{\mathbf{E}^2 + \mathbf{H}^2}{2} - \mathbf{A} \cdot \mathbf{j} + \frac{\hbar c}{2i} \{ (\mathbf{f} \cdot \boldsymbol{\alpha} v^*) - (v \boldsymbol{\alpha} \cdot \mathbf{f}^*) \} + mc^2 (v \beta v^*) \right) d\tau. \quad (2.7)$$

(b) The field equations in space:

$$\text{grad } v = \mathbf{f}, \quad \text{curl } \mathbf{A} = \mathbf{H}, \quad \text{div } \mathbf{E} = \rho, \quad (2.8), (2.9), (2.10)$$

(c) The field equations in time:

$$-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E} + \text{grad } \phi, \quad \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{H} - \mathbf{j}, \quad (2.11), (2.12)$$

$$-\frac{\hbar}{i} \frac{\partial v}{\partial t} = -ev\phi + \frac{\hbar c}{i} \mathbf{f} \cdot \boldsymbol{\alpha} + ev\boldsymbol{\alpha} \cdot \mathbf{A} + mc^2 v \beta. \quad (2.13)$$

(d) The equation of continuity which is a consequence of (2.4), (2.5), (2.8), and (2.13):

$$\operatorname{div} \mathbf{j} + \frac{1}{c} \frac{\partial \rho}{\partial t} = 0. \quad (2.14)$$

In addition, in order to avoid the arbitrary gradient which can be added to the vector potential  $\mathbf{A}$  according to the gauge transformation (1.9), it is practical to adopt the restriction

$$\operatorname{div} \mathbf{A} = 0 \quad (2.15)$$

for any time  $t$ . This restriction fixes the electromagnetic potentials and leaves only a constant phase factor undetermined for the electron field. The scalar potential is then determined by the density of charge according to Poisson's equation

$$\operatorname{div} \operatorname{grad} \phi = -\rho, \quad (2.16)$$

which follows easily from (2.15), (2.11), and (2.10).

### 3. SEPARATION OF THE ELECTROMAGNETIC FIELD INTO A TRANSVERSAL AND A LONGITUDINAL PART IN THE WAVE-VECTOR REPRESENTATION

It is a well-known theorem that any vector field can be decomposed into a divergence-free part plus a curl-free part. For the electromagnetic field  $\mathbf{H}$  is, by (2.9), divergence-free and  $\mathbf{A}$  is, by (2.15), chosen to be so.  $\mathbf{E}$  is not divergence-free, but its divergence is determined, by (2.10) and (2.5), by the fundamental variables of the electron field.

Hence as the fundamental variables of the total field we shall take those of the electron field plus those of the divergence-free part of the electromagnetic field, because the curl-free part of the electromagnetic field can be expressed in terms of the variables of the electron field. This can be done explicitly by resolving all field variables into their Fourier coefficients.

We enclose the field in a rectangular box and expand all the field components into three-dimensional Fourier series, assuming the usual periodic boundary conditions. For the electron field the field components are complex quantities. Let †

$$v(\mathbf{r}) = \sum_{\mathbf{l}} v_{\mathbf{l}} e^{i\mathbf{l} \cdot \mathbf{r}}, \quad \mathbf{f}^*(\mathbf{r}) = \sum_{\mathbf{l}} \mathbf{f}_{\mathbf{l}}^* e^{-i\mathbf{l} \cdot \mathbf{r}}, \quad \text{etc.} \quad (3.1)$$

The wave-vector  $\mathbf{l}$  covers the whole reciprocal space. For the electromagnetic field the field components are real. Therefore † we write

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}} (\mathbf{A}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} + \mathbf{A}_{\mathbf{k}}^* e^{-i\mathbf{k} \cdot \mathbf{r}}), \quad \text{etc.} \quad (3.2)$$

Here the wave-vector  $\mathbf{k}$  covers only half the reciprocal space in order that all the Fourier coefficients  $\mathbf{A}_{\mathbf{k}}$ 's and  $\mathbf{A}_{\mathbf{k}}^*$ 's may be independent. Then the Fourier coefficients of the divergence-free part of any vector field are perpendicular to the wave-vectors and will be called transversal, while those of the curl-free part are parallel to the wave-vectors and will be called longitudinal.

By inserting the Fourier series for  $\mathbf{E}$  and  $\rho$  in (2.10) and equating the coefficients of  $\exp(i\mathbf{k} \cdot \mathbf{r})$  on both sides we get

$$i\mathbf{k} \cdot \mathbf{E}_{\mathbf{k}} = \rho_{\mathbf{k}}. \quad (3.3)$$

Thus the longitudinal part of  $\mathbf{E}_{\mathbf{k}}$  is, by projection along the direction parallel to  $\mathbf{k}$ ,

$$\mathbf{E}_{\mathbf{k}, \text{long.}} = \mathbf{k} \frac{\mathbf{k} \cdot \mathbf{E}_{\mathbf{k}}}{k^2} = -i\mathbf{k} \frac{\rho_{\mathbf{k}}}{k^2}. \quad (3.4)$$

The transversal part of  $\mathbf{E}_{\mathbf{k}}$  is therefore

$$\mathbf{E}_{\mathbf{k}, \text{tr.}} = \mathbf{E}_{\mathbf{k}} - \mathbf{E}_{\mathbf{k}, \text{long.}} = \mathbf{E}_{\mathbf{k}} + i\mathbf{k} \rho_{\mathbf{k}} / k^2. \quad (3.5)$$

† Unlike the usual practice, no normalisation factor is introduced in the Fourier analysis. The Fourier coefficients of a quantity are then of the same physical dimension as that of the quantity itself.



We shall, however, regard this equation as the definition of  $\mathbf{E}_k$  in terms of  $\mathbf{E}_{k, \text{tr}}$  and take the latter as a fundamental variable. In fact, as will be shown immediately, the field equations for the electromagnetic field can be expressed in terms of the variables of the transversal part alone.

The Fourier coefficients of the scalar potential are determined by virtue of (2.16), by the electron field

$$\phi_k = \rho_k / k^2. \quad (3.6)$$

By this and (3.5), (2.11) yields simply

$$-\frac{1}{c} \frac{\partial \mathbf{A}_k}{\partial t} = \mathbf{E}_k + i\mathbf{k}\phi_k = \mathbf{E}_k + i\mathbf{k} \frac{\rho_k}{k^2} = \mathbf{E}_{k, \text{tr}}. \quad (3.7)$$

From (2.12) follows the vector equation

$$\frac{1}{c} \frac{\partial \mathbf{E}_k}{\partial t} = i\mathbf{k} \wedge \mathbf{H}_k - \mathbf{j}_k, \quad (3.8)$$

but the longitudinal part of this is, by (3.3), nothing but the equation of continuity

$$\frac{1}{c} \frac{\partial \rho_k}{\partial t} = -i\mathbf{k} \cdot \mathbf{j}_k, \quad (3.9)$$

which involves only the variables of the electron field, and is a consequence of the field equations for the latter. Hence (3.8) is, by (3.9), equivalent to

$$\frac{1}{c} \frac{\partial \mathbf{E}_{k, \text{tr}}}{\partial t} = i\mathbf{k} \wedge \mathbf{H}_k - \mathbf{j}_k + \mathbf{k} \frac{\mathbf{k} \cdot \mathbf{j}_k}{k^2}. \quad (3.10)$$

The other variables,  $\mathbf{H}_k$  and  $\mathbf{A}_k$ , are entirely transversal. Either may be taken as a fundamental variable, the other can then be expressed in terms of it. By (2.9),  $\mathbf{H}_k$  can be expressed in terms of  $\mathbf{A}_k$  thus

$$\mathbf{H}_k = i\mathbf{k} \wedge \mathbf{A}_k. \quad (3.11)$$

Conversely, this can be solved for  $\mathbf{A}_k$  (because the latter is transversal) and yields

$$\mathbf{A}_k = i\mathbf{k} \wedge \mathbf{H}_k / k^2. \quad (3.12)$$

After inserting in (2.6) the Fourier series for the field quantities the integration gives for the total momentum

$$\mathbf{p} = \Omega \left( \sum_k \frac{1}{c} \{ \mathbf{E}_k \wedge \mathbf{H}_k^* + \mathbf{E}_k^* \wedge \mathbf{H}_k - \mathbf{A}_k \rho_k^* - \mathbf{A}_k^* \rho_k \} + \frac{\hbar}{2i} \sum_l \{ (\mathbf{f}_l \mathbf{v}_l^*) - (\mathbf{v}_l \mathbf{f}_l^*) \} \right). \quad (3.13)$$

By (3.12) and (3.5) this becomes the sum of contributions by the transversal electromagnetic field and the electron field without any interaction term,

$$\mathbf{p} = \Omega \left( \sum_k \frac{1}{c} \{ \mathbf{E}_{k, \text{tr}} \wedge \mathbf{H}_k^* + \mathbf{E}_{k, \text{tr}}^* \wedge \mathbf{H}_k \} + \frac{\hbar}{2i} \sum_l \{ (\mathbf{f}_l \mathbf{v}_l^*) - (\mathbf{v}_l \mathbf{f}_l^*) \} \right). \quad (3.14)$$

However, the total energy  $E$  still contains interaction terms:

$$E = \Omega \left( \sum_k \{ \mathbf{E}_{k, \text{tr}} \cdot \mathbf{E}_{k, \text{tr}}^* + \mathbf{H}_k \cdot \mathbf{H}_k^* + \rho_k \rho_k^* / k^2 - \mathbf{A}_k \cdot \mathbf{j}_k^* - \mathbf{A}_k^* \cdot \mathbf{j}_k \} + \frac{\hbar c}{2i} \sum_l \{ (\mathbf{f}_l \cdot \boldsymbol{\alpha} \mathbf{v}_l^*) - (\mathbf{v}_l \boldsymbol{\alpha} \cdot \mathbf{f}_l^*) \} + mc^2 \sum_l (\mathbf{v}_l \beta \mathbf{v}_l^*) \right). \quad (3.15)$$

The equations (2.4), (2.5), (2.8), and (2.13), which are mainly concerned with the electron field, become, in terms of the Fourier coefficients,

$$\mathbf{j}_k = -e \sum_l (\mathbf{v}_{l+k} \boldsymbol{\alpha} \mathbf{v}_l^*), \quad \rho_k = -e \sum_l (\mathbf{v}_{l+k} \mathbf{v}_l^*), \quad (3.16), (3.17)$$

$$i\mathbf{l} \mathbf{v}_l = \mathbf{f}_l, \quad (3.18)$$

$$-\frac{\hbar}{i} \frac{\partial \mathbf{v}_l}{\partial t} = -e \sum_k \{ \phi_k \mathbf{v}_{l-k} + \mathbf{v}_{l+k} \phi_k^* \} + \frac{\hbar c}{i} \mathbf{f}_l \cdot \boldsymbol{\alpha} + e \sum_k \{ \mathbf{v}_{l-k} \mathbf{A}_k \cdot \boldsymbol{\alpha} + \mathbf{v}_{l+k} \mathbf{A}_k^* \cdot \boldsymbol{\alpha} \} + mc^2 \mathbf{v}_l \beta. \quad (3.19)$$

The total charge  $q$  contained in the volume  $\Omega$  is, by integrating (2.5),

$$q = -e\Omega \sum_l (v_l v_l^*). \quad (3.20)$$

#### 4. QUANTISATION OF MAXWELL'S FIELD AND DIRAC'S FIELD IN INTERACTION BY THE METHOD OF HEISENBERG AND PAULI

##### (a) Simple Treatment

A simple and practical way of applying the method of Heisenberg and Pauli for the quantisation of a field—in our case the combined field of Maxwell and Dirac—is the following: after the field quantities have been resolved into their Fourier coefficients (what we have done in § 3), the field is treated as an assembly of oscillators characterised by the wave-vectors. Then the field equations in time (§ 2 (c)) give rise to the equations of motion of the oscillators, viz. in our case (3.7), (3.10), (3.19) and their adjoint equations. The canonical variables, which can be read off from the time-derivative terms of these equations, are the following conjugate pairs:—

$$v_l, v_l^*; \quad \mathbf{A}_k, \mathbf{E}_{k, \text{tr.}}^*; \quad \mathbf{E}_{k, \text{tr.}}, \mathbf{A}_k^*.$$

The equations which arise from space-derivatives are, according to this view, to be regarded partly as definitions of some auxiliary variables (namely, (3.11) as the definition of  $\mathbf{H}_k$  and (3.18) of  $\mathbf{f}_l$ ) and partly as constraints among the above canonical variables (namely,

$$i\mathbf{k} \cdot \mathbf{A}_k = 0, \quad i\mathbf{k} \cdot \mathbf{E}_{k, \text{tr.}} = 0 \quad (4.1)$$

and their adjoint equations). Quantisation consists in considering all the canonically conjugate pairs of variables as  $q$ -numbers which satisfy simple commutation or anti-commutation laws, as follows.

For the spinor components ( $s$  denoting the spinor index) anti-commutation laws hold †

$$[v_{ls}, v_{ls}^*]_+ \equiv v_{ls} v_{ls}^* + v_{ls}^* v_{ls} = 1/\Omega. \quad (4.2)$$

All other anti-commutators vanish.

For the vector components let the axes be chosen so that  $\mathbf{k}$  lies on the  $x$ -axis. The  $x$ - and  $y$ -components of  $\mathbf{A}_k$  and  $\mathbf{E}_{k, \text{tr.}}^*$  satisfy the commutation laws

$$[A_{kx}, E_{kx, \text{tr.}}^*] \equiv E_{kx, \text{tr.}}^* A_{kx} - A_{kx} E_{kx, \text{tr.}}^* = i\hbar c/\Omega, \quad [A_{ky}, E_{ky, \text{tr.}}^*] = i\hbar c/\Omega, \quad (k_x = k_y = 0). \quad (4.3)$$

All other commutators vanish.

By an arbitrary rotation of the co-ordinate axes (4.3) becomes in general ‡

$$[A_{kx}, E_{ky, \text{tr.}}^*] = \frac{i\hbar c}{\Omega} \left( \delta_{xy} - \frac{k_x k_y}{k^2} \right). \quad (4.4)$$

All the vector components of  $\mathbf{A}_k$ ,  $\mathbf{E}_{k, \text{tr.}}^*$ ,  $\mathbf{E}_{k, \text{tr.}}$  and  $\mathbf{A}_k^*$  commute with all the spinor components of  $v_l$  and  $v_l^*$ .

The quantised equations of motion are

$$\frac{\partial F}{\partial t} = -\frac{i}{\hbar} [E, F], \quad F = \mathbf{A}_k, \mathbf{E}_{k, \text{tr.}}^*; \mathbf{E}_{k, \text{tr.}}, \mathbf{A}_k^*; v_l, v_l^*; \quad (4.5)$$

where the total energy  $E$  of the assembly of oscillators is given by the  $q$ -number expression § of (3.15). The close formal analogy between the classical and quantum equations of motion can be demonstrated by working out the commutators of (4.5). If  $F$  is a component of  $\mathbf{A}_k$ , in (3.15) only the term  $\mathbf{E}_{k, \text{tr.}} \cdot \mathbf{E}_{k, \text{tr.}}^*$  does not commute with  $\mathbf{A}_k$ . By (4.4) and (4.1) the

† Jordan and Wigner, *Zeits. f. Physik*, XLVII, 1928, 631.

‡ Novobatzky, *loc. cit.*

§ Since the order of factors can be arbitrarily changed in a  $c$ -number expression but not so in a  $q$ -number expression the  $q$ -number expression of (3.15) (as well as that of (3.14), etc.) is slightly ambiguous. This has no effect on the anti-commutation laws or commutation laws. The zero-point momentum and charge, however, can be avoided by taking the mean of the two possible  $q$ -number expressions as discussed in Part I.

commutator has the value

$$[E, A_k] = -i\hbar c E_{k, \text{tr}}. \quad (4.6)$$

Hence (4.5) with  $F = A_k$  is formally identical with (3.7). If  $F = E_{k, \text{tr}}$ , the relevant term of (3.15) can be written, using the adjoint of (3.11), in the form

$$H_k \cdot H_k^* - A_k^* \cdot j_k = (ik \wedge H_k) \cdot A_k^* - A_k^* \cdot j_k. \quad (4.7)$$

With the help of the adjoint of (4.4), (4.5) with  $F = E_{k, \text{tr}}$  is seen to be formally identical with (3.10). Making use of (3.6), (3.16), (3.17), (3.18), their adjoint equations and (4.2) the quantised equation of motion (4.5) for  $v_i$  is formally identical with (3.19), if the order of the factors  $\phi$  and  $v$  in the latter equation is properly adjusted.

Instead of working with the Heisenberg representation (4.5) one can consider the variables of (4.5) as operators satisfying (4.2) and (4.4) and describe the field (which is here treated as an assembly of oscillators) by the Schrodinger wave function  $\Psi$ .  $\Psi$  contains, besides the independent variable  $t$ , the variables on which the operators of (4.5) act. One then replaces (4.5) by the wave equation

$$\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = -E\Psi, \quad (4.8)$$

where the Hamiltonian  $E$  is the operator expression of (3.15).

#### (b) Complete Treatment

In the complete and rigorous application of Heisenberg and Pauli's method for the quantisation of a field the equations (2.8), (2.9), (2.10), and (2.15), where only space derivatives appear, are to be treated on the same footing as the equations containing time-derivatives. The quantised equations of motion in space supplement those in time (4.5), namely

$$\text{grad } F = \frac{i}{\hbar} [p, F]; \quad F = F(\mathbf{r}), \text{ the field variables at the point } \mathbf{r}. \quad (4.9)$$

$p$  denotes the total momentum of the field contained in the volume  $\Omega$ . If the Fourier series (3.1) and (3.2) for the field variables are used and their Fourier coefficients considered as  $q$ -numbers (the  $\mathbf{r}$  in field theories is a  $c$ -number which may be compared to the time  $t$  in the quantum mechanics of particles) (4.9) becomes

$$[p, F_i] = \hbar l F_i, \quad F_i = v_i, f_i; \quad [p, F_k] = \hbar k F_k, \quad F_k = A_k, E_{k, \text{tr}}, H_k, j_k, \rho_k. \quad (4.10)$$

One has now to add to the canonical variables considered in the simple treatment the quantities  $f_i, H_k$  and their adjoints as fundamental field variables. The following additional commutation and anti-commutation laws containing the additional field variables have to hold and to be considered as fundamental as those given above, (4.2) and (4.4):

$$[f_{is}, v_{is}^*]_+ = -[f_{is}^*, v_{is}]_+ = i1/\Omega, \quad [f_{is} f_{is}^*]_+ = I_{is} I_{is}^*/\Omega; \quad (4.11)$$

$$[H_{kx}, E_{ky, \text{tr}}^*] = -[H_{ky}, E_{kx, \text{tr}}^*] = [E_{ky, \text{tr}}, H_{kx}^*] = -[E_{kx, \text{tr}}, H_{ky}^*] = \hbar c k_z/\Omega. \quad (4.12)$$

It is understood that the vector indices  $x, y, z$  in (4.12) may be cyclically permuted. With  $p$  given by the  $q$ -number expression of (3.14), (4.10) follows directly from the totality of commutation and anti-commutation laws if one chooses  $F_i = v_i, f_i$  and  $F_k = A_k, E_{k, \text{tr}}, H_k$ . By using the definition (3.16) and (3.17), and also (4.10) with  $F_i = v_i$  together with the adjoint equation, one sees that (4.9) holds for  $F_k = j_k$  and  $\rho_k$ ; for instance

$$\begin{aligned} [p, \rho_k] &= -e \sum_i \{ ([p, v_{i+k}] v_i^*) + (v_{i+k} [p, v_i^*]) \} \\ &= -e \sum_i \{ \hbar(1+k)(v_{i+k} v_i^*) - (v_{i+k} v_i^*) \hbar 1 \} = \hbar k \rho_k. \end{aligned} \quad (4.13)$$

By (4.2) and (4.11) the combination  $f_{is} - i1 v_{is}$  anti-commutes with all the field variables of the electron field and commutes with those of the electromagnetic field. Hence it vanishes. By (4.4) and (4.10) the combinations  $H_k - ik \wedge A_k, ik \cdot E_{k, \text{tr}}$  and  $ik \cdot A_k$  commute with all

field variables. Hence these expressions commute with the total momentum and the total energy of the field; therefore they vanish in virtue of (4.9) and properly chosen boundary conditions, or in virtue of (4.5) and proper initial conditions.

Thus in the complete treatment (3.18), (3.11), and (4.1) as  $q$ -number equations result from the commutation and anti-commutation laws. In virtue of (4.10) and (4.9) these equations can be written in a way formally identical with the classical equations (2.8), (2.9), (2.10), and (2.15).

The demonstration of the formal identity between the classical and quantum equation of motion in time can now be carried out in the same manner as in the simple treatment.

It follows from (4.10) and its adjoint that the components of the momentum and the energy all commute with each other.

$$[\dot{p}_x, \dot{p}_y] = 0, \text{ etc.}, \quad [\mathbf{p}, E] = 0. \quad (4.14)$$

In the Schroedinger representation (4.5) and (4.9) are to be replaced by the equations

$$\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = -E\Psi, \quad \frac{\hbar}{i} \text{grad } \Psi = \mathbf{p}\Psi, \quad (4.15)$$

where  $E$  and  $\mathbf{p}$  are the operator expressions of (3.15) and (3.14). Because of (4.14) the wave equations (4.15) are compatible.

## 5. NEW METHOD OF QUANTISATION

The Heisenberg-Pauli method developed in the last section can be considered as semi-classical; while it uses the classical representation of the field variables by Fourier series (3.1) and (3.2) which are ordinary functions of the position vector  $\mathbf{r}$ , it considers the Fourier coefficients not as functions of time, but as  $q$ -numbers. Now we proceed to a method of complete quantisation in which space and time are treated on the same footing. We discard the Fourier series as a sum of terms and replace for each field variable the set of its Fourier coefficients by an array of elements which form a more complicated matrix † than that needed to represent the individual  $q$ -number Fourier coefficients by matrices in the Heisenberg and Pauli quantisation. The total matrix belongs to the volume  $\Omega$  as a whole and contains, as will soon appear, the totality of information about the field variable it represents throughout this volume. The only non-vanishing anti-commutation and commutation laws for the total matrices are those given in Part I, § 5 and § 6 (equations (5.13), (5.18), and (6.28)); they are the following ( $s$  denoting the spinor index):—

$$[v_s, v_s^*]_+ = 1/\Omega, \quad (5.1)$$

$$[f_s, v_s^*]_+ = -[f_s^*, v_s]_+ \equiv i1/\Omega, \quad f_{sz} f_y^*]_+ = l_x l_y \Omega, \quad (5.2)$$

$$[H_x, E_{y, \text{tr}}^*] = -[H_y, E_{x, \text{tr}}^*] = [E_{y, \text{tr}}, H_x^*] = -[E_{x, \text{tr}}, H_y^*] \equiv \hbar c k_x / \Omega, \quad (5.3)$$

where  $x, y, z$  may be cyclically permuted;

$$[A_x, E_{y, \text{tr}}^*] = [A_x^*, E_{y, \text{tr}}] = \frac{i\hbar c}{\Omega} \{\delta_{xy} - (k_x^2 + k_y^2 + k_z^2)^{-1} k_x k_y\}. \quad (5.4)$$

(5.2) and (5.3) contain also the definitions of the new self-adjoint variables  $\mathbf{l}$  and  $\mathbf{k}$  thus introduced. These variables, like all the field variables, are represented by total matrices and are not to be confused with the  $c$ -number wave-vectors used in § 3 or § 4 for Fourier analysis. As shown in Part I, § 6, the three components of  $\mathbf{l}$  commute with all the field variables of the electron field by virtue of the anti-commutation laws (5.1) and (5.2) only. By (5.2) again they commute among themselves. In the present case, because the field variables of the electron field all commute with those of the transversal electromagnetic field,  $l_x, l_y$ , and  $l_z$  commute also with all variables of the transversal electromagnetic field. Similarly, as a consequence of (5.3) and (5.4),  $k_x, k_y$ , and  $k_z$  commute with all field variables, among themselves and also with  $l_x, l_y, l_z$ .

† Such a matrix will be referred to in what follows as a "total matrix."

Since all the field variables commute with  $\hat{k}_x, \hat{k}_y, \hat{k}_z, \hat{l}_x, \hat{l}_y$ , and  $\hat{l}_z$  the set of matrices which represent them is reducible. In the representation where  $\hat{k}_x, \hat{k}_y, \hat{k}_z, \hat{l}_x, \hat{l}_y$ , and  $\hat{l}_z$  are simultaneously diagonal the total matrices for all the field variables will appear as being composed of submatrices placed along the diagonal. The representation is the *direct product* of the two representations for the two pure fields considered in Part I, § 5 and § 6, separately. In this representation we have

$$\mathbf{k}_{k'v, k''v} = \mathbf{k}' \delta_{k'k''} \delta_{vv'}, \quad \mathbf{l}_{k'v, k''v} = \delta_{k'k''} \mathbf{l}' \delta_{vv'}; \quad (5.5)$$

$$\mathbf{F}_{k'v, k''v} = \mathbf{F}_k \delta_{k'k''} \delta_{vv'} \quad \text{for } \mathbf{F} = \mathbf{H}, \mathbf{E}_{\text{tr}}, \mathbf{A}, \mathbf{j}, \rho \quad \text{and their adjoints}; \quad (5.6)$$

$$\mathbf{F}_{k'v, k''v} = \delta_{k'k''} \mathbf{F}_v \delta_{vv'} \quad \text{for } \mathbf{F} = v, \mathbf{f} \quad \text{and their adjoints}; \quad (5.7)$$

where the submatrices  $\hat{k}_x, \hat{k}_y, \hat{k}_z, \hat{l}_x, \hat{l}_y$ , and  $\hat{l}_z$  are scalar (*i.e.* a number multiplied by a unit matrix) and the submatrices  $\mathbf{F}_k$  or  $\mathbf{F}_v$  will be shown to be the same as those matrices used in the Heisenberg and Pauli quantisation to represent the corresponding  $q$ -number Fourier coefficients.

As in the Heisenberg and Pauli quantisation where it is convenient to treat the field as an assembly of oscillators each of which is described by the Fourier coefficients belonging to a wave-vector, so in the new method of quantisation it is convenient to treat the field as an assembly of *apeirons* each of which is described by the submatrices belonging to an eigenvalue of the total matrices  $\mathbf{k}$  or  $\mathbf{l}$  introduced by (5.3) and (5.2). Yet in the new method of quantisation the eigenvalue  $\mathbf{k}'$  or  $\mathbf{l}'$  needs only to assume a *selection* of the possible values of the wave-vector, while all of them are automatically included in the Heisenberg and Pauli quantisation where the wave-vector is introduced by means of Fourier analysis.

The non-vanishing anti-commutation laws and commutation laws obtained by taking the diagonal submatrices  $\hat{k}'l', \hat{k}l'$  of (5.1), (5.2), (5.3), and (5.4) coincide exactly with those of Heisenberg and Pauli's Fourier coefficients, namely (4.2), (4.10), (4.11), and (4.4), except that now  $\hat{k}'$  and  $l'$  are written in place of the former  $\hat{k}$  and  $l$ . The vanishing anti-commutation and commutation brackets obtained by taking the non-diagonal submatrices  $\hat{k}'l', \hat{k}l''$  (either  $\mathbf{k}' \neq \mathbf{k}''$  or  $l' \neq l''$  or both) of (5.1), (5.2), (5.3), and (5.4) are trivial identities because no field quantity, by virtue of its being reducible, contains any non-vanishing non-diagonal submatrices.

In order to obtain the full correspondence of the commutation laws and anti-commutation laws for the submatrices with those for the Fourier coefficients we have to supplement (5.1) by

$$[\mathbf{P}v, \mathbf{P}^*, v_s^*]_+ = 0, \quad (5.8)$$

where  $\mathbf{P}$  denotes any permutation matrix permuting all the  $l'$ -apeirons. It is sufficient, however, to take  $\mathbf{P}$  to be the set of cyclic permutations. (5.2), (5.3), and (5.4) are to be supplemented in a similar way.

Let  $\mathbf{I}$  denote the unit matrix  $\delta_{k'k''}$  and  $\text{trace}_{(k)}$   $\mathbf{A}$  denote the sum  $\sum_{k'} \mathbf{A}_{k'k'}$ . Then the operation  $\mathbf{I} \text{ trace}_{(k)}$  produces a scalar matrix of the same number of rows and columns as the matrix on which it operates. Let  $\mathbf{J} \text{ trace}_{(l)}$  have a similar significance where  $\mathbf{J}$  denotes the unit matrix  $\delta_{l'l''}$ . Corresponding to (3.14) and (3.15), which express the total momentum and energy as summations over the Fourier coefficients, we use now the summations over the submatrices, namely

$$\mathbf{p} = \Omega \left( \frac{1}{c} \mathbf{I} \text{ trace}_{(k)} \{ \mathbf{E}_{\text{tr}} \wedge \mathbf{H}^* + \mathbf{E}_{\text{tr}}^* \wedge \mathbf{H} \} + \frac{\hbar}{2z} \mathbf{J} \text{ trace}_{(l)} \{ (f v^*) - (v f^*) \} \right). \quad (5.9)$$

$$\begin{aligned} \mathbf{E} = \Omega \left( \mathbf{I} \text{ trace}_{(k)} \{ \mathbf{E}_{\text{tr}} \cdot \mathbf{E}_{\text{tr}}^* + \mathbf{H} \cdot \mathbf{H}^* + \rho (\hat{k}_x^2 + \hat{k}_y^2 + \hat{k}_z^2)^{-1} \rho^* - \mathbf{A} \cdot \mathbf{j}^* - \mathbf{A}^* \cdot \mathbf{j} \} \right. \\ \left. + \frac{\hbar c}{2z} \mathbf{J} \text{ trace}_{(l)} \{ (f \cdot \alpha v^*) - (v \alpha \cdot f^*) \} + m c^2 \mathbf{J} \text{ trace}_{(l)} (v \beta v^*) \right), \end{aligned} \quad (5.10)$$

The density  $\rho$  and current  $\mathbf{j}$  are now defined by

$$\mathbf{j} = -e \mathbf{J} \text{ trace}_{(l)} (C v C^* \alpha v^*), \quad \rho = -e \mathbf{J} \text{ trace}_{(l)} (C v C^* v^*), \quad (5.11), (5.12)$$

where the matrix  $C$  satisfies the following conditions:—

$$\mathbf{k}C = C\mathbf{k}, \quad C\mathbf{l} - \mathbf{l}C = \mathbf{k}C, \quad CC^* = \mathbf{1}. \quad (5.13)$$

It follows from (5.13) that  $C_{\mathbf{k}'\mathbf{l}', \mathbf{k}''\mathbf{l}''}$  vanishes except if  $\mathbf{k}' = \mathbf{k}''$  and  $\mathbf{l}'' - \mathbf{l}' = \mathbf{k}'$  when it is of modulus unity. Hence we have

$$(CvC^*)_{\mathbf{k}'\mathbf{l}', \mathbf{k}''\mathbf{l}''} = \delta_{\mathbf{k}'\mathbf{l}', \mathbf{k}''\mathbf{l}''} + \delta_{\mathbf{k}'\mathbf{l}' + \mathbf{k}''\mathbf{l}''} \quad (5.14)$$

and the correspondence of (5.11) and (5.12) with (3.16) and (3.17) is apparent.

The total charge is

$$q = -e\Omega J \text{ trace}_{(1)}(vv^*). \quad (5.15)$$

The correspondence of the submatrices of the new method of quantisation with the Fourier coefficients of the Heisenberg and Pauli quantisation is so close that the demonstration of the field equations given in § 4 for the Fourier coefficients may be now taken over for the submatrices with corresponding formal change, which is too obvious to be repeated here.

In the Schroedinger representation (4.15) holds with  $E$  and  $\mathbf{p}$  now considered as the operator expression of (5.10) and (5.9).

Since the total matrices representing  $\mathbf{k}$  and  $\mathbf{l}$  commute with those representing the energy  $E$  and the momentum  $\mathbf{p}$ ,  $\mathbf{k}$  and  $\mathbf{l}$  are constants of motion in time and space. That is, the distribution of both the electromagnetic and the electronic apeirons (which is described by the values  $\mathbf{k}'$ ,  $\mathbf{k}''$ , etc., and  $\mathbf{l}'$ ,  $\mathbf{l}''$ , etc., that actually occur as submatrices of  $\mathbf{k}$  and  $\mathbf{l}$ , cf. Part II) remains the same throughout space and time in spite of the interaction between Maxwell's field and Dirac's field. The interaction affects only the number of quanta occupying the apeirons, the quanta being electrons and photons respectively. Hence our attempt made in Part II to determine the apeiron distribution by statistical considerations cannot be based on the theory in its present form, which is very likely only provisional. Then there may be another way, also mentioned in Part II, to determine this distribution, namely by studying its effect on the self-energies of the quanta and the transition probabilities of collision processes.

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XVI.—Studies in Practical Mathematics. IV. On Linear Approximation by Least Squares. By A. C. Aitken, D.Sc., F.R.S., Mathematical Institute, University of Edinburgh.

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I. INTRODUCTORY

R. FRISCH, in a paper (Frisch, 1928) on correlation and scatter in statistical variables, made an extensive use of matrices, and in particular of the *moment matrix*, as he called it, of a set of variables. The matrices were square arrays, with an equal number of rows and columns. This paper of Frisch pointed the way to an even more extensive use of the algebra of matrices in problems of statistics.

What Frisch called the *moment matrix* may perhaps be more suitably called, nowadays, the *variance matrix* of a set or vector of variates, since the moments in question are all variances or covariances. In the present paper, which is illustrative of matrix methods, we explore the familiar ground of linear approximation by Least Squares, making full use of the properties of the variance matrix. We also study the linear transformations that convert crude data into smoothed or graduated values, or into residuals, or into coefficients in a linear representation by chosen functions.

The problems (i) of obtaining by Least Squares the best solution of a set of inconsistent linear equations

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = u_i, \quad i = 1, 2, \dots, m, \quad m > n, \quad (1)$$

where the  $u_i$  are subject to error, and (ii) of representing a set of data  $u(x_i)$  by Least Squares in the form

$$y(x_i) = a_0 + a_1p_1(x_i) + a_2p_2(x_i) + \dots + a_kp_k(x_i), \quad i = 1, 2, \dots, n, \quad n \geq k+1, \quad (2)$$

where the  $p_j(x_i)$  are values of  $k$  prescribed functions, are in essence the same, though differences of notation are apt to conceal this fact and to perplex the beginner. In (1) the  $a_{ij}$  are known constants, the  $u_i$  are the observations affected by error, the  $x_i$  are the unknowns. In (2) the functional values  $p_j(x_i)$  are known constants, being for example polynomial, or harmonic, or arbitrary, but in every case belonging to some prescribed basis of  $k+1$  linearly independent functions  $1, p_1(x), p_2(x), \dots, p_k(x)$ ; the  $u(x_i)$  are the observations over the set of values of  $x$ , the coefficients  $a_j$  are the unknowns. The values  $u(x)$  may be imagined as plotted ordinates, the  $y(x)$  as corresponding ordinates on the approximating or *graduated* curve.

In the notation of matrices the observational equations in (1) are  $Ax = u$  and in (2) are  $Pa = u$ . Here  $A$  is a matrix  $[a_{ij}]$ ,  $x$  and  $u$  are column vectors, while  $P$  is a matrix  $[p_j(x_i)]$ , and  $a$  is a column vector of elements  $a_j$ . We shall first consider  $Ax = u$ , always observing that by a simple change of lettering we have analogues for the case  $Pa = u$ .

2. THE VARIANCE MATRIX AND FUNDAMENTAL LEMMAS

Let  $x$  be a column vector having  $n$  variates  $x_i$  as its elements. Let the mean value of each  $x_i$  be taken as origin, so that the  $x_i$  are deviations from means. Then  $xx'$  is a symmetric matrix  $[x_ix_j]$  of order  $n \times n$  and rank 1. Since the mean value of  $x_ix_j$  is the product moment or covariance  $\rho_{ij}\sigma_i\sigma_j$ , where  $\rho_{ij}$  is the correlation coefficient of  $x_i$  and  $x_j$ , we construct the mean value of the matrix  $[xx']$ , namely

$$E(xx') = [\rho_{ij}\sigma_i\sigma_j] = V \quad (1)$$

and call it the *variance matrix*  $V$  of the  $x_i$ , or simply of the vector  $x$ . In general  $V$  is positive definite, but in certain cases it may be non-negative definite of less than full rank.

**Lemma 1.**—Let the  $x_i$  be linearly transformed to new variates  $y_j$  by  $y = Hx$ , where  $H$  is in general a rectangular matrix. Then the variance matrix of  $y$  is

$$E(Hxx'H) = HVH'. \quad (2)$$

Thus  $V$  is transformed like the matrix of a quadratic form, except that  $H$  and  $H'$  appear in reversed order. This simple Lemma is of great value.

We make use also of the important

**Lemma 2.**—Let  $B$  be a variable rectangular matrix with fewer rows than columns, let  $P$  and  $A$  be given matrices such that  $BA = P$ , and let  $V$  be a positive definite symmetric matrix of such an order that  $A'V^{-1}A$  can be constructed. Then the trace (sum of diagonal elements) of  $BVB'$  is a minimum (or, alternatively, the diagonal elements of  $BVB'$  attain independently their minimum values) provided that

$$B = P(A'V^{-1}A)^{-1}A'V^{-1}. \quad (3)$$

This was proved by the author in an earlier paper (Aitken, 1934).

By Lemma 2 we may treat the inconsistent equations  $Ax = u$ , postulating that the optimal value of each  $x_i$  is a consistent linear function of the  $u_i$ , of minimum variance. Let us at this stage generalize the problem by supposing that the  $u_i$  are not uncorrelated but have variance matrix  $V$ . We have then to express  $x$  in the form  $Bu$ , and so the variance matrix of  $x$  is  $BVB'$ . The diagonal elements of  $BVB'$  are thus each to be made a minimum subject to the condition of consistency, and this condition, since  $Ax = u$  gives  $Bu = BAx$ , must be that  $BA = I$ . Hence, by Lemma 2,

$$B = (A'V^{-1}A)^{-1}A'V^{-1}, \quad (4)$$

and so the vector of solutions  $x$  is obtained by solving

$$A'V^{-1}Ax = A'V^{-1}u. \quad (5)$$

These are indeed, in matrix notation, the *normal equations* derived from correlated observations. They also give the vector  $x$  which makes the positive definite quadratic form  $(u - Ax)'V^{-1}(u - Ax)$  a minimum. We shall call this the *residual quadratic*, since  $u - Ax$  is the vector of residuals. For observations of unit weight and uncorrelated we have  $V = I$ , and the normal equations take the simple shape  $A'Ax = A'u$ . For observations uncorrelated and of weights  $w_i = 1/\sigma_i^2$  the equations are  $A'WAx = A'Wu$ , where  $W$  is a diagonal matrix with elements  $w_i$  in the diagonal.

**The Variance Matrix of the Solutions.**—By Lemma 1 the variance matrix of the solutions  $x$  is

$$(A'V^{-1}A)^{-1}A'V^{-1}.V.V^{-1}A(A'V^{-1}A)^{-1} = (A'V^{-1}A)^{-1}. \quad (6)$$

This result, that the variance matrix of the solutions is the reciprocal of the matrix of the normal equations, contains the classic result of Gauss on the weights of the solutions, a result that in modern notation would be expressed thus: the weights of the solutions (in the uncorrelated case) are the reciprocals of the diagonal elements of  $(A'WA)^{-1}$ .

**The Variance Matrix of the Residuals.**—The vector of residuals is

$$u - Ax = \{I - A(A'V^{-1}A)^{-1}A'V^{-1}\}u, \quad (7)$$

and so by Lemma 1 the variance matrix of the residuals is

$$\{I - A(A'V^{-1}A)^{-1}A'V^{-1}\}V\{I - A(A'V^{-1}A)^{-1}A'V^{-1}\}' = V - A(A'V^{-1}A)^{-1}A'. \quad (8)$$

For uncorrelated observations all of unit weight it is

$$I - A(A'A)^{-1}A'. \quad (9)$$

In this latter case we may easily find the mean value of the sum of the squared residuals. For if the variance matrix just given is  $T$ , we note at once that  $T^2 = T$ , so that  $T$  is idempotent and satisfies the reduced characteristic or minimal equation  $\lambda^2 - \lambda = 0$ . Its latent roots are thus exclusively 1 and 0. But  $A(A'A)^{-1}A'$  is also idempotent, and of rank  $n$ . Hence its trace, being equal to the sum of its latent roots, is  $n$ , and so the trace of  $T$  is  $m - n$ . It



follows that if each  $u$  is of variance  $\sigma^2$ , then the mean value of the sum of squared residuals is  $(m-n)\sigma^2$ . If  $\sigma^2$  is to be estimated from the sum of squared residuals, then  $m-n$  is the divisor to use. This is a classical result, first proved by Gauss.

*The Residual Quadratic.*—The residual quadratic itself, namely  $(u - Ax)' V^{-1}(u - Ax)$ , can be expressed in a variety of ways, for by referring to

$$A' V^{-1} A x = A' V^{-1} u$$

we may transform it to

$$u' V^{-1}(u - Ax), \text{ or } u' V^{-1} u - x' A' V^{-1} A x, \quad (10)$$

or

$$u' V^{-1} u - u' V^{-1} A (A' V^{-1} A)^{-1} A' V^{-1} u. \quad (11)$$

The uncorrelated cases of these are obtained by putting  $V=I$ , and are well known, though differently expressed, in the classical literature of Least Squares. The last form given above can be written as a quotient of two symmetric determinants and can be expanded in a series of some interest. We take  $V=I$ . Since a quadratic form involving a reciprocal matrix can be written as the quotient of a bordered determinant by the cofactor of its leading element, we have here

$$u'u - u'A(A'A)^{-1}A'u = \frac{\begin{vmatrix} u'u & u'a & u'b & \dots & u'h \\ a'u & a'a & a'b & \dots & a'h \\ b'u & b'a & b'b & \dots & b'h \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ h'u & h'a & h'b & \dots & h'h \end{vmatrix}}{\begin{vmatrix} a'a & a'b & \dots & a'h \\ b'a & b'b & \dots & b'h \\ \cdot & \cdot & \cdot & \cdot \\ h'a & h'b & \dots & h'h \end{vmatrix}}, \quad (12)$$

where  $a, b, \dots, h$  denote the successive columns of  $A$ . (In the notation of Gauss the elements of the determinants would appear as  $[uu]$ ,  $[au]$ ,  $[bu]$ ,  $[aa]$ ,  $[ab]$  and the like.) Such a quotient of determinants may be expanded (*cf.* Aitken, 1942) as a Schweinsian series, giving the sum of squared residuals as

$$s^2 = u'u - \frac{(u'a)^2}{1 \cdot a'a} - \frac{\begin{vmatrix} u'a & u'b \\ a'a & a'b \\ a'a & a'b \\ b'a & b'b \end{vmatrix}^2}{\begin{vmatrix} a'a & a'b \\ b'a & b'b \end{vmatrix}} - \frac{\begin{vmatrix} u'a & u'b & u'c \\ a'a & a'b & a'c \\ b'a & b'b & b'c \end{vmatrix}^2}{\begin{vmatrix} a'a & a'b \\ b'a & b'b \end{vmatrix} \begin{vmatrix} a'a & a'b & a'c \\ b'a & b'b & b'c \\ c'a & c'b & c'c \end{vmatrix}} - \dots, \quad (13)$$

the series terminating after  $n+1$  terms. It will appear in the sequel that the terms after the first, with negative sign, represent decrements in the sum of squared residuals, produced *pari passu* with a set of orthogonal vector increments by which the vector of residuals is built up. But this will be more in place in the discussion of  $Pa=u$ .

The residual quadratic in the correlated case can be expressed in the same way as a quotient of determinants and can be expanded as a Schweinsian series, differing from (12) merely in that  $u'u$  becomes  $u' V^{-1} u$ , and similarly for other elements. But the increments of the vector of residuals are then not orthogonal, and the matter becomes somewhat academic and remote from practice.

### 3. THE CURVE-FITTING PROBLEM

We turn now to the observational equations in what may be termed the *curve-fitting* problem,  $Pa=u$ . We are interested here, certainly in the coefficients  $a_j$  of the representation, but equally in the graduated values  $y=Pa$ . In this case the vector  $a$  is given by the normal equations

$$P' V^{-1} P a = P' V^{-1} u, \quad (1)$$

and so

$$y = P(P' V^{-1} P)^{-1} P' V^{-1} u = G u, \quad (2)$$

where  $G$  may be called the *graduating matrix*. Though the fitting of correlated data has been studied (Aitken, 1933) it seldom arises in practice, and is difficult. Accordingly we confine ourselves to the cases  $V=I$  and  $V=W^{-1}$ . When  $V=I$ , the commonly occurring case of equally weighted data, the graduating matrix is

$$G = P(P'P)^{-1}P', \quad (3)$$

a matrix obviously symmetric, idempotent and of rank  $k+1$ . The sum of squared residuals, by § 2 (10), is  $u'u - a'P'Pa$ . If the data  $u$  are spaced at equal intervals in  $x$ , as is often the case, they can be graduated in reversed order, and so, since  $G$  is independent of  $u$ , we see that  $G$  is unaltered when its rows and columns are reversed. Thus  $G$  is in this case symmetric about both its diagonals. Again, from its rank and idempotency, its trace is  $k+1$ . We have thus various useful checks upon its evaluation.

For a polynomial basis, or for a basis of harmonic functions of the usual kind, these matrices  $G$  are not difficult to construct for moderate values of  $n$  and  $k$ . Especially is this the case when the basis of functions is orthogonal and normal. Let the functions  $1, p_1(x), p_2(x), \dots, p_k(x)$  be transformed into a set of functions  $q_j(x)$  orthonormal over the values of  $x$ , by linear combination

$$q_j(x) = c_{0j} + c_{1j}p_1(x) + c_{2j}p_2(x) + \dots + c_{kj}p_k(x), \quad j=0, 1, \dots, k. \quad (4)$$

In other words,  $Q=PC$ , where  $C$  is a matrix of triangular shape, all elements above the diagonal being zero. From the orthonormality of the  $q_j(x)$  we then have  $Q'Q=I$ . Also  $QQ'=G$ , since

$$P(P'P)^{-1}P' = PC(C'P'P)^{-1}C'P' = Q(Q'Q)^{-1}Q' = QQ'. \quad (5)$$

If the coefficients in such an orthonormal representation are  $a_j$ , we have the sum of squared residuals given as  $u'u - a'a$ , a standard result.

As an example, consider the construction of a graduating matrix  $G$  for  $n=6$  and a polynomial basis with  $k=3$ , in fact for fitting cubic polynomial values to 6 equally spaced data. We take the four columns of values, for  $k=0, 1, 2, 3$  from some table of orthogonal polynomials (e.g. Fisher and Yates, 1943), normalize them if necessary, and thus we have  $Q$ , and so  $QQ'$ . The actual matrices in this case are

$$Q = \begin{bmatrix} 1 & -5 & 5 & -5 \\ 1 & -3 & -1 & 7 \\ 1 & -1 & -4 & 4 \\ 1 & 1 & -4 & -4 \\ 1 & 3 & -1 & -7 \\ 1 & 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & & & \\ & 1/\sqrt{70} & & \\ & & 1/\sqrt{84} & \\ & & & 1/\sqrt{180} \end{bmatrix}, \quad (6)$$

$$G = QQ' = \frac{1}{126} \begin{bmatrix} 121 & 16 & -14 & -4 & 11 & -4 \\ 16 & 73 & 52 & 2 & -28 & 11 \\ -14 & 52 & 58 & 32 & 2 & -4 \\ -4 & 2 & 32 & 58 & 52 & -14 \\ 11 & -28 & 2 & 52 & 73 & 16 \\ -4 & 11 & -4 & -14 & 16 & 121 \end{bmatrix}. \quad (7)$$

Or again, for fitting a harmonic function

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x \quad (8)$$

to 12 data  $u_x$ , equally spaced over one complete oscillation of the angular variable, we take for the columns of  $Q$  the values of  $1, \cos x, \sin x, \cos 2x, \sin 2x$  at the phases  $0^\circ, 30^\circ, 60^\circ, \dots, 330^\circ$ , normalize these columns and so construct

$$G = QQ' = \frac{1}{12} \text{circ} \begin{bmatrix} 5 & 2 + \sqrt{3} & 1 & -1 & -1 & 2 - \sqrt{3} & 1 & 2 - \sqrt{3} & -1 & -1 & 1 & 2 + \sqrt{3} \end{bmatrix}, \quad (9)$$

where  $\text{circ} [\dots]$  is used to denote a symmetric circulant matrix, completely determined by its first row, successive later rows being written down by cyclically permuting the elements of the first row. This circulant property could have been deduced at once from the consideration that a cycle of arbitrary periodic data can be graduated in any one of its cyclic orders.

These graduating matrices  $G$ , evaluable once and for all, and applicable to any proposed set of  $n$  data, are of considerable practical use, quite apart from their interesting theoretical properties. Tables of  $G$  for polynomials up to  $n=15$ ,  $k=5$  form the Appendix of a thesis by F. Mary Harding (Edinburgh, 1934); and tables of  $G$  for the harmonic fitting of  $2n$  data,  $n=2, 3, 4, 5, 6, 8, 12$  and  $k=1, 2, \dots, n$  occur in the Appendix of a thesis by A. F. Buchan (Edinburgh, 1939).

In the case of unequal weights,  $V=W^{-1}$ , we may graduate similarly by *weighted* orthogonal functions such that  $Q'WQ=I$ , the graduating matrix being then  $QQ'W$ . For example, if the weights followed a binomial distribution, the polynomials to use in polynomial fitting would be the orthogonal polynomials of Gram.

#### 4. THE COEFFICIENTS OF TERMS IN THE REPRESENTATION

The coefficients  $a_j$  in the representation . . .

$$y = a_0 + a_1 p_1(x) + \dots + a_k p_k(x) \quad (1)$$

are found by the transformation  $a = (P'P)^{-1}P'u$ , but here again it is of the greatest advantage to choose if possible the orthonormal representation

$$y = a_0 + a_1 q_1(x) + \dots + a_k q_k(x), \quad (2)$$

for then  $a = (Q'Q)^{-1}Q'u = Q'u$ , a familiar and central result in orthogonal representation. If a good calculating machine is available we can use  $Q'$  directly upon the data  $u$ . For example to find the Fourier coefficients for 6 data  $u_x$  over one oscillation, we have

$$Q' = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 & -1 & 1 \\ \cdot & \sqrt{3} & \sqrt{3} & \cdot & -\sqrt{3} & -\sqrt{3} \\ 2 & -1 & 1 & 2 & -1 & -1 \\ \cdot & \sqrt{3} & -\sqrt{3} & \cdot & \sqrt{3} & -\sqrt{3} \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \quad (3)$$

which is merely a matrix of values of  $\cos kx$  and  $\sin kx$  suitably normalized. It is our experience that the use of such matrices with the machine is much more rapid than any of the computing schemes of harmonic analysis.

#### 5. THE ORTHOGONAL COMPONENTS OF THE GRADUATED VECTOR

The graduated vector  $y = P(P'P)^{-1}P'u$  can be expressed as the sum of orthogonal component vectors. Each of these is at the same time an increment of the graduated vector and a decrement of the vector of residuals.

Let  $P_j$  denote the matrix comprised by the first  $j+1$  columns of  $P$ . Then

$$y_j = P_j(P_j'P_j)^{-1}P_j'u, \quad (1)$$

where  $y_j$  is the vector of values graduated in accordance with the partial or truncated representation

$$y_j(x) = a_0 + a_1 p_1(x) + \dots + a_j p_j(x). \quad (2)$$

Let  $j$  take the values 0, 1, 2, . . . ,  $k$  in succession. Each vector  $y_j$  may then be regarded as derived from its predecessor by the incremental vector

$$y_j - y_{j-1} = \{P_j(P_j'P_j)^{-1}P_j' - P_{j-1}(P_{j-1}'P_{j-1})^{-1}P_{j-1}'\}u. \quad (3)$$

It is easy to see that these incremental vectors are always orthogonal to each other—that is, that their scalar product is 0. For suppose the basis transformed as in § 3 to an orthogonal basis with matrix  $Q$ , then in a manner similar to § 3 (5), with  $C$  replaced by its leading sub-matrix of order  $j \times j$ , we have

$$P_j(P_j'P_j)^{-1}P_j' = Q_jQ_j', \quad j=0, 1, \dots, k. \quad (4)$$

Thus the incremental vectors are  $(Q_jQ_j' - Q_{j-1}Q_{j-1}')u$ ; but these produce the individual terms  $a_jq_j(x)$  of the orthonormal representation; since it is well known, and almost intuitive, that to graduate  $u(x)$  by means of a basis obtained from a previous orthogonal basis by annexing a further orthogonal function  $q_j(x)$  is merely to add a term  $a_jq_j(x)$  to the existing representation. The orthogonality of the incremental vectors is thus established.

These orthogonalities lead to many interesting properties of graduating matrices. For example, since

$$\{P_j(P_j'P_j)^{-1}P_j' - P_{j-1}(P_{j-1}'P_{j-1})^{-1}P_{j-1}'\}u \quad \text{and} \quad \{P_i(P_i'P_i)^{-1}P_i' - P_{i-1}(P_{i-1}'P_{i-1})^{-1}P_{i-1}'\}u \quad (5)$$

are orthogonal for  $i < j$ , and  $u$  is arbitrary, we deduce that

$$\begin{aligned} & \{P_j(P_j'P_j)^{-1}P_j' - P_{j-1}(P_{j-1}'P_{j-1})^{-1}P_{j-1}'\} \{P_i(P_i'P_i)^{-1}P_i' - P_{i-1}(P_{i-1}'P_{i-1})^{-1}P_{i-1}'\} \\ &= \{P_i(P_i'P_i)^{-1}P_i' - P_{i-1}(P_{i-1}'P_{i-1})^{-1}P_{i-1}'\} \{P_j(P_j'P_j)^{-1}P_j' - P_{j-1}(P_{j-1}'P_{j-1})^{-1}P_{j-1}'\} \\ &= 0. \end{aligned} \quad (6)$$

In particular we deduce

$$A_i(A_i'A_i)^{-1}A_i' \cdot A_j(A_j'A_j)^{-1}A_j' = A_j(A_j'A_j)^{-1}A_j' \cdot A_i(A_i'A_i)^{-1}A_i' = A_i(A_i'A_i)^{-1}A_i', \quad i \leq j, \quad (7)$$

where we have written  $A$  instead of  $P$  to emphasize the fact that the result is a general theorem of matrices,  $A$  being an arbitrary matrix having linearly independent columns, and  $A_j$  being the sub-matrix composed of the first  $j+1$  of those columns.

Translated into the language of fitting by Least Squares such an identity as the above has a very simple interpretation. To take an example, it means that if we graduate a set of data by fitting values of a quintic polynomial to them, and then, treating the graduated values as new data, graduate them in turn by fitting a cubic polynomial, we obtain exactly the same final values as if we had graduated the original data directly by a cubic polynomial. Such results are evident at once if set out in terms of an orthogonal basis. They have their use, however, in the numerical verification of elements of the graduating matrices  $G$ . Thus the graduating matrices  $G_2$  and  $G_3$  for fitting harmonics up to the second and third respectively to 8 data are the symmetric circulant matrices

$$G_2 = \frac{1}{8} \text{circ} [5 \quad 1 + \sqrt{2} \quad -1 \quad 1 - \sqrt{2} \quad 1 \quad 1 - \sqrt{2} \quad -1 \quad 1 + \sqrt{2}] \quad (8)$$

and

$$G_3 = \frac{1}{8} \text{circ} [7 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1]. \quad (9)$$

It is readily verified here that  $G_2G_3 = G_3G_2 = G_2$ .

We return now to the Schweinsian series of § 2 for the sum of squared residuals. A Schweinsian series (*cf.* Aitken, 1942) for the quotient of two determinants, the numerator being obtained by bordering the denominator, arises thus. Form the following sequence: first, 0; second, the leading element in the numerator; third, the leading minor of order 2 in the numerator divided by the leading element in the denominator; fourth, the leading minor of order 3 in the numerator divided by the leading minor of order 2 in the denominator; and so on. The first differences of this sequence may be proved (*loc. cit.*) to be equal to the respective terms of the Schweinsian series. Now in our present problem the leading sub-matrices of  $A'A$  or of  $P'P$ , as the case may be, are  $A_0'A_0$ ,  $A_1'A_1$ ,  $\dots$  or  $P_0'P_0$ ,  $P_1'P_1$ ,  $\dots$ . These are the matrices of successive normal equations in  $x$  or in  $a$ , of higher and higher order. Thus we see that the decrements in the sum of squared residuals, as given by successive terms in the Schweinsian series, correspond precisely to the orthogonal component vectors by which the graduated vector is augmented as  $j$  increases step by step to  $k$ . It follows from the invariance of  $P(P'P)^{-1}P'$  that every term of such a Schweinsian expansion is invariant

under a transformation  $Q=PC$  of the basis, where  $C$  is a matrix of triangular type with all elements above the diagonal equal to zero; and this, whether the new basis  $Q$  is orthogonal or not. In fact the  $(j+2)$ th term of the expansion is equal to  $-a_j^2$ , where  $a_j$  is the  $(j+1)$ th coefficient in the orthonormal representation.

These and many other properties of Least Square representation were first established by Tchebychef (Tchebychef, 1858). From the present point of view they arise as immediate consequences of the relations  $P(P'P)^{-1}P'=QQ'$ ,  $P_j(P_j'P_j)^{-1}P_j'=Q_jQ_j'$ , where  $Q$  is orthogonal.

## 6. LEAST SQUARES UNDER EXACT LINEAR CONDITIONS

We turn now to a different topic, the solution of a linear problem in Least Squares subject to extraneous restrictions, which we shall assume to be linear. For example, the estimates of the three angles of a measured plane triangle must have the sum  $180^\circ$ . In the curve-fitting problem, also, it might conceivably be the case that the coefficients  $a_j$  of the representation had to satisfy some linear condition or conditions.

The customary procedure (*cf.* Brunt, 1931, p. 127) is direct and summary: it is, to invoke a principle of conditioned Least Squares, and to determine the conditioned minimum by introducing as many Lagrange multipliers as may be required. But it is of some interest to arrive at this procedure from a different approach, always from the standpoint of consistent linear representation and minimum variance.

The restrictions themselves are, of their very nature, the results of prior experiment and measurement. For example, the fact that the three angles of a plane triangle sum to  $180^\circ$ , when regarded from the point of view of mensuration and not from that of deductive geometry, is the conclusion arrived at, and then not finally, from an immense number of measurements of such triangles. The linear restrictions are to be viewed, therefore, as equivalent to so many more *observational* equations, of large (and ultimately indefinitely large) weights  $w_j$ . In the sequel we shall make  $w_j \rightarrow \infty$ , and so for convenience we may regard all the  $w_j$  as equal to  $w$ ; further, by "preparing" the linear conditions in the usual way by multiplying by  $\sqrt{w}$  we can treat them as all of weight 1.

Thus in the non-correlated case the solution consists in minimizing the quadratic form

$$(u - Ax)'(u - Ax) + w(k - Cx)'(k - Cx), \quad (1)$$

where  $Cx=k$  are the  $s$  linear restrictions. Let us suppose that the solutions of these normal equations lead to  $Cx=k+\epsilon$ . Then the solving vector  $x$  is of course the same as would be obtained by minimizing  $(u - Ax)'(u - Ax)$ , subject to  $Cx=k+\epsilon$ . If now  $w \rightarrow \infty$ , then  $\epsilon \rightarrow 0$ , and the desired values  $x$  are obtained by minimizing  $(u - Ax)'(u - Ax)$  subject to  $Cx=k$ . Here we make contact with the usual direct approach; a vector  $\lambda$  of Lagrange multipliers can be introduced, and by minimizing

$$\frac{1}{2}(u - Ax)'(u - Ax) - (Cx - k)'\lambda \quad (2)$$

we derive the normal equations

$$A'Ax = A'h + C'\lambda, \quad Cx = k, \quad (3)$$

these being  $n+s$  equations in the  $n$  unknowns  $x$  and  $s$  unknowns  $\lambda$ .

The above may be called the  $\lambda$ -method. Actually the method of very large weights, which may be called the  $w$ -method, is of some practical value. Suppose, for example, that we are satisfied with an accuracy of 1 in 1000 in our residuals. We might then take  $w=10,000$ , and introducing the linear conditions as *observations* having this weight we could solve for  $n$  unknowns, not  $n+s$ , in the ordinary way. We choose the weight just so large that the error induced by not proceeding to the limit,  $w \rightarrow \infty$ , may be negligible in the number of significant digits retained. A practical disadvantage is perhaps that the matrix of the normal equations is sensitive to small relative changes in its elements.

By regarding exact conditions as observations of indefinitely great weight we obtain at once the mean value of the sum of squared residuals as  $(m+s-n)\sigma^2$ , since there are  $m+s$  observations in  $n$  unknowns. This is of course well known from another point of view,

namely that from  $s$  linearly independent equations we can express  $s$  of the unknowns in terms of the remaining  $n-s$  and so, entering the results in each observational equation, we have  $m$  observational equations in  $n-s$  unknowns, whence the mean value of the sum of squared residuals is  $\{m-(n-s)\}\sigma^2$ .

It is of interest at this point to work out a very simple example. It is one in which the restrictive condition makes a conspicuous difference in the solutions.

*Example.*—Observational equations :

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ & 1 & 2 \\ & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \\ 13 \\ 13 \\ 10 \end{bmatrix}.$$

Exact linear condition :

$$2x_1 + 3x_2 + x_3 = 20.$$

Normal equations:

$$\begin{bmatrix} 6+4w & 5+6w & 6+2w \\ 5+6w & 11+9w & 9+3w \\ 6+2w & 9+3w & 11+w \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 55+40w \\ 91+60w \\ 94+20w \end{bmatrix}. \quad (4)$$

Solutions by reciprocal matrix :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{109+231w} \begin{bmatrix} 40+56w & -1-35w & -21-7w \\ -1-35w & 30+26w & -24-8w \\ -21-7w & -24-8w & 41+38w \end{bmatrix} \begin{bmatrix} 55+40w \\ 91+60w \\ 94+20w \end{bmatrix} \quad (5)$$

$$= \frac{1}{109+231w} \begin{bmatrix} 135+357w \\ 419+969w \\ 515+999w \end{bmatrix}. \quad (6)$$

Thus the solutions with and without the linear condition are respectively

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{231} \begin{bmatrix} 357 \\ 969 \\ 999 \end{bmatrix} = \begin{bmatrix} 1.545 \\ 4.195 \\ 4.325 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{109} \begin{bmatrix} 135 \\ 419 \\ 515 \end{bmatrix} = \begin{bmatrix} 1.239 \\ 3.844 \\ 4.725 \end{bmatrix}.$$

Now let us try the moderate value  $w=100$ . The normal equations are

$$\begin{bmatrix} 406 & 605 & 206 \\ 605 & 911 & 309 \\ 206 & 309 & 111 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4055 \\ 6091 \\ 2094 \end{bmatrix}.$$

The solutions by the reciprocal matrix are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{23209} \begin{bmatrix} 5640 & -3501 & -721 \\ -3501 & 2630 & -824 \\ -721 & -824 & 3841 \end{bmatrix} \begin{bmatrix} 4055 \\ 6091 \\ 2094 \end{bmatrix} = \frac{1}{23209} \begin{bmatrix} 35835 \\ 97319 \\ 100415 \end{bmatrix} = \begin{bmatrix} 1.544 \\ 4.193 \\ 4.327 \end{bmatrix}.$$

These differ only in the third place of decimals from the accurate solutions, and they are such that  $2x_1 + 3x_2 + x_3 = 19.994$ . Indeed they are stated with excessive accuracy, for the size of the alteration produced in the solutions by imposing the linear condition indicates relatively large observational error; but the example is purely illustrative.

It may be noted, in (5) and (6) above, that in the determinant  $|A'A + wC'C|$ , in the elements of the adjugate,  $\text{adj}(A'A + wC'C)$ , and in  $\{\text{adj}(A'A + wC'C)\}(A'u + wC'k)$  no higher powers of  $w$  than the first have occurred. In the general case of  $s$  restrictive conditions the highest power of  $w$  that appears in this determinant and matrices is  $w^s$ . This is a simple consequence of the fact that  $C'C$  and  $C'$  are of rank  $s$ , so that in the expansion of any minor

of the augmented matrix  $[A'A + wC'C : A'u + wC'k]$  in powers of  $w$  no higher power than  $w^2$  can occur. The practical bearing of this, as we have mentioned above, is that the dominant part of the matrix is of less than full rank, and so the matrices in question are sensitive to small relative changes in their elements, and more significant digits have to be retained in the arithmetical workings.

#### 7. THE RELATION BETWEEN THE $\lambda$ -METHOD AND THE $w$ -METHOD

It is of interest to set out the relationship of the  $\lambda$ -method to the  $w$ -method. The  $\lambda$ -method leads to the normal equations, in partitioned matrix form,

$$\begin{bmatrix} A'A & C' \\ C & . \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A'u \\ k \end{bmatrix}, \quad (1)$$

while the  $w$ -method leads to the normal equations

$$(A'A + wC'C)x = A'u + wC'k. \quad (2)$$

Now equations (1) may be regarded as the limit, when  $w \rightarrow \infty$ , of

$$\begin{bmatrix} A'A & C' \\ C & -w^{-1}I \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A'u \\ k \end{bmatrix}. \quad (3)$$

Premultiplying both sides by

$$\begin{bmatrix} I & wC' \\ . & I \end{bmatrix}$$

we obtain

$$\begin{bmatrix} A'A + wC'C & . \\ C & -w^{-1}I \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A'u + wC'k \\ k \end{bmatrix}, \quad (4)$$

that is to say,

$$(A'A + wC'C)x = A'u + wC'k \quad (5)$$

together with

$$Cx = k + w^{-1}\lambda.$$

In a word, the normal equations for  $x$  as given by the  $\lambda$ -method under the restrictions  $Cx = k + \epsilon$  are the normal equations of the  $w$ -method, provided that  $\epsilon = w^{-1}\lambda$ . When  $w \rightarrow \infty$ , the restricting conditions tend to  $Cx = k$  and the two methods tend to equivalence.

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## XVII.—The Regraduation of Clocks in Spherically Symmetric Space-times of General Relativity. By G. C. McVittie, Ph.D.

(MS. received October 7, 1944. Revised MS. received February 2, 1945. Read May 7, 1945)

### I. INTRODUCTION

IN an earlier paper in these *Proceedings* \* I was able to show that the principle of the regraduation of clocks, combined with the definitions of coordinates and of velocity and acceleration used in kinematical relativity, had far-reaching consequences in that theory. In particular, if it were postulated that the "fundamental" observers had a linear velocity-function, it followed that the non-fundamental or "free" particles obeyed Milne's acceleration formula. This investigation dealt only with motion "in the line of sight" of an observer, all orientations around him being regarded as equivalent.

In this note I have attempted to investigate the question of clock regraduation in general relativity under the same limitation (spherical symmetry around an observer) as before. It was of interest to discover whether clock regraduation would have as important consequences in general as in kinematical relativity, and the investigation was prompted by a remark † of Dr A. G. Walker's which reads: "... in general relativity the time which is identified with that measured by the physicist is the interval  $\int ds$ . Whatever is done to the coordinates, this can on no account be altered, for otherwise the metric value would be changed and the associated Riemannian space deformed. In all work on general relativity which I have seen, there occurs not one example of a material system being described by more than one Riemannian space and yet this is necessary if clocks were to be regraduated in the sense of Milne."

If therefore I have understood Dr Walker aright, clock regraduation should change the metric of space-time intrinsically and this should be universally true, *i.e.* if *any* observer (defined according to the postulates of general relativity) in *any* space-time were to regraduate his clock which measures  $\int ds$ , the space-time would be altered to one of different curvature. A preliminary examination of the question—which is all that is attempted here—may therefore be carried out by discussing a special class of space-times. The method which we shall use is, we believe, of more general application and could be extended to other types of space-time.

Clock regraduation is defined as follows:—

An observer has a clock which reads time  $T_0$ . He alters the graduations on the clock-face in any manner so that the clock reads time  $T_1$  where

$$T_0 = F(T_1) \quad (1)$$

and  $F$  is a function expressing the arbitrary change in the marks on the clock-face.

If, however, the observer is of a scientific turn of mind he need not *alter* the graduations on the face of the  $T_0$  clock, but secure a second clock (the  $T_1$  clock) with the *same* graduations as the first, this clock working at a different *rate* relative to the original one. The function  $F$  then expresses the relationship between the rates of working of the two clocks. In this way the observer could refer his description of events at will to either clock and discover whether any significant change was produced by so doing.

### 2. THE POSTULATES OF GENERAL RELATIVITY

We require to know exactly what postulates are the basis of general relativity and the following are a sufficient set:—

(a) The equations of mechanics, and of mathematical physics generally, are expressible in tensor form. This provides a rule by which an observer may rewrite his equations when he changes from one coordinate-system to another.

\* *Proc. Roy. Soc. Edin.*, LXI, A, 210, 1942.

† *Observatory*, LXIV, 18, 1941.



(b) Events are represented by points in a 4-dimensional space-time with metric

$$ds^2 = g_{ij} dx^i dx^j \quad (i, j = 1, 2, 3, 4), \quad (2)$$

one coordinate being time-like and the other three space-like in one at least of the coordinate-systems employed by the observer, and the  $g_{ij}$  being functions of the  $(x^i)$ . The interval  $ds$ , between two events  $(x^i)$  and  $(x^i + dx^i)$ , is not integrable unless a curve joining the two events is also specified.

(c) Material particles (and observers) trace out the non-null geodesics of space-time. The integrated interval along the geodesic is defined as the time kept by a clock travelling with the observer and is called his *proper-time*.

(d) The paths of light-rays are the null-geodesics of space-time, *i.e.* those for which  $ds = 0$ .

(e) The dynamical properties of the material content of space-time are reflected in the geometrical properties of space-time, the precise relationship being defined thus: The energy-tensor ( $T_{ij}$ ) of the material content is related to the curvature-tensor ( $G_{ij}$ ) of space-time and its invariant curvature ( $G$ ) by Einstein's gravitational equations

$$-\kappa T_{ij} = G_{ij} - \frac{1}{2}(G - 2\lambda)g_{ij}, \quad (3)$$

where  $\kappa$  is proportional to the constant of gravitation and  $\lambda$  is the cosmical constant, these being regarded as physical constants independent of the particular space-time used.

A word of explanation regarding these postulates is perhaps necessary. There is firstly the question of the determination of the  $g_{ij}$ . It is usual to say that they are given by equations (3), but a little consideration will show that we have here a vicious circle. For an observer cannot measure the values of the components  $T_{ij}$  until he has set up his coordinate-system and knows the geometry of space-time, *i.e.* unless he knows the  $g_{ij}$  as explicit functions of the coordinates. But he cannot know the  $g_{ij}$  until he has introduced specific values for the  $T_{ij}$  in (3) and solved these differential equations. In practice this difficulty is overcome by assigning *a priori* values to the  $T_{ij}$  which are deemed on general grounds to correspond to the physical situation under contemplation. More exactly, postulate (e) is applied in two stages. In the first stage it is used in a "qualitative" way to secure agreement between the general qualitative properties of the material distribution and the geometrical properties of space-time. In the second stage it is employed in a "quantitative" manner by giving explicit values to the  $T_{ij}$  and solving (3). Examples may make these points clearer. In finding the gravitational field of the Sun the first stage is represented by the choice of a static and spherically symmetric metric for space-time which determines all the  $g_{ij}$  except two. These two are, however, restricted to being functions of the radial coordinate alone. The second stage is represented by the assigning of zero values to the  $T_{ij}$  and to  $\lambda$  and the consequent solution of equations (3) for the surviving  $g_{ij}$ . In the Expanding Universe theory, the application of postulate (e) does not go beyond the first stage. General considerations of homogeneity, uniform motion of matter in all directions, etc., determine the metric to within one arbitrary function of the time and one arbitrary constant. Equations (3) are then merely used to calculate the corresponding components of the energy-tensor in a space-time regarded as known.

Two points also require consideration regarding the geodesic postulate (c). In the first place, we have identified observers with freely moving (in the sense of general relativity) material particles. This we do because we assume that our observers, like terrestrial ones, have material bodies and carry material instruments with them and that they are not subject to constraints outside the scope of general relativity (*e.g.* to the electromagnetic forces of "unified" field theories). Though the amount of matter represented by the observer and his instruments may be small compared with that of the system he is contemplating, it cannot be taken as strictly zero. In the second place, we regard the postulate as independent of the rest, as an expression of the principle of equivalence.\* It is true that Eddington † has sought to deduce it from equations (3) by using a special form of  $T_{ij}$  and proceeding to the limit.

\* See, for example, A. Einstein, *Ann. de Physik*, XLIX, 1916: French translation in *Les Fondements de la théorie de la relativité générale*, etc., § 9, Hermann et Cie, Paris, 1933; or R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, § 74, Oxford, 1934.

† *The Mathematical Theory of Relativity*, § 56, 2nd ed., Cambridge, 1924.

But this investigation, in our opinion, shows no more than that (3) is consistent with the postulate. And indeed Eddington's concluding remark that his proof "does not add very much to the argument of § 17" (where he discusses the principle of equivalence) lends colour to this interpretation of his work. Before leaving the subject of postulate (c), we remind the reader that the differential equations of the geodesics are (in terms of an arbitrary parameter  $\mu$ )

$$\frac{d^2 x^i}{d\mu^2} + \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} \frac{dx^k}{d\mu} \frac{dx^j}{d\mu} - \frac{dx^i}{d\mu} \frac{d^2 s}{d\mu^2} \bigg/ \frac{ds}{d\mu} = 0, \quad (4)$$

and that they reduce to

$$\frac{d^2 x^i}{ds^2} + \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} \frac{dx^k}{ds} \frac{dx^j}{ds} = 0 \quad (5)$$

when the arc-length  $s$  along the geodesic is used as parameter.

Finally, it is mathematically obvious that these postulates could be replaced by others not logically reducible to them. Examples of theories in which most of the postulates are abandoned are provided by the many "unified" field theories of gravitation and electromagnetism, by kinematical relativity or by G. D. Birkhoff's \* "relativity in flat space-time." Less generally, Levinson and Zeisler † have investigated the possibility of altering equations (3) alone, for the case when the  $T_{ij}$  and  $\lambda$  are all zero. None of these theories can, however, claim to be general relativity, which is based on the simultaneous use of all five postulates or on others equivalent to them. Recognition of this fact does not preclude us from analysing the consequences of each postulate in turn. We shall therefore discuss proper-time regraduation firstly using postulates (a) to (d) and postulate (e) in its qualitative aspect, and then showing the effect of introducing the quantitative aspect of (e).

### 3. ORDINARY REGRAUATION

Before proceeding with the general problem we point out that the regraduation of clocks has, in one sense, always been a feature of general relativity. This is a consequence of the principle that an observer may use a time-coordinate other than his proper-time for describing the events which he sees around him. For example, let the observer have determined the metric of space-time ‡ to be

$$ds^2 = g(t, r) dt^2 - h(t, r) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,$$

where  $g, h$  now stand for known functions of  $t$  and  $r$ . For definiteness, suppose that the observer's geodesic § is  $r=0, \theta=0, \phi=0$ . Then the relation between his proper-time  $s$  and coordinate-time  $t$  is

$$s = \int \sqrt{g(t, 0)} dt,$$

which is precisely of the form (1), so that the observer may be said to regraduate his proper-time clock when he describes events along his geodesic in terms of the same time-variable as he uses for all other events. Alternatively, it follows that the observer carries more than one clock with him, one of which reads proper-time and the rest read one or other of the coordinate-times.

### 4. REGRAUATION OF PROPER-TIME

We now turn to the discussion of the effect of applying formula (1) to proper-time directly and not, as in the last paragraph, through the intermediary of an already established coordinate-

\* G. D. Birkhoff, *Proc. Nat. Acad. Sci.*, xxix, 231, 1943, and LX, 324, 1944.

† *The Law of Gravitation in Relativity*, Chicago Univ. Press, 1929.

‡ Purely as a matter of convenience  $r$  is measured in the same units as  $t$ . To convert to units of length,  $r$  must be multiplied by the velocity of light,  $c$ .

§ By evaluating the equations (5) for this space-time, the condition  $\left( \frac{\partial g}{\partial r} \right)_{r=0} = 0$  must be imposed on  $g$ .

time. We shall consider only the particular class of space-times \* with metric

$$ds^2 = dt^2 - \frac{1}{n^2(t)} \{ m^2(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \}, \quad (6)$$

and we shall assume that this form has been arrived at by our observer through the use of postulates (a) to (d) and postulate (e) in its qualitative aspect only. We also assume that our observer's geodesic is

$$s = t, \quad r = 0, \quad \theta = 0, \quad \phi = 0. \quad (7)$$

These assumptions are equivalent to saying that the observer has decided that the material distribution he is observing (i) possesses spherical symmetry round himself so that for a given value of a time-parameter  $t$ , the distribution changes only with a distance-parameter  $r$ ; and (ii) that in terms of  $t$ , the distribution remains similar to itself, but on a different scale, at successive instants  $t$ . In this way he has concluded that the metric of space-time contains two functions  $m, n$  of  $r, t$  respectively, which enter into the coefficients of the metric in the way exhibited in formula (6) but which are otherwise unspecified.

Regraduation of clocks in kinematical relativity plays an important rôle because the time and radial coordinates used in that theory are defined by means of clock-readings on the observer's clock. We can introduce coordinates of this type in general relativity also, which we call *light-signal* coordinates and which we shall use in preference to  $r$  and  $t$ . These coordinates are defined as follows:—

Let the observer whose geodesic is (7) be called the observer  $A$  and let there be another observer,  $B$ , who is described by  $A$  as having the coordinates †  $t, r$ . Let  $B$  have a mirror which he uses to flash back to  $A$  light-signals received from  $A$ . We write

$$N(x) = \int n(x) dx, \quad M(x) = \int m(x) dx. \quad (8)$$

Then a light-signal incoming at  $B$  which left  $A$  at  $A$ 's proper-time  $s_1$  has the equation

$$N(t) = N(s_1) + M(r) - M(0), \quad (9)$$

and the outgoing signal from  $B$  which reaches  $A$  at  $A$ 's proper-time  $s_2$  is

$$N(t) = N(s_2) - M(r) + M(0). \quad (10)$$

We define  $(s_1, s_2)$  as the light-signal coordinates ‡ of  $B$ : they are linked to his  $t, r$  coordinates by the equations §

$$t = N^{-1}[\frac{1}{2}\{N(s_2) + N(s_1)\}], \quad (11)$$

$$r = M^{-1}[\frac{1}{2}\{N(s_2) - N(s_1)\} + M(0)]. \quad (12)$$

We also require the expression of the metric (6) in terms of light-signal coordinates which is obtained by performing the coordinate transformation (11), (12). It is

$$ds^2 = \frac{n(s_2)n(s_1)ds_1ds_2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2}{n^2(t)}, \quad (13)$$

where, for brevity,  $t$  and  $r$  stand respectively for the right-hand sides of equations (11) and (12). Hence the statement that the metric contains the unspecified functions  $m(r)$  and  $n(t)$  is now replaced by the statement that the metric contains the unspecified functions  $n$  and  $r$  of  $s_1, s_2$  which enter into the coefficients in the way exhibited in formula (13).

Suppose now that  $A$  regraduates his proper-time clock to read  $S$  instead of  $s$  where

$$s = F(S). \quad (14)$$

\* The "expanding universes" of general relativity are the special cases of (6) in which

$$m^2(r) = 1/(1 - kr^2/R_0^2), \quad k = +1, 0, -1.$$

† The angular coordinates  $\theta, \phi$  of  $B$  do not enter into this discussion.

‡ In kinematical relativity, coordinates  $t_m, r_m$  are used which are the linear combinations  $t_m = \frac{1}{2}(s_1 + s_2)$ ,  $r_m = \frac{1}{2}(s_2 - s_1)$  of  $s_1, s_2$ .

§ We use the index  $-1$  throughout to denote an *inverse function*, not a negative power.

Transforming the equation of  $A$ 's geodesic to the parameter  $S$ , it becomes

$$F(S) = t, \quad r = 0, \quad \theta = 0, \quad \phi = 0.$$

It is, of course, open to  $A$  to regraduate his  $t$ -clock as well as his proper-time clock. If he applies the *same* regraduation to both his clocks, *i.e.* if he regraduates  $t$  to  $\tau$ , where

$$t = F(\tau), \quad (15)$$

then the equation of his geodesic becomes

$$S = \tau, \quad r = 0, \quad \theta = 0, \quad \phi = 0. \quad (16)$$

Thus a simultaneous regraduation of the two clocks reduces the equation of  $A$ 's geodesic to its original form (7) so that, as far as this equation goes, the regraduated proper-time clock is "just as good as" the original one.

Next consider the effect on the expression for the interval  $ds$  between the two events  $(s_2, s_1, \theta, \phi)$  and  $(s_2 + ds_2, s_1 + ds_1, \theta + d\theta, \phi + d\phi)$ . We write

$$\bar{N}(x) = N[F(x)], \quad \bar{N}(x) = \int \bar{n}(x) dx, \quad (17)$$

so that

$$\bar{n}(x) = n[F(x)] \cdot \frac{dF(x)}{dx} = n[F(x)] \cdot F'(x).$$

Hence using (15), equations (11) and (12) become

$$\tau = \bar{N}^{-1}[\frac{1}{2}\{\bar{N}(S_2) + \bar{N}(S_1)\}], \quad (18)$$

$$r = M^{-1}[\frac{1}{2}\{\bar{N}(S_2) - \bar{N}(S_1)\} + M(0)]. \quad (19)$$

The expression for the metric (13) is then

$$ds^2 = \{F'(\tau)\}^2 dS^2, \quad (20)$$

where

$$dS^2 = \frac{\bar{n}(S_2)\bar{n}(S_1)dS_2dS_1 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2}{\bar{n}^2(\tau)}, \quad (21)$$

and  $\tau, r$  stand for the expressions on the right-hand sides of (18) and (19) respectively.

At this point, having used the postulates of general relativity *except for postulate (e) in its quantitative aspect* (equations (3)), we might construct the argument given below to justify the proposition that the regraduation of a proper-time clock altered the space-time—from (13) to (21) in our case. We do not claim that the reader will be convinced by this argument; we are only concerned to show how the idea that the space-time is changed may have arisen. The argument would run as follows:—

(i) Every event  $(s_1, s_2, \theta, \phi)$  in (13) is represented by an event  $(S_1, S_2, \theta, \phi)$  in (21).

(ii) On the one hand, all that was known of the space-time (13) was that its metric had the form exhibited in (13) in which two undetermined functions,  $n$  and  $r$ , of  $s_1, s_2$  entered in a certain way. These functions involved  $s_1, s_2$  in the particular combinations given by (11), (8), and (12), (8) respectively. On the other hand, the metric (21) has the same form as (13), and its coefficients also contain two unspecified functions  $\bar{n}, r$  of  $S_1, S_2$ . These functions are the combinations of  $S_1, S_2$  given by (18), (17), and (19), (17) respectively, in which the unspecified functions  $\bar{n}, m$  play exactly the same rôle as did previously the unspecified functions  $n, m$ .

(iii) The finite equation of the observer's geodesic is formally the same in the space-time (21) as it was in the space-time (13) (equations (7) and (16)).

(iv) The differential equations of all other geodesics (null and non-null) in the space-time (21) are obtained by rewriting those of (13), replacing  $s_1, s_2$  by  $S_1, S_2$ , and the functions  $n(s_1, s_2), r(s_1, s_2)$  by  $\bar{n}(S_1, S_2), r(S_1, S_2)$  respectively.

(v) Hence finally, the observer might conclude that the space-times with metrics (13) and (21) were indistinguishable as far as his description of the material system had so far gone, and that therefore regraduation of proper-time had produced a transition from one space-time to another of, in general, intrinsically different curvature.

We note that we have so far used only the *kinematical* postulates of general relativity, together with the postulate that there is a qualitative parallelism between the properties of the material system and the geometrical properties of space-time.

### 5. PROPER-TIME REGRADUATION AND EINSTEIN'S GRAVITATIONAL EQUATIONS

The remark at the end of the last paragraph makes it clear that the theory we have considered so far is not, strictly speaking, general relativity, an integral feature of which is equations (3). We now examine the effect of taking them into account.

For brevity we write

$$S_1 = x_1, \quad S_2 = x_2, \quad \theta = x_3, \quad \phi = x_4, \quad \sigma = \log [F'(\tau)], \quad (22)$$

so that (20), (21) now become

$$dS^2 = e^{2\sigma} d\tau^2, \quad (23)$$

$$dS^2 = 2g_{12}dx_1dx_2 + g_{33}dx_3^2 + g_{44}dx_4^2, \quad (24)$$

$$\text{where} \quad g_{12} = g_{21} = \frac{1}{2}\bar{n}(x_1)\bar{n}(x_2)/\bar{n}^2(\tau), \quad g_{33} = g_{44}/\sin^2 x_3 = -r^2/\bar{n}^2(\tau), \quad (25)$$

and the remaining  $g_{ij}$  are all zero. In these equations  $\tau, r$  are given by (18) and (19) with  $S_1, S_2$  replaced by  $x_1, x_2$  respectively.

We denote by  $T_{ij}$  the energy-tensor calculated for the metric (23) from the right-hand sides of equations (3) and by  $t_{ij}$  the energy-tensor similarly calculated for (24). Using formulæ given by Eisenhart\* we can express  $T_{ij}$  in terms of  $t_{ij}$  as follows:—

$$-\kappa T_{ij} = -\kappa t_{ij} + 2\sigma_{ij} + g_{ij}\{(e^{2\sigma} - 1)\lambda - 2\Delta_2\sigma - \Delta_1\sigma\}, \quad (26)$$

where

$$\sigma_{ij} = \frac{\partial^2 \sigma}{\partial x_i \partial x_j} - \frac{\partial \sigma}{\partial x_k} \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} - \frac{\partial \sigma}{\partial x_i} \frac{\partial \sigma}{\partial x_j}, \quad (27)$$

$$\Delta_2\sigma = g^{ij} \left[ \frac{\partial^2 \sigma}{\partial x_i \partial x_j} - \frac{\partial \sigma}{\partial x_k} \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \right], \quad (28)$$

$$\Delta_1\sigma = g^{ij} \frac{\partial \sigma}{\partial x_i} \frac{\partial \sigma}{\partial x_j}, \quad (29)$$

and the  $\left\{ \begin{matrix} k \\ ij \end{matrix} \right\}$  are the Christoffel symbols for (24). In obtaining (26) we have assumed that scales are chosen so that the numerical values of  $\epsilon, \kappa$ , and  $\lambda$  do not alter in passing from the space-time (23) to the space-time (24).

If now it could be shown that

$$T_{ij} \equiv t_{ij}, \quad (30)$$

the argument at the end of the preceding section would be considerably reinforced. The observer would then indeed be justified in thinking that regraduation of proper-time had produced a change from the space-time (23) to (24) and that this change had left the energy-tensor (in its covariant form at least) unchanged in value. But we can show that (30) is, in general, impossible except for two trivial types of regraduation, as follows:—

Consider the infinitesimal regraduation in which

$$\sigma = \epsilon q(x_1, x_2), \quad (31)$$

where powers of  $\epsilon$  higher than the first are negligible. We then have that  $\Delta_1\sigma = 0$ , and that the third term on the right-hand side of (27) is also negligible. If (30) be true, then  $T_{11} = t_{11}$  and  $T_{22} = t_{22}$ , and these equations, by (25) and (26), reduce to  $\sigma_{11} = 0$  and  $\sigma_{22} = 0$ . Calculating the Christoffel symbols for (24), the last two equations become

$$\frac{\partial^2 q}{\partial x_1^2} - \frac{1}{g_{12}} \frac{\partial g_{12}}{\partial x_1} \frac{\partial q}{\partial x_1} = 0, \quad \frac{\partial^2 q}{\partial x_2^2} - \frac{1}{g_{12}} \frac{\partial g_{12}}{\partial x_2} \frac{\partial q}{\partial x_2} = 0,$$

\* L. P. Eisenhart, *Riemannian Geometry*, § 28, Princeton Univ. Press, 1926.

whence we obtain

$$\frac{\partial q}{\partial x_1} = Q_2(x_2)g_{12}, \quad \frac{\partial q}{\partial x_2} = Q_1(x_1)g_{12}, \quad (32)$$

where  $Q_2, Q_1$  are arbitrary functions of  $x_2$  and  $x_1$  respectively. Now let  $F_\epsilon$  be the regraduation function corresponding to (31). Then

$$e^\sigma = F'_\epsilon(\tau) = 1 + \epsilon q(x_1, x_2).$$

But, by (18),  $\bar{N}(\tau) = \frac{1}{2}\{\bar{N}(x_1) + \bar{N}(x_2)\}$ , and hence, using (17) and (25), the equations (32) become

$$F''_\epsilon(\tau)\bar{n}(\tau) = \epsilon Q_2(x_2)\bar{n}(x_2) = \epsilon Q_1(x_1)\bar{n}(x_1) = \epsilon a. \quad (32a)$$

Since these must be true for all values of  $x_1, x_2$ ,  $a$  must be a constant.

Again, to our order of approximation,

$$\Delta_2\sigma = g^{ij}\sigma_{ij}.$$

If therefore  $T_{ij} = t_{ij}$ , we must have

$$2\frac{\sigma_{ij}}{g_{ij}} = -\{(e^{2\sigma} - 1)\lambda - 2\Delta_2\sigma\},$$

so that

$$\frac{\sigma_{12}}{g_{12}} = \frac{\sigma_{33}}{g_{33}} = \frac{\sigma_{44}}{g_{44}}.$$

These last equations yield

$$\Delta_2\sigma = 4\frac{\sigma_{12}}{g_{12}},$$

and the equation

$$2\sigma_{12} + g_{12}\{(e^{2\sigma} - 1)\lambda - 2\Delta_2\sigma\} = 0$$

then reduces to

$$3\frac{\partial^2 q}{\partial x_1 \partial x_2} - \lambda q g_{12} = 0.$$

On using (32), (32a), and (18) we find

$$+ 3a\frac{d}{d\tau}\left(\frac{1}{\bar{n}(\tau)}\right) - \lambda q = 0.$$

Differentiating this equation with respect to  $\tau$  and using (32a) again, we finally obtain

$$a\left\{3\frac{d^2}{d\tau^2}\left(\frac{1}{\bar{n}(\tau)}\right) - \frac{\lambda}{\bar{n}(\tau)}\right\} = 0.$$

If therefore we do not impose any restrictions on  $\bar{n}(\tau)$ , the only solution of this equation is  $a = 0$ . We shall consider this general case in a moment. We can, however, also satisfy the foregoing equation for  $a \neq 0$  in the special case for which

$$3\frac{d^2}{d\tau^2}\left(\frac{1}{\bar{n}(\tau)}\right) - \frac{\lambda}{\bar{n}(\tau)} = 0.$$

This means that

$$\bar{n}(\tau) = e^{-\sqrt{\frac{\lambda}{3}}\tau}.$$

There is only one permissible regraduation given by (32a), viz.:

$$F_\epsilon(\tau) = c + b\tau + \epsilon\frac{3}{\lambda}ae^{\sqrt{\frac{\lambda}{3}}\tau},$$

where  $a, b, c$  are constants, and also only one permissible function  $n$  in (13), viz.:

$$n(x) = \frac{1}{b}e^{-\sqrt{\frac{\lambda}{3}}(x-a)/b}.$$

This last result follows from the relation

$$\bar{n}(x) = n[F_*(x)] \cdot F'_*(x).$$

Hence, for a metric in which the function  $n$  has this particular form, there is a regraduation function—depending on  $n$ —which makes  $T_{ij} = t_{ij}$  and for which  $a \neq 0$ . This metric is of “de Sitter” type.\*

Returning to the general case  $a = 0$ , we have by (32a)

$$0 = Q_1 = Q_2 = F''_*(\tau).$$

Hence we have: if  $T_{ij} = t_{ij}$  for an infinitesimal regraduation  $F_*$  from  $s$  to  $S$ , then

$$s = F_*(S) = (1 + \alpha)S + \beta,$$

where  $\alpha, \beta$  are constants proportional to  $\epsilon$ . But this regraduation is trivial as it corresponds to a change of origin ( $\beta$ ) of time and to a change of scale in the unit used for time measurement ( $\alpha$ ). Changes of this type do not produce a transition from one space-time to another, as is obvious from the general formula (2) for the metric in which the constant  $a$  can be absorbed into the coordinates by a change of scale.

Summarising the results of this and the last section, we may say:

Regraduation of proper-time from  $s$  to  $S$  might be regarded as producing a change from one space-time to another provided that Einstein's gravitational equations did not form part of general relativity. But since they do, the transition from one space-time to another is, in general, impossible because the material energy-tensors are not equivalent. Regraduation then reduces itself to a change of coordinates, as is most easily seen by expressing the metric in terms of light-signal coordinates, the analytical expressions of the metric, the energy-tensor, etc., being altered according to the well-known rules of the tensor calculus.

It is tempting to go further and conclude that regraduation of proper-time is not possible in general relativity, a conclusion which is not justified, in our view, for the following reason. Regraduation is defined, both here and in kinematical relativity, by equation (1) and by nothing else. There is in particular no stipulation in the definition that the “regraduated” time-variable must be the arc-length of a geodesic. An observer in general relativity must, as we have seen in section 3, carry more than one clock with him (or keep on regraduating his proper-time clock!). Application of the transformation (1) to proper-time therefore merely produces a change from one of the observer's clocks to another or, alternatively, to one of the parameters  $\mu$  in terms of which the equations of the geodesics have the form (4) instead of (5).

## 6. A PARTICULAR REGRADUATION

Regraduation of clocks came into prominence in kinematical relativity but, as we have seen, it is possible in general relativity also where it appears as a coordinate transformation. To illustrate this we consider a generalised case of the regraduation from “kinematical” to “dynamical” time of kinematical relativity which also has the effect of making the space-time (6) conformal to a static space-time.

Since the equations which express  $\tau$  and  $r$  in terms of  $S_1, S_2$  ((18) and (19)) are identical in form with those which express  $t$  and  $r$  in terms of  $s_1, s_2$  ((11) and (12)), equation (20) may also be written

$$ds^2 = \{F'(\tau)\}^2 \left\{ d\tau^2 - \frac{m^2(r)d\tau^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}{\bar{n}^2(\tau)} \right\}.$$

It is possible to choose a regraduation function and a corresponding coordinate  $\tau_m$  so that

$$\bar{n}(\tau_m) \equiv 1. \quad (33)$$

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\* In the de Sitter universe  $1/\bar{n}^2(t) = e^{2\sqrt{\frac{\lambda}{3}}t}$ ,  $m(r) = 1$ . See, for example, G. C. McVittie, *Cosmological Theory*, p. 64, Methuen, 1937.

Let  $f$  be the regraduation function in question; then by (33) we have

$$n\{f(\tau_m)\} \cdot f'(\tau_m) = 1,$$

and since  $t = f(\tau_m)$ , it follows that

$$\tau_m = \int n(t) dt + (\text{constant}).$$

Hence  $f$  is defined by its inverse  $f^{-1}$  by

$$f^{-1}(x) = \int n(x) dx + (\text{constant}).$$

Again  $\bar{N}(x) = \int \bar{n}(x) dx$ , so that, for the regraduation which produces (33), we have

$$\bar{N}(\tau_m) \equiv \tau_m.$$

Hence, by (18) and (19),  $\tau_m$  and  $r$  are given in terms of the light-signal coordinates corresponding to the regraduation  $s = f(S)$  by

$$\tau_m = \frac{1}{2}(S_2 + S_1), \quad M(r) = \frac{1}{2}(S_2 - S_1) + M(o). \quad (34)$$

If therefore we define a new "radial" coordinate by  $dR_m = m(r)dr$  and adjust the constant of integration so that  $R_m = 0$  when  $r = 0$ , we can write (34) in the form

$$\tau_m = \frac{1}{2}(S_2 + S_1), \quad R_m = \frac{1}{2}(S_2 - S_1).$$

Hence  $\tau_m, R_m$  are coordinates of the kind used in kinematical relativity. We thus reach the conclusion:

In the space-time (6) the observer at the origin of spatial coordinates can regraduate his proper-time and his  $t$ -clock by

$$s = f(S), \quad t = f(\tau_m),$$

where

$$f^{-1}(x) = \int n(x) dx + (\text{constant}),$$

so that the expression for the metric becomes

$$ds^2 = \{f'(\tau_m)\}^2 \{d\tau_m^2 - dR_m^2 - [M^{-1}(R_m + M(o))]^2 (d\theta^2 + \sin^2 \theta d\phi^2)\},$$

where  $\tau_m, R_m$  are "kinematical" coordinates. The metric is thus reduced to a form conformal to a static metric.

The coordinate  $\tau_m$  is a generalisation of the "dynamical" time of kinematical relativity for any pair of functions  $n, m$ . It will be remembered that Milne's theory throws up a particular form of (6) in which

$$n(t) = t_0/t, \quad m^2(r) = 1/(1 + r^2/R_0^2),$$

so that  $\tau_m = t_0 + t_0 \log(t/t_0)$ , which is Milne's transformation from dynamical to kinematical time.

I wish to express my thanks to Dr A. G. Walker for many helpful suggestions and criticisms.

#### SUMMARY

The changes in his description of events brought about by an arbitrary regraduation of an observer's clock are examined, taking the axioms of general relativity as fundamental. It is shown that regraduation does not imply a change from one Riemannian space-time to another but merely a coordinate transformation within space-time. A generalisation of the "dynamical time" of kinematical relativity is a by-product of the investigation.



XVIII.—The Riemann Tensor in a Completely Harmonic  $V_4$ . By H. S. Ruse,  
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(MS. received April 5, 1945. Read June 4, 1945)

I. STATEMENT OF THE PROBLEM

If  $(x_0^i)$  is a fixed point of a Riemannian  $V_n$  of fundamental tensor  $g_{ij}$ , and if  $s$  is the geodesic distance between it and a variable point  $(x^i)$ , then the  $V_n$  has been called *centrally harmonic* with respect to the *base-point*  $(x_0^i)$  if

$$\Delta_2 s \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial s}{\partial x^j} \right)$$

is a function of  $s$  only, and *completely harmonic* if this holds for every choice of base-point  $(x_0^i)$ . A flat  $V_n$  ( $g_{ij} = \delta_{ij}$ ) is obviously completely harmonic, since for such a space  $s = \sqrt{\{\Sigma (x^i - x_0^i)^2\}}$  and

$$\Delta_2 s \equiv \Sigma \frac{\partial^2 s}{(\partial x^i)^2} = \frac{n-1}{s}.$$

The concept of such spaces arose out of an attempt to find a single general formula for Hadamard's "elementary" solution of the tensor generalisation  $\Delta_2 V = 0$  of Laplace's equation (Ruse, 1930-31; Copson and Ruse, 1940). In the second of these papers it was shown how to obtain the conditions, in terms of the Riemann tensor, that a  $V_n$  should be completely harmonic. The first condition was

$$R_{ij} = k g_{ij} \quad (k \text{ const.}), \quad (\text{A})$$

showing that a completely harmonic space is an Einstein space. The second was a condition that may be written

$$\Sigma' g^{pr} g^{qs} R_{pijq} R_{rkl s} = \theta \Sigma' g_{ij} g_{kl}, \quad (\text{B})$$

where  $\Sigma'$  denotes the sum taken over all permutations of the free suffixes  $i, j, k, l$ .  $\theta$  is a scalar and  $R_{ijkl}$  the Riemann tensor (skew in  $i, j$  and in  $k, l$ ). The remaining conditions were infinite in number and involved the covariant derivatives of  $R_{ijkl}$ . They were too complicated to be given in terms of the Riemann tensor itself, but were expressed instead in terms of the normal tensors. Walker (1942) has given another method of obtaining them.

Now every  $V_n$  of constant curvature is completely harmonic, and it seems quite probable, as Copson and I suggested (*loc. cit.*), that every completely harmonic  $V_n$  is of constant curvature. It is certainly so for  $n=2$  and  $n=3$ , and Walker has shown it also to be so when

(a)  $V_n$  is conformal to a flat space;

(b)  $n=4$  and  $V_n$  is of signature  $\pm 2$ .

Condition (A) is alone sufficient to prove the result for  $n=3$  and for Walker's case (a). His (b) requires the use of both (A) and (B). The purpose of the present paper is to obtain all types of  $V_4$  that are algebraically possible when the Riemann tensor satisfies both (A) and (B). It is found that, when the signature is not  $\pm 2$ , there is no algebraic necessity for the  $V_4$  to be of constant curvature. *A fortiori*, the same is true for a  $V_n$  with  $n \geq 5$ . It therefore seems probable that, even when (A) and (B) are regarded as partial differential equations in the  $g_{ij}$  and not merely as algebraic conditions imposed upon the Riemann tensor, they cannot alone require the  $V_n$  to be of constant curvature, though this is a point that is still unsettled. So also is the question of what limitation is placed on  $V_n$  by the remaining infinite sequence of conditions for a completely harmonic space.

This paper therefore makes comparatively small though definite headway in the problem

of determining the nature of completely harmonic spaces. What may be of greater interest than the result itself is the illustration it provides of methods developed in four recent papers (Ruse, 1944, 1945 *a*, 1945 *b*, 1946), in which the Riemann tensor for a  $V_4$  is regarded as defining, at each point ( $x^i$ ) of  $V_4$ , a quadratic complex of lines in the projective 3-space associated with the point ( $x^i$ ). The problem of solving (A) and (B) completely as simultaneous equations in  $R_{ijkl}$ —for that is what it amounts to—is by no means a trivial one even for the case  $n=4$ , and has provided an entirely unexpected application of the theory of the Riemann complex.

## 2. GEOMETRICAL SIGNIFICANCE OF CONDITIONS (A) AND (B)

It is a consequence of conditions (A) and (B) that, at every point of  $V_4$ , the equations

$$R_{ij} \xi^i \xi^j = 0, \quad (2.1)$$

$$g^{pr} g^{qs} R_{pijq} R_{rkls} \xi^i \xi^j \xi^k \xi^l = 0 \quad (2.2)$$

hold for every vector  $\xi^i$  such that

$$g_{ij} \xi^i \xi^j = 0, \quad (2.3)$$

that is, for every null vector  $\xi^i$ .

In the projective  $S_3$  at infinity in the tangent-space  $T_4$  at any point ( $x^i$ ) of  $V_4$ , the equation

$$R_{ijkl} p^{ij} p^{kl} = 0$$

represents the *Riemann complex* (Ruse, 1944), the  $p^{ij}$  being current Plücker co-ordinates. It will be assumed in the first place that this is a proper quadratic complex—that is, that it is not a pair of linear complexes. This restriction will be removed later.

If we write

$$S_{pq} \equiv S_{qp} \equiv R_{pijq} \xi^i \xi^j,$$

then, in  $S_3$ ,  $S_{pq} X^p X^q = 0$  is the equation of the complex cone of the point  $\xi^i$ , which, by (2.3), is any point on the fundamental quadric. Conditions (2.1) and (2.2) may now be written

$$g^{kl} S_{kl} = 0, \quad (2.4)$$

$$g^{pr} g^{qs} S_{pq} S_{rs} = 0. \quad (2.5)$$

Equation (2.4) states that the complex cone of any point  $\xi^i$  on the fundamental quadric is outpolar to that quadric, or, taken in its original form (A), that the quadratic complex is self-polar (of the first kind) with respect to the fundamental quadric (Ruse, 1944, p. 71; and 1946).

Now

$$(g^{ik} g^{jl} - g^{il} g^{jk})(S_{ik} S_{jl} - S_{il} S_{jk}) \equiv 2(g^{pq} S_{pq})^2 - 2g^{pr} g^{qs} S_{pq} S_{rs} = 0 \quad (2.6)$$

by (2.4) and (2.5).

Consider the pencil of quadrics  $S_{ij} + \sigma g_{ij}$  defined by the cone  $S_{ij}$  and the non-degenerate quadric  $g_{ij}$ . In the usual notation, its characteristic equation  $|S_{ij} + \sigma g_{ij}| = 0$  is

$$\Delta \sigma^4 + \Theta \sigma^3 + \Phi \sigma^2 + \Theta' \sigma + \Delta' = 0. \quad (2.7)$$

All the coefficients except  $\Delta$ , which is equal to the determinant  $g$ , are zero;  $\Delta'$  because it is equal to the determinant  $|S_{ij}|$ , which is zero because  $S_{ij}$  is a cone;  $\Theta'$  because the vertex  $\xi^i$  of this cone lies upon the quadric (Sommerville, 1934, p. 319);  $\Phi$  because it is equal to  $g$  times the left-hand side of (2.6); and  $\Theta$  because it is equal to  $g$  times the left-hand side of (2.4).

To find the precise nature of the relationship of the cone  $S_{ij}$  to the fundamental quadric, take a set of non-homogeneous co-ordinates  $(x, y, z)$  with origin  $O$  at the vertex of the cone, and with  $x$ - and  $y$ -axes along the generators to the quadric through  $O$ . The tangent-plane to the quadric at  $O$  is then  $z=0$ , and the equations of the quadric ( $g$ ) and cone ( $S$ ) are respectively of the forms

$$(g) \quad cz^2 + 2fyz + 2gzx + 2hxy + 2rz = 0, \quad (2.8)$$

$$(S) \quad a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0. \quad (2.9)$$

The characteristic equation (2.7) is now

$$h^2 r^2 \sigma^4 + 2h h' r^2 \sigma^3 - r^2 (a' b' - h'^2) \sigma^2 = 0. \quad (2.10)$$

As the coefficient  $\Delta$  of  $\sigma^4$  is not zero, neither  $h$  nor  $r$  is zero. Hence  $\Theta = 0$  gives  $h' = 0$ . Consequently  $\Phi = 0$  gives  $a' b' = 0$ , so one at least of  $a'$  and  $b'$  is zero. Take  $b' = 0$ . Then the equation of the cone is reduced to the form

$$(S) \quad a' x^2 + c' z^2 + 2f' yz + 2g' zx = 0, \quad (2.11)$$

where  $a'$  may or may not be zero. The Segre characteristic of the pencil is in general [4] (Sommerville, 1934, p. 271), but may have one of the forms [(31)], [(22)], [(211)] if one or more of the coefficients  $a', c', f', g'$  are zero.

Suppose  $a' \neq 0$ . Then the cone cuts the tangent-plane  $z = 0$  of the quadric where  $x^2 = 0$ —that is, it touches it along the generator  $z = 0 = x$  of the quadric. This generator therefore lies upon the cone and so belongs to the quadratic complex  $R_{ijkl}$ .

If  $a' = 0$ , then the cone degenerates into a pair of planes of which one is the tangent-plane  $z = 0$  to the quadric. The tangent-plane contains the two generators of the quadric that pass through the vertex  $O$  of the degenerate cone, so both belong to the quadratic complex. We therefore have the following theorem:—

*The first two of the conditions for a  $V_4$  to be completely harmonic, namely conditions (A) and (B), require in  $S_3$  that one or both of the generators through every point  $O$  of the fundamental quadric should belong to the quadratic complex  $R_{ijkl}$ .*

Conversely, if at least one generator through every point of the fundamental quadric belongs to the quadratic complex, and if the complex is self-polar with respect to the fundamental quadric (thereby satisfying the Einstein condition (A)), then it also satisfies condition (B). For take any point  $O$  on the quadric and the generators through it as  $x$ - and  $y$ -axes. Its equation is then of the form (2.8), and the complex cone of  $O$ , being of vertex  $O$ , has an equation of the form (2.9). If the generator  $z = 0 = x$  of the quadric belongs to the complex, it is also a generator of the cone, so  $b' = 0$ . As the complex is self-polar with respect to the quadric, we also have  $h' = 0$ , as seen above, so by (2.10) the coefficient  $\Phi$  of  $\sigma^2$  is zero. Thus  $\Phi = 0$  for every point on the quadric, which is equivalent to condition (B) when taken in conjunction with condition (A).

### 3. CONSEQUENCES OF THE THEOREM

It follows from the theorem of § 2 that an infinite number of the generators of the fundamental quadric—at least one through each point upon it—belong to the quadratic complex. Therefore at least one whole regulus belongs to the complex, because a regulus has either all of its lines in common with any quadratic complex or else only four of them, the four being not necessarily all distinct. This fact is obvious from the well-known representation of the lines of a projective 3-space by points in a projective 5-space (see, e.g., Ruse, 1944, § 7). Thus either both systems of generators of the fundamental quadric belong to the Riemann complex  $R_{ijkl}$ , or else all the lines of one belong to it, and only four of the other. In the former case the complex is necessarily harmonic\* with the fundamental quadric as one of its defining quadrics (Hudson, 1905, p. 97; Jessop, 1903, p. 358, ex. 66). So in this case  $R_{ijkl}$  is of the form

$$R_{ijkl} \equiv g_{ik} \beta_{jl} + g_{jl} \beta_{ik} - g_{il} \beta_{jk} - g_{jk} \beta_{il},$$

$\beta_{ij}$  being a symmetric tensor. From the fact that

$$R_{ij} = \frac{1}{2} g_{ij} R \quad (3.1)$$

for an Einstein space, it follows at once that

$$R_{ijkl} = \frac{1}{2} \beta (g_{ik} g_{jl} - g_{il} g_{jk}) \quad (\beta = g^{ij} \beta_{ij}), \quad (3.2)$$

and the  $V_4$  is thus of constant curvature.

\* This use of the word *harmonic* has no connection with its use in the phrase “completely harmonic  $V_4$ ”. When Copson and I adopted the latter terminology it was quite impossible to foresee that the theory would have any connection with that of harmonic quadratic complexes in classical projective geometry.

Now suppose that one whole regulus belongs to the quadratic complex, but only four lines of the other.

At  $(x^i)$  in  $V_4$  take an orthogonal ennuple  $h_a^i \equiv (h_1^i, \dots, h_4^i)$  in such a way that

$$g_{ij} h_a^i h_b^j = \delta_{ab}$$

whether  $ds^2$  is positive definite at  $(x^i)$  or not. If it is not, then some of the  $h_a^i$  will be purely imaginary. Form the ennuplet components of all tensors at  $(x^i)$ ; for example, let

$$R_{abcd} \equiv R_{ijkl} h_a^i h_b^j h_c^k h_d^l.$$

Let  $p^{ab}$  be the ennuplet components of a simple bivector at  $(x^i)$ —that is, the Plücker co-ordinates of a line in  $S_3$  with the tetrahedron  $h_a^i (a=1, 2, 3, 4)$  as tetrahedron of reference. Then if we write

$$\begin{aligned} \chi^1 &= \frac{1}{\sqrt{2}}(p^{23} + p^{14}), & \chi^2 &= \frac{1}{\sqrt{2}}(p^{31} + p^{24}), & \chi^3 &= \frac{1}{\sqrt{2}}(p^{12} + p^{34}), \\ \chi^4 &= \frac{1}{\sqrt{2}}(p^{23} - p^{14}), & \chi^5 &= \frac{1}{\sqrt{2}}(p^{31} - p^{24}), & \chi^6 &= \frac{1}{\sqrt{2}}(p^{12} - p^{34}), \end{aligned}$$

we get the usual representation of the lines of  $S_3$  by points of a projective space  $S_5$  in terms of the co-ordinates  $\chi^a \equiv (\chi^1, \dots, \chi^6)$ , which are effectively those of Klein. Greek suffixes will always run from 1 to 6 and will refer to  $S_5$ . (For a more detailed account of the notation, see Ruse, 1946, §§ 1, 2.) Then the lines of  $S_3$  are represented in  $S_5$  by the points of the 4-quadric

$$\epsilon_{\alpha\beta} \chi^\alpha \chi^\beta = 0,$$

where (*ibid.* (2.7))

$$\epsilon_{\alpha\beta} \chi^\alpha \chi^\beta \equiv \frac{1}{4} \epsilon_{abcd} p^{ab} p^{cd} = (\chi^1)^2 + (\chi^2)^2 + (\chi^3)^2 - (\chi^4)^2 - (\chi^5)^2 - (\chi^6)^2, \quad (3.3)$$

$\epsilon_{abcd}$  being the ennuplet dualising tensor of components  $\pm 1, 0$ . The lines of  $S_3$  that touch the fundamental quadric form a special quadratic complex, which, in  $S_5$ , is represented by the intersection of the  $\epsilon$ -quadric (3.3) with the quadric

$$g_{\alpha\beta} \chi^\alpha \chi^\beta \equiv \sum_{\alpha=1}^6 (\chi^\alpha)^2 = 0 \quad (3.4)$$

(*ibid.* (2.8)). The lines of the Riemann complex likewise correspond in  $S_5$  to the intersection of the  $\epsilon$ -quadric with the 4-quadric

$$R_{\alpha\beta} \chi^\alpha \chi^\beta \equiv \frac{1}{4} R_{abcd} p^{ab} p^{cd} = 0.$$

Also, as the Riemann complex for a completely harmonic space is self-polar of the first kind, the  $6 \times 6$  matrix  $[R_{\alpha\beta}]$  has the form

$$R_{\alpha\beta} = \begin{bmatrix} U & O \\ O & V \end{bmatrix}, \quad (3.5)$$

where  $U, V$  are symmetric  $3 \times 3$  matrices and  $O$  is the null  $3 \times 3$  matrix (*ibid.* (3.8)).

Now the regulus in  $S_3$  that we are supposing to belong to the Riemann complex corresponds in  $S_5$  to the conic in which one of the 2-planes  $\chi^1 = 0 = \chi^2 = \chi^3$  and  $\chi^4 = 0 = \chi^5 = \chi^6$ , say the former, cuts the  $\epsilon$ -quadric (*cf.* Jessop, 1903, p. 211, § 170). Thus every point  $(0, 0, 0, \chi^4, \chi^5, \chi^6)$  on the  $\epsilon$ -quadric—that is, by (3.3), every point  $(0, 0, 0, \chi^4, \chi^5, \chi^6)$  for which

$$(\chi^4)^2 + (\chi^5)^2 + (\chi^6)^2 = 0,$$

must lie on the 4-quadric  $R_{\alpha\beta}$ . Consequently we must have

$$R_{\alpha\beta} = \gamma [(\chi^4)^2 + (\chi^5)^2 + (\chi^6)^2] \quad \text{when} \quad \chi^1 = 0 = \chi^2 = \chi^3,$$

$\gamma$  being a scalar. This means that the matrix  $V$  of (3.5) must be equal to  $\gamma I$ , where  $I$  is the unit  $3 \times 3$  matrix. Thus  $R_{\alpha\beta}$  has the form

$$R_{\alpha\beta} = \begin{bmatrix} U & O \\ O & \gamma I \end{bmatrix} \equiv \begin{bmatrix} a & h & g & . & . & . \\ h & b & f & . & . & . \\ g & f & c & . & . & . \\ . & . & . & \gamma & . & . \\ . & . & . & . & \gamma & . \\ . & . & . & . & . & \gamma \end{bmatrix}, \quad (3.6)$$

say. Hence, by (3.3), the matrix that determines the  $\epsilon$ -characteristic (Ruse, 1945 a, § 4) of the Riemann complex—that is, its Segre characteristic in the ordinary sense (Jessop, 1903, ch. xi; Zindler, 1922, pp. 1129–31)—is

$$R_{\alpha\beta} - \lambda \epsilon_{\alpha\beta} = \begin{bmatrix} U - \lambda I & O \\ O & (\gamma + \lambda)I \end{bmatrix} = \begin{bmatrix} a - \lambda & h & g & . & . & . \\ h & b - \lambda & f & . & . & . \\ g & f & c - \lambda & . & . & . \\ . & . & . & \gamma + \lambda & . & . \\ . & . & . & . & \gamma + \lambda & . \\ . & . & . & . & . & \gamma + \lambda \end{bmatrix}. \quad (3.7)$$

When the signature of  $d_s^2$  at  $(x^i)$  in  $V_4$  is either  $(+++ -)$  or  $(--- +)$ , then  $U$  and  $V$  are complex conjugate matrices (Ruse, 1946, § 4). Therefore, since  $V = \gamma I$ ,  $U$  is equal to  $\gamma^* I$ , the asterisk denoting the complex conjugate. So in this case

$$[R_{\alpha\beta} - \lambda \epsilon_{\alpha\beta}] = \begin{bmatrix} (\gamma^* - \lambda)I & O \\ O & (\gamma + \lambda)I \end{bmatrix},$$

and the  $\epsilon$ -characteristic is therefore  $[(111)(111)]$ . Hence the Riemann complex consists of the tangents to a quadric (Jessop, 1903, p. 211), and this can only be the fundamental quadric itself. We are in fact back again to the case considered above when both reguli of the fundamental quadric belong to the Riemann complex, and the  $V_4$  is of constant curvature. The cases  $(+++ -)$ ,  $(--- +)$  cover all those of one minus or one plus sign, since it is a matter of choice which of the four co-ordinates  $x^i$  of  $V_4$  is initially called  $x_4$ . Thus we have re-established Walker's result that any completely harmonic  $V_4$  of signature  $\pm 2$  is of constant curvature.

In general, however, (3.7) is not of characteristic  $[(111)(111)]$ . Assuming for the moment that none of the latent roots of the matrix  $U$  is equal to  $-\gamma$ , we obviously obtain all the possible characteristics of (3.7) by combining with the Segre characteristic  $[(111)]$  of the  $3 \times 3$  matrix  $(\gamma + \lambda)I$  all the possible characteristics of the pencil  $U - \lambda I$ . These are

$$[(111)], \quad [(11)1], \quad [21], \quad [(21)], \quad [3], \quad [(111)] \quad (3.8)$$

(Bôcher, 1936, p. 309). The following is therefore an exhaustive list of the  $\epsilon$ -characteristics that are possible for a non-degenerate Riemann complex satisfying conditions (A) and (B) for a completely harmonic space:—

$$\left. \begin{array}{lll} [(111)111], & [(111)(11)1], & [(111)21] \\ [(111)(21)], & [(111)3], & [(111)(111)] \end{array} \right\}. \quad (3.9)$$

Of these, the last has already been found and corresponds to the case when  $V_4$  is of constant curvature. The others are the characteristics of all the non-special and non-degenerate complexes that have a repeated quadric, namely the fundamental quadric itself, as singular surface (Jessop, 1903, p. 231; Zindler, 1922, pp. 1130–31).

#### 4. THE DEGENERATE CASES

Enough has already been done to show that conditions (A) and (B) do not impose upon  $V_4$  any algebraic necessity that it should be of constant curvature. This result was established on the assumption that the Riemann complex was non-degenerate—that is, that it did not consist of a pair of linear complexes. This assumption is equivalent to the supposition made above that none of the latent roots of  $U$  is equal to  $-\gamma$ . We can now remove this restriction.

A direct if slightly laborious calculation shows that, when  $R_{ijkl}$  is reducible to the form (3.6), it satisfies conditions (A) and (B) *whatever the values of the elements  $a, b, c, f, g, h$  of  $\mathcal{U}$* . Therefore there is no bar to the assumption that some or all of the latent roots of  $\mathcal{U}$  are equal to  $-\gamma$ , and we may therefore combine the characteristics (3.8) with [(111)] by enclosing any individual index, or any round-bracketed pair or triplet of indices, with the 111 in the round brackets of [(111)]. Omitting the trivial case [(11111)], which corresponds to  $R_{ijkl} \equiv 0$ , we therefore have the following possible  $\epsilon$ -characteristics in addition to those given above:—

$$\left. \begin{aligned} &[(1111)11]; \quad [(11111)1], \quad [(1111)(11)]; \\ &[(2111)1], \quad [(1111)2]; \quad [(21111)]; \quad [(3111)] \end{aligned} \right\} \quad (4.1)$$

## 5. ANALYSIS OF THE DEGENERATE CASES

The list (4.1) includes all cases except one, namely [(2211)], in which the quadratic complex is a pair of linear complexes (Zindler, 1922, p. 1133). If the linear complexes are  $m_{ij}$ ,  $n_{ij}$ , where, from the point of view of the underlying  $V_4$ ,  $m_{ij}$  and  $n_{ij}$  are bivectors, then the equation of the quadratic complex is

$$(m_{ij}p^{ij})(n_{kl}p^{kl}) = 0,$$

and so

$$R_{ijkl} = \frac{1}{2}\{m_{ij}n_{kl} + n_{ij}m_{kl}\} + \rho\epsilon_{ijkl}.$$

The identity

$$R_{ijkl} + R_{iklj} + R_{iljk} \equiv 0,$$

which may be written

$$\frac{1}{4}\epsilon^{ijkl}R_{ijkl} \equiv 0,$$

at once gives

$$\begin{aligned} \rho &= -\frac{1}{2}\epsilon^{ijkl}m_{ij}n_{kl} \\ &= -\frac{1}{2}\epsilon^{ijkl}m_{kl}n_{ij}, \end{aligned}$$

where

$${}^\circ m^{kl} \equiv \frac{1}{2}\epsilon^{ijkl}m_{ij}$$

is the dual of  $m_{ij}$ . So if for brevity the inner product  $r^{ij}s_{ij}$  of any contravariant bivector  $r^{ij}$  and any covariant bivector  $s_{ij}$  is denoted by  $(rs)$ , then

$$R_{ijkl} = \frac{1}{2}(m_{ij}n_{kl} + n_{ij}m_{kl}) - \frac{1}{2}({}^\circ mn)\epsilon_{ijkl}. \quad (5.1)$$

Detailed analysis similar to that given above for the non-degenerate cases shows that, if  $R_{ijkl}$  satisfies conditions (A) and (B), then

$${}^\circ m_{ij} = em_{ij}, \quad {}^\circ n_{ij} = en_{ij}, \quad (5.2)$$

where

$$e = \pm 1.$$

Geometrically these equations mean that each of the linear complexes  $m_{ij}$ ,  $n_{ij}$  is self-polar with respect to the fundamental quadric. Condition (A) by itself merely requires the Riemann complex to be self-polar of the first kind, which does not rule out the possibility that the linear complexes  $m_{ij}$ ,  $n_{ij}$  in  $S_3$  should be polar to one another even when they are non-coincident; it is condition (B) that restricts each to be self-polar. In  $V_4$ , equations (5.2) mean that the bivectors are either both self-dual or both anti-self-dual.

If  $m_{ij}$  and  $n_{ij}$  happen both to be special linear complexes, then, by (5.2), their directrices are generators of the same system of the fundamental quadric. That they could not belong to opposite systems is otherwise evident. For, if they did, we should have  $({}^\circ mn) = 0$  because they intersect, while, instead of (5.2), we should have

$${}^\circ m_{ij} = em_{ij}, \quad {}^\circ n_{ij} = -en_{ij},$$

and hence, by (5.1),

$$\begin{aligned} {}^\circ R_{ijkl} &\equiv \frac{1}{4}\epsilon_{ijpq}\epsilon_{klrs}R^{pqrs} \\ &\equiv \frac{1}{2}({}^\circ m_{ij}{}^\circ n_{kl} + {}^\circ n_{ij}{}^\circ m_{kl}) \\ &= -R_{ijkl}, \end{aligned}$$

and the Riemann complex would be self-polar of the second kind (Ruse, 1944 (6.12)). But as it is known to be self-polar of the first kind, we already have  ${}^{\circ}R_{ijkl} = R_{ijkl}$ , and hence the Riemann tensor would be identically zero and the Riemann complex non-existent. Hence the case when the Riemann complex is a pair of special linear complexes with intersecting directrices is ruled out, this being the case [(2211)] referred to above. All the other degenerate cases appear to be algebraically possible.

Consider in particular the case [(1111)1]. This is the case in which the Riemann complex is a repeated linear complex, obtained from (5.1) by putting  $n_{ij} = m_{ij}$ . We have, therefore,

$$R_{ijkl} = m_{ij}m_{kl} - \frac{1}{12}({}^{\circ}mm)\epsilon_{ijkl}, \quad (5.3)$$

where, by (5.2),  $m_{ij}$  is such that

$${}^{\circ}m_{ij} = \pm m_{ij}. \quad (5.4)$$

But, as is well known, any skew-symmetric tensor  $m_{ij}$  in  $V_4$  satisfies the identity

$${}^{\circ}m^{ik}m_{kl} \equiv -\frac{1}{4}({}^{\circ}mm)\delta_l^i,$$

and so, by (5.4),

$$m^{ik}m_{kl} = -\frac{1}{4}(mm)\delta_l^i. \quad (5.5)$$

Multiplying (5.3) by  $g^{jk}$ , summing for  $j, k$ , and raising the suffix  $i$ , we obtain, with the help of (5.5),

$$R_i^j = -\frac{1}{4}(mm)\delta_i^j, \quad (5.6)$$

which verifies that the  $V_4$  is an Einstein space. Further,

$$\begin{aligned} R^{rjls}R_{rkls} &= [m^{ri}m^{js} - \frac{1}{12}({}^{\circ}mm)\epsilon^{rjs}] [m_{rk}m_{ls} - \frac{1}{12}({}^{\circ}mm)\epsilon_{rkls}] \\ &= m^{ri}m_{rk}m^{js}m_{ls} + \text{terms involving the } \epsilon\text{-tensor} \\ &= \frac{1}{16}(mm)^2\delta_k^i\delta_l^j + \dots \end{aligned}$$

by (5.5), whence, lowering  $i, j$ ,

$$g^{pr}g^{qs}R_{pijq}R_{rkls} = \frac{1}{16}(mm)^2g_{ik}g_{jl} + \dots$$

If the sum is now taken over all permutations of  $i, j, k, l$ , the terms involving the  $\epsilon$ -tensor (represented by dots in the last equation) disappear on account of the skewness of  $\epsilon_{ijkl}$  in all its suffixes, and we obtain

$$\Sigma' g^{pr}g^{qs}R_{pijq}R_{rkls} = \frac{1}{16}(mm)^2\Sigma' g_{ik}g_{jl}, \quad (5.7)$$

which verifies that, if the Riemann tensor has the form (5.3), it satisfies condition (B) for a completely harmonic space. This example is alone sufficient to show that conditions (A) and (B), treated purely algebraically, do not require the  $V_4$  to be of constant curvature.

If in (5.3) we take  $({}^{\circ}mm) = 0$ , then the linear complex  $m_{ij}$  in  $S_3$  is special; and, as it is self-polar, its directrix is a generator of the fundamental quadric. The Riemann tensor now has the form

$$R_{ijkl} = m_{ij}m_{kl}$$

and the right-hand sides of (5.6) and (5.7) are both zero. So in this case the Riemann tensor satisfies conditions (A) and (B) with  $k$  and  $\theta$  both zero. The  $\epsilon$ -characteristic is [(2111)] (Zindler, 1922, p. 1133, no. 57). An example of a non-Einstein  $V_4$  with this  $\epsilon$ -characteristic has been given elsewhere (Ruse, 1945 a, § 5), but it is open to doubt whether an Einstein  $V_4$  can in fact be as specialised as this.

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*Note added in proof, 10th September 1945.*—Since this paper was written, a paper "Sur les espaces riemanniens complètement harmoniques," by A. Lichnerowicz, has appeared in the current volume (1945) of the *Bull. Soc. Math. de France*. Professor Lichnerowicz, with whom I have lately been in correspondence, has also made some headway in the problem of determining the nature of completely harmonic spaces, though by methods entirely different from those of this paper. The question whether such spaces are all of constant curvature remains open.

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XIX.—A Theory of Regraduation in General Relativity. By A. G. Walker,  
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by Sir EDMUND WHITTAKER, F.R.S.

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1. It was remarked by me a few years ago that temporal regratuations, other than trivial changes of zero and unit, had not so far been considered in General Relativity. An interesting paper by Dr G. C. McVittie \* has now appeared in which regratuations are examined in certain spherically symmetric space-times. Under the assumptions made by McVittie it is shown that regratuations can exist for some but not all space-times, those for which they can exist being of a very special form which excludes many space-times generally regarded as significant or interesting. In the present paper I take the matter further and discuss the problem with more generality. It will be shown that the existence of non-trivial regratuations depends firstly upon which theory is being assumed for the derivation of the conservation equations  $T_{ij,j} = 0$ . There are two alternatives, and regratuations are found to be excluded by one, the "geodesic" theory, but not necessarily by the other, the "equivalence" theory.

When the equivalence theory is adopted, the general problem of regratuation depends upon what we mean by physically equivalent models, *i.e.* models transformable into each other by regratuations. A precise definition is given in § 5, and this I shall call the "theory of regratuation", since it is additional to what is usually understood as the General Theory of Relativity. It is, however, consistent with, and in the spirit of, General Relativity, and leads to remarkable results in connection with the Lemaitre universes. For example, a non-static Lemaitre model can in general be regratuated to become static; and, even more remarkable, a regratuation can always be found which will "transform away" the cosmical constant. Lemaitre models are the only systems examined in detail, for it will be shown that with few exceptions these are the only systems in which pressure is isotropic and which admit non-trivial regratuations.

In order to explain the present theory it is necessary to examine closely the relations between the whole and the local space-times, and the constructions of the tensors  $g_{ij}$  and  $T_{ij}$ . This is done in §§ 1, 2, 3, and a discussion of the field equations is given in § 4. Many standard results are given without proof or reference, and a knowledge of the main features of General Relativity is assumed.

2. The concept of a coordinate system is fundamental in General Relativity, and an *event* may be defined as an ordered set of four numbers  $x^i$ . In order to avoid confusion between regratuations and coordinate transformations, the coordinate system will at first be regarded as invariable, the coordinates of an event being permanent labels attached to the event. The physical system ( $P_4$ ) under consideration is the set of all events, and space-time ( $V_4$ ) appears when a Riemannian structure is associated with  $P_4$ .

Associated with an *observer* is a one-dimensional continuous set of events called the observer's *world-line*; an event on this line is *at* the observer, and the world-line may be described as the observer's "history". At each event  $E$  are infinitely many observers, and each observer possesses a *local coordinate system*,  $X^a$ , for the description of his infinitesimal neighbourhood in  $P_4$ . Events for which  $X^a = 0$  are at the observer, and in particular,  $E$  is the event  $X^a = 0$ . For any other event  $X^a$ ,  $X^0$  is the interval of time and  $X^\lambda$  the cartesian components of distance from  $E$  to the event. The  $X^a$  are infinitesimals and the ratios  $X^\lambda/X^0$  are components of velocity. The scalar velocity of light at  $E$  is the same in all directions,

\* 1945. I am indebted to Dr McVittie for permitting me to refer to his MS.

† Roman suffixes will take values 0, 1, 2, 3, and Greek suffixes will take values 1, 2, 3. The summation convention will apply only when a suffix is repeated and occurs once as a subscript and once as a superscript.

say  $c$  in the units of  $X^a$ . The *proper vector* of an observer at  $E$  is the vector  $\delta_0^a$  in his local system, and the *proper fundamental tensor* is  $\delta_{ab}$ , where  $\delta_{00} = 1$ ,  $\delta_{\lambda\lambda} = -c^{-2}$ , and  $\delta_{ab} = 0$  ( $a \neq b$ ).

Each other observer at  $E$  has a similar local system, and his units can be chosen so that his local system is related to the above system by a Lorentz transformation. Transforming this observer's proper vector, it becomes say  $h^a$  in the first observer's local system. If now  $E'(X)$  is an event near  $E$ , then the temporal and distance measures of the interval  $EE'$  as given by the second observer at  $E$  are the projections of the vector  $X^a$  along and orthogonal to  $h^a$ , the latter projection being then multiplied by  $c$ ; here "projection" and "orthogonal" are defined with respect to  $\delta_{ab}$ . In particular, if  $X^a = \epsilon h^a$ , then  $E'$  occurs at the second observer, and  $\epsilon$  is his interval of proper time between  $E$  and  $E'$ .

Consider now the relations between local systems and the basic system  $x^i$ . Then for each event  $E(x)$ , the  $X^a$  are infinitesimals localized at  $E$  and will correspond to differentials  $dx^i$  by relations of the form

$$dx^i = A_a^i(x) X^a \quad (|A_a^i| \neq 0). \quad (1)$$

The coefficients  $A_a^i$  at each event are used to transform tensors from local to basic coordinates. An observer's proper-vector is now  $h^i = A_a^i h^a$ , and from the local tensor  $\delta_{ab}$  we get the field tensor

$$g_{ij}(x) = \delta_{ab} A_i^a(x) A_j^b(x), \quad (2)$$

the matrix  $(A_i^a)$  being the reciprocal of  $(A_a^i)$ . These new tensors are independent of the particular local system used in the transformation (1), the  $A$ 's transforming with the  $X$ 's in passing from one observer to another so that  $dx^i$  in (1) are unaltered.

We now have a fundamental tensor field  $g_{ij}(x)$ , and a proper vector  $h^i$  for each observer at each event. The differential quadratic

$$ds^2 = g_{ij} dx^i dx^j \quad (3)$$

is taken to be the metric of space-time  $V_4$ ; from (2) it has signature  $-2$ . From (2) and the definition of  $h^i$ , it follows that  $h^i$  is a unit time-like vector in  $V_4$ , i.e. satisfies

$$g_{ij} h^i h^j = 1. \quad (4)$$

It also follows that:

*For an observer whose proper vector is  $h^i$  at an event  $x^i$ , the temporal and distance measures of the interval between the events  $x^i$  and  $x^i + dx^i$  are  $g_{ij} h^i dx^j$ , and  $c$  times the component of the vector  $dx^i$  orthogonal to  $h^i$ , respectively.*

An observer's world-line is now a curve in  $V_4$ , and from the definition of proper-vector it follows that the unit tangent vector to this curve at any event is the observer's proper-vector at that event. Also, from the above theorem, the interval of proper-time between any two events of the world-line is  $\int ds$  where  $ds$  is given by (3).

From the definitions of  $c$  and  $\delta_{ab}$  it follows that intervals along light paths at an event  $E$  satisfy  $\delta_{ab} X^a X^b = 0$ . In basic coordinates, therefore, we see from (1) and (2) that a light path satisfies  $g_{ij} dx^i dx^j = 0$ , i.e. is a null curve. The adoption of Fermat's principle now leads to the result that light paths are null geodesics in  $V_4$ .

This completes the description of the purely kinematical properties of space-time.

3. The dynamical properties of a physical system and their representation in space-time concern the energy tensor  $T^{ij}$ . This is primarily a local tensor, and is best described at an event  $E$  in terms of the local system  $X^a$  of an observer having the mean motion of the matter in the neighbourhood of  $E$ . If  $\rho$  is the mass-density and  $p_\lambda^*$  the principal pressures measured by this observer, and if the axial planes  $dX^\lambda = 0$  are taken to be the principal pressure planes, then  $T^{ab}$  is defined by

$$T^{00} = \rho, \quad T^{\lambda\lambda} = p_\lambda^*, \quad T^{ab} = 0 \quad (a \neq b). \quad (5)$$

Transforming now to basic coordinates by means of  $T^{ij} = T^{ab} A_a^i A_b^j$ , we find

$$T^{ij} = \rho h^i h^j + c^{-2} \sum p_\lambda^* h_\lambda^i h_\lambda^j, \quad (6)$$

where  $h^i$  is the proper-vector of the above observer and  $h_\lambda^i$ ,  $\lambda = 1, 2, 3$ , are certain unit space-like vectors orthogonal to each other and to  $h^i$ .

This vector  $h^i$  will be called the *stream-vector* at E. It is unique except when  $\rho = -c^{-2}p_\lambda^*$  for some  $\lambda$ , and since this relation implies either empty space or large negative pressure, it is not satisfied in any sensible physical system. An important property, which follows from the form of (6), is that *the stream-vector is the unit time-like principal vector of the energy tensor*. The world-lines whose tangent vectors are stream-vectors will be called *stream-lines*. In general, therefore, there is just one stream-line through each event. Observers having these curves as world-lines will be called *fundamental observers*.

The pressure at E is isotropic if  $p_1^* = p_2^* = p_3^* = p^*$ , say. It is convenient to write  $p = c^{-2}p^*$  and call  $p$  the pressure at E. It is in fact the equivalent mass-density, the mass-equivalent of energy  $V$  being  $Vc^{-2}$ . We now see from (6) and the properties of orthogonal vectors that when pressure is isotropic,

$$T^{ij} = (\rho + p)h^i h^j - p g^{ij}. \quad (7)$$

An important quantity which gives the density of matter at E apart from radiation-mass (which is included in  $\rho$ ) is  $\rho_0 = g_{ij}T^{ij} = T$ . From (7) this is equal to  $\rho - 3p$ .

4. It is part of the general theory of relativity that the equations expressing conservation of energy and momentum are taken to be

$$T^{ij}_{;j} = 0. \quad (8)$$

There are two standard derivations: the first is from the hypothesis (Synge, 1934) that the paths of free particles are geodesics in  $V_4$ , and the second is from the classical equations of motion and conservation by means of the Principle of Equivalence (Eddington, 1930). It can be deduced from (8) that the path of a particle under certain conditions of symmetry and isolation is a geodesic. On the whole, however, the geodesic hypothesis is the more restrictive, as will appear from the results of the present paper since this hypothesis excludes regraduations whereas the other does not.

Another part of the general theory is the identification of  $T^{ij}$  with a tensor involving only  $g_{ij}$  and its derivatives apart from constants. The possible forms of this tensor are restricted by (8), and the field equations adopted by Einstein are

$$-\kappa T_{ij} = G_{ij} - \frac{1}{2} G g_{ij} + \lambda g_{ij}, \quad (9)$$

where  $T_{ij}$  is the covariant form of  $T^{ij}$ ,  $G_{ij}$  is the contracted curvature tensor of  $V_4$ ,  $G = g^{ij}G_{ij}$ , and  $\kappa$  and  $\lambda$  are constants ( $\kappa \neq 0$ ).

The simplest interpretation of the field equations is that if  $g_{ij}(x)$ ,  $\kappa$  and  $\lambda$  are known, then  $T_{ij}$  and hence the physical state at every event is known, and we have a complete description of some physical system. This, however, is usually regarded as logically unsatisfactory because it subordinates physics to geometry. Alternatively it has been suggested that knowledge of the physical system gives  $T_{ij}(x)$  and that the field equations then serve to determine  $g_{ij}$ , the basic coordinate system having been previously fixed in some primitive way, and  $\kappa$  and  $\lambda$  being supposed known. This again is not logical, for knowledge of the physical state at an event E gives both  $T_{ij}$  and  $g_{ij}$  at E. This follows from the fact that if  $X^a$  are the special local coordinates described in § 3, then the relations (1) as well as the density and pressures are presumed known at E, and these lead to  $g_{ij}$  in (2) as well as to  $T_{ij}$ .

We conclude, therefore, that the field equations are in the nature of a test. Given the totality of local measures of a physical system, including the relations (1) for some basic coordinate system, then  $g_{ij}$  and  $T_{ij}$  are determined at every event, and the physical system possesses the structure of general relativity if (9) is satisfied for some constants  $\kappa$  and  $\lambda$ . This point of view introduces two new ideas. Firstly,  $\kappa$  and  $\lambda$  are now regarded as determined by the field equations rather than put into them, and any physical significance which these constants may have (e.g.  $\kappa = 8\pi\gamma$ ) is presumed to be implicit in the field equations. Secondly, all local measures and hence also  $g_{ij}$  and  $T_{ij}$  depend upon the units of time, length, and mass adopted at each event. There are no *a priori* restrictions upon these units, but the field equations prevent them from being completely arbitrary. These equations thus serve to determine preferential scales of time, length, and mass for any particular system.

When the field equations permit changes of scales, we say that the system admits a

regraduation. For the remainder of this paper we shall be concerned with the study of such systems.

Returning to the constants  $\kappa$  and  $\lambda$ , these are dimensional and may therefore alter in value with a regraduation. A novel feature, however, is that the present interpretation of the field equations allows  $\lambda$  to vanish for some systems even when  $\lambda \neq 0$  for others. In a Lemaitre model, for example, it will be found that if  $\lambda$  is non-zero, then there exists a regraduation which has the effect of making the new  $\lambda$  zero. This fact should go some way towards answering the philosophical question concerning the significance of the cosmical constant.

5. A regraduation must first be considered as a local affair. If  $X^a$  are local coordinates and  $m$  represents mass at an event  $E$ , then a regraduation at  $E$  is a transformation of the form

$$\bar{X}^0 = uX^0, \quad \bar{X}^\lambda = vX^\lambda, \quad \bar{m} = wm \quad (10)$$

for some positive numbers  $u, v, w$  and all  $X^a, m$ . If new values arising from this regraduation are denoted by a bar, then we have at once  $\bar{c} = cvu^{-1}$ ,  $\bar{\rho} = \rho wv^{-3}$ ,  $\bar{p}_\lambda^* = p_\lambda^* wv^{-1}u^{-2}$ , and from (1),

$$\bar{A}_0^i = u^{-1}A_0^i, \quad \bar{A}_\lambda^i = v^{-1}A_\lambda^i. \quad (11)$$

Hence from (5),  $\bar{T}^{00} = wv^{-3}T^{00}$  and  $\bar{T}^{\lambda\lambda} = wv^{-1}u^{-2}T^{\lambda\lambda}$ , so that from (11) and the definitions of  $\bar{g}_{ij}$ ,  $\bar{T}^{ij}$  and  $\bar{T}_{ij}$ , we find

$$\bar{g}_{ij} = u^2 g_{ij}, \quad \bar{T}^{ij} = wv^{-3}u^{-2}T^{ij}, \quad \bar{T}_{ij} = wv^{-3}u^2T_{ij}. \quad (12)$$

If now a regraduation is supposed to take place at every event, then  $u, v, w$  exist for all  $x^i$  and we have equations of the form

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}, \quad \bar{T}_{ij} = e^\psi T_{ij} \quad (13)$$

relating old and new tensor fields. Here,  $\sigma$  and  $\psi$  can be functions of the  $x$ 's for we do not presuppose that  $u, v, w$  are the same for all events, i.e.  $u, v, w$  may be functions of the  $x$ 's. The relations  $\bar{g}_{ij} \propto g_{ij}$  are not surprising when we remember that null directions in  $V_4$  are significant physically and must therefore correspond to null directions in  $\bar{V}_4$ , the space-time with metric

$$d\bar{s}^2 = \bar{g}_{ij} d\bar{x}^i d\bar{x}^j. \quad (14)$$

It follows from  $\bar{g}_{ij} \propto g_{ij}$  that null geodesics in  $V_4$  correspond to null geodesics in  $\bar{V}_4$ , as required since these are light paths.

The relations between old and new proper-times depends generally upon the world-line under consideration. Integrating along the same curve and between the same two events, the old and new intervals of time are  $\int ds$  and  $\int d\bar{s} = e^\sigma \int ds$ .

It must be remembered that a regraduation is *not* a coordinate transformation, the *basic* coordinates  $x^i$  attached to an event remaining unaltered while *local* coordinates are changed as in (10). We have seen that a regraduation changes the structure of space-time,  $V_4$  being replaced by  $\bar{V}_4$ , but this should not be confused with the effect of a coordinate transformation. In (13), both old and new tensors are functions of the same basic coordinates, and the relations between old and new are not those which arise when coordinates are transformed.

Considering now the physical system as a whole, a *uniform regraduation* is defined as one in which  $u, v$  and  $w$  are constants. Thus  $\sigma$  and  $\psi$  are constants, and  $\int d\bar{s} = e^\sigma \int ds$ . Calculating  $\bar{G}_{ij}$  and  $\bar{G}$  for  $\bar{V}_4$ , we see at once that if the old tensors satisfy (9), then the new tensors satisfy

$$-\bar{\kappa}\bar{T}_{ij} = \bar{G}_{ij} - \frac{1}{2}\bar{G}\bar{g}_{ij} + \bar{\lambda}\bar{g}_{ij} \quad (15)$$

where  $\bar{\kappa} = \kappa e^{-\psi}$  and  $\bar{\lambda} = \lambda e^{-2\sigma}$ . These are field equations, and we have simply shown that every system admits all uniform regraduations.

As explained in § 4, there are in our theory no restrictions on  $u, v, w$  except those resulting from the field equations and from the geodesic hypothesis if this is adopted (see § 7). Excluding this hypothesis, then a physical system together with a basic coordinate system, tensors  $g_{ij}$  and  $T_{ij}$ , and constants  $\kappa$  and  $\lambda$  satisfying (9) will be called a *model* ( $M$ ), and we shall say that *models*  $M$  and  $\bar{M}$  are *equivalent* if their tensors satisfy equations of the form (13). It is understood here that  $M$  satisfies (9) and  $\bar{M}$  satisfies (15), but it is not assumed that  $\bar{\kappa}$  and  $\bar{\lambda}$  are related in any definite way to  $\kappa$  and  $\lambda$ . If  $M$  and  $\bar{M}$  are equivalent, then

each can be obtained from the other by a regraduation, and, in our view, both are models of the same physical system.

6. For non-uniform regraduations,  $u$ ,  $v$  and  $w$  are not all constants, and certain frequencies, lengths, velocities, masses, etc., which are constant (as the event varies) in the old scales may become variable in the new. We thus get essentially different descriptions of the same physical system although no conservation laws are broken, these being obeyed in each model in consequence of the appropriate field equations.

If for convenience we adopt a definable universal method of measuring some physical object, then regraduations are restricted accordingly. The simplest example of this is to measure length in relation to time so that the velocity of light has the same value at all events. This gives  $v/u = \text{const.}$  but does not restrict  $\sigma$  or  $\psi$ . Another example is for fundamental observers to define scales of time by means of the frequency of a particular Fraunhofer line, so that  $u = \text{const.}$  and  $\sigma = \text{const.}$  Still another example is to define a unit of mass everywhere by means of a recognisable elementary particle. This gives  $w = \text{const.}$ , and when taken together with the " $c$ " restriction gives  $\psi = -\sigma + \text{const.}$  Other relations between  $\sigma$  and  $\psi$  result from other restrictions. One of the assumptions made by McVittie is that  $\psi = 0$ , but no physical reason was given; an interpretation is that mass is everywhere measured in relation to time and length so that the Newtonian constant of gravitation has a fixed value.

In the present paper we shall consider regraduations in general until § 9, when we shall impose the restriction  $\psi = \psi(\sigma)$  and later (§ 12),  $\psi = n\sigma + a$  where  $n$  and  $a$  are constants. This last includes many of the interesting physical restrictions.

7. According to the geodesic hypothesis, time-like geodesics in  $V_4$  are paths of free particles and therefore significant physically. It follows that a permissible regraduation must be such that these same curves are geodesics in  $\bar{V}_4$ . The coordinates  $x^i$  therefore give a geodesic correspondence between  $V_4$  and  $\bar{V}_4$ , the conditions for which are known to be (Eisenhart, 1926, § 40)

$$\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} = \delta_j^i \chi_{k,i} + \delta_k^i \chi_{j,i}$$

for some vector  $\chi_i$ , where  $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ ,  $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$  are the Christoffel symbols for  $V_4$ ,  $\bar{V}_4$  respectively.

From (13) we find

$$\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} = \delta_j^i \sigma_{,k} + \delta_k^i \sigma_{,j} - g_{jk} g^{il} \sigma_{,l} \quad \left( \sigma_{,i} = \frac{\partial \sigma}{\partial x^i} \right) \quad (16)$$

whence  $\sigma$  must be such that

$$\delta_j^i \sigma_{,k} + \delta_k^i \sigma_{,j} - g_{jk} g^{il} \sigma_{,l} = \delta_j^i \chi_{k,i} + \delta_k^i \chi_{j,i} \quad (17)$$

for some  $\chi_i$ . Multiplying by  $g^{jk} g_{ih}$  and contracting, we find  $\chi_h = -\sigma_{,h}$ . Substituting in (17), we get  $\sigma_{,i} = 0$ , so that  $\sigma$  must be constant.

When  $\sigma$  is constant,  $\bar{G}_{ij} = G_{ij}$  and from (9) and (15) we deduce that either  $\psi$  is also constant or  $T_{ij} \propto g_{ij}$ . From (9), the latter implies  $G_{ij} = \frac{1}{2} G g_{ij}$  and  $V_4$  is therefore an Einstein space, with  $G = \text{constant}$ . This again leads to  $\psi = \text{constant}$ , from (13) and (15).

Without any substantial loss of generality we can assume that a regraduation for which  $\sigma$  and  $\psi$  are both constant is uniform.\* Hence, the only regraduations consistent with the geodesic hypothesis are uniform.

8. For the remainder of this paper the geodesic hypothesis will be excluded, and a regraduation is valid if (13) hold for some functions  $\sigma$ ,  $\psi$ , the respective field equations being (9) and (15). From the first part of (13) we have †

$$\bar{G}_{ij} = G_{ij} + 2\sigma_{,ij} - 2\sigma_{,i}\sigma_{,j} + g_{ij}(\Delta_2\sigma + 2\Delta_1\sigma),$$

where a comma denotes covariant differentiation in  $V_4$ , and

$$\Delta_1\sigma = g^{ij}\sigma_{,i}\sigma_{,j}, \quad \Delta_2\sigma = g^{ij}\sigma_{,ij}. \quad (18)$$

\* This is exact if the velocity of light is everywhere taken to have the same value.

† Eisenhart, 1926, § 28. Eisenhart's  $R_{ij}$  is our  $G_{ij}$ .

Substituting in (15), then from (9) and (13) we find

$$(\kappa - \bar{\kappa}e^\psi)T_{ij} = 2\sigma_{,ij} - 2\sigma_{,i}\sigma_{,j} - g_{ij}(2\Delta_2\sigma + \Delta_1\sigma + \lambda - \bar{\lambda}e^{2\sigma}). \quad (19)$$

A given model can therefore be regraduated if it satisfies (19) for some functions  $\sigma$ ,  $\psi$  and some constants  $\bar{\kappa}$ ,  $\bar{\lambda}$ , and the new model is given by these constants and (13).

Regarding (19) as equations for  $\sigma$  and  $\psi$ , then certain conditions of integrability must be satisfied, and these will ultimately give in explicit form the conditions which a model must satisfy if it is to admit regraduation. This general problem has not been considered, but it is clear from the form of (19) that not every model admits a non-uniform regraduation. This extends McVittie's conclusions, which concern only the case  $\psi=0$ ,  $\bar{\kappa}=\kappa$ ,  $\bar{\lambda}=\lambda$ .

We shall now consider only those regraduations for which  $\psi=\psi(\sigma)$ , and the models which admit them, because these are of some physical interest and because (19) become more tractable.

9. We shall first prove in the case  $\psi=\psi(\sigma)$ ,  $\sigma \neq \text{constant}$ , that the surfaces  $\sigma=\text{constant}$  in  $V_4$  are geodesic parallels, except possibly when  $\psi = -2\sigma + a$  where  $a$  is constant. The condition for this is that  $\Delta_1\sigma$  should be a function of  $\sigma$ , i.e.  $(\Delta_1\sigma)_{,i} \propto \sigma_{,i}$ , this being necessary and sufficient.\* From (19) we have

$$(\Delta_1\sigma)_{,i} = 2g^{jk}\sigma_{,j}\sigma_{,ki} = (\kappa - \bar{\kappa}e^\psi)T_i^j\sigma_{,j} + \sigma_{,i}(2\Delta_2\sigma + 3\Delta_1\sigma + \lambda - \bar{\lambda}e^{2\sigma}),$$

so that it is sufficient to prove that  $T_i^j\sigma_{,j} \propto \sigma_{,i}$ .

From (8), which are consequences of (9), and from the corresponding equations in  $\bar{V}_4$ , we get

$$T_i^j(\psi_{,j} - 2\sigma_{,j}) + T_i^k\left(\left\{\begin{matrix} j \\ jk \end{matrix}\right\} - \left\{\begin{matrix} j \\ jk \end{matrix}\right\}\right) - T_j^k\left(\left\{\begin{matrix} j \\ ik \end{matrix}\right\} - \left\{\begin{matrix} j \\ ik \end{matrix}\right\}\right) = 0.$$

Writing  $\psi_1 = d\psi/d\sigma$ , and substituting from (16), we get

$$(\psi_1 + 2)T_i^j\sigma_{,j} = T\sigma_{,i}. \quad (20)$$

Hence  $T_i^j\sigma_{,j} \propto \sigma_{,i}$  except possibly when  $\psi_1 = -2$ .

When  $\psi_1 = -2$ , then  $T=0$  since  $\sigma$  is not constant. Thus  $\rho_0=0$ , and this case can only apply to a model which is either empty or contains only radiation. Even in this case it is possible that the above theorem still holds in consequence of the conditions of integrability of (19). In the following discussion, regraduations will be restricted if necessary so that the theorem holds when  $\psi_1 = -2$ .

Another consequence of  $T_i^j\sigma_{,j} \propto \sigma_{,i}$  is that  $g^{ij}\sigma_{,j}$  is in a principal direction of  $T_{ij}$ , and is therefore along or orthogonal to the stream-vector at each event, the former being the case when  $g^{ij}\sigma_{,j}$  is time-like. If this vector is space-like, then it at once follows that  $d\sigma=0$  along every stream-line and the temporal regraduation is uniform for each fundamental observer, although not necessarily the same for all such observers. Such a situation promises to be interesting, but is not the kind with which we are primarily concerned in the present paper. It will therefore be assumed that  $g^{ij}\sigma_{,j}$  is time-like, and so is in the direction of the stream-vector.

It follows from this and the first theorem that in a model which admits regraduations of the kind here considered, *the stream lines are geodesics and the surfaces  $\sigma=\text{constant}$  are orthogonal to them.* We see therefore that basic coordinates  $t, x^\lambda$  can be chosen so that the stream-lines are  $x^\lambda=\text{constant}$  and so that the metric of  $V_4$  takes the form

$$ds^2 = dt^2 + g_{\lambda\mu}dx^\lambda dx^\mu. \quad (21)$$

Every regraduation of the kind here considered now satisfies  $\sigma=\sigma(t)$  and hence also  $\psi=\psi(t)$ .

10. The physical systems which we intend to consider in detail are those in which pressure is everywhere isotropic. From the definitions of  $t, x^\lambda$  above and from (7), we have

$$h^i = \delta_0^i, \quad T_{0\lambda} = 0, \quad T_{\lambda\mu} \propto g_{\lambda\mu}.$$

Also  $\sigma_{,i} = \sigma' \delta_i^0$ , where a prime denotes differentiation with respect to  $t$ , and (19) gives

$$\sigma_{,\lambda\mu} \propto g_{\lambda\mu}. \quad \text{But } \sigma_{,\lambda\mu} = -\sigma' \left\{ \begin{matrix} 0 \\ \lambda\mu \end{matrix} \right\} = \frac{1}{2}\sigma' g'_{\lambda\mu}, \text{ whence } g_{\lambda\mu} = F a_{\lambda\mu} \text{ for some } F(t, x) \text{ and } a_{\lambda\mu}(x).$$

\* This follows from Eisenhart. 1926, § 19.

From (9) and (21),  $T_{0\lambda} = 0$  implies  $G_{0\lambda} = 0$ . Calculating  $G_{0\lambda}$ , these equations become  $\frac{\partial^2 \log F}{\partial t \partial x^\lambda} = 0$ , so that  $F$  is of the form  $f(t) \cdot g(x)$ . Writing  $f(t) = -R^2$ , and absorbing  $g(x)$  in  $a_{\lambda\mu}$ , we finally see that (21) becomes

$$ds^2 = dt^2 - R^2 a_{\lambda\mu} dx^\lambda dx^\mu, \quad (22)$$

where  $R = R(t)$  and  $a_{\lambda\mu}$  are independent of  $t$ .

Writing  $G_{\lambda\mu}^*$  for the contracted curvature tensor in the space  $V_3$  with metric  $a_{\lambda\mu} dx^\lambda dx^\mu$ , then from (22) we find

$$G_{\lambda\mu} = G_{\lambda\mu}^* + \theta a_{\lambda\mu}$$

for some  $\theta$ . Substituting in (9) and remembering that  $T_{\lambda\mu} \propto g_{\lambda\mu}$ , we have that  $G_{\lambda\mu}^* \propto a_{\lambda\mu}$ , and  $V_3$  is therefore an Einstein space. But every Einstein 3-space has constant curvature (Eisenhart, 1926, p. 92), whence  $V_3$  is a space of constant curvature. The metric (22) is now recognized as that of a Lemaitre universe, and we have

*The only physical systems in which pressure is everywhere isotropic and which admit non-uniform regradautions of the restricted kind  $\psi = \psi(\sigma)$  are those described by Lemaitre models.*

By modifying  $R$ , the constant curvature of  $V_3$  can be taken to be  $k = 1, 0$ , or  $-1$ , and it will be understood that this has been done. We shall now be concerned with Lemaitre models only, the elements of such a model being  $R(t)$ ,  $k$ ,  $\kappa$  and  $\lambda$ , with  $T_{ij}$  given by (9) and (22), and  $\rho$  and  $p$  by (7) with  $h^i = \delta_0^i$ . We find, as is well known,\*

$$\kappa\rho = 3R^{-2}(R'^2 + k) - \lambda, \quad (23)$$

$$\kappa p = -R^{-2}(2RR'' + R'^2 + k) + \lambda. \quad (24)$$

11. We now have

$$\sigma_{,t} = \sigma' \delta_t^0, \quad \sigma_{,00} = \sigma'', \quad \sigma_{,0\lambda} = 0, \quad \sigma_{,\lambda\mu} = -\sigma' R R' a_{\lambda\mu},$$

and

$$\Delta_1 \sigma = \sigma'^2, \quad \Delta_2 \sigma = \sigma'' + 3R^{-1} R' \sigma'.$$

From (7) and (22), equations (19) therefore reduce to the two equations

$$(\kappa - \bar{\kappa} e^\psi) \rho = -3\sigma'^2 - 6R^{-1} R' \sigma' - \lambda + \bar{\lambda} e^{2\sigma}, \quad (25)$$

$$(\kappa - \bar{\kappa} e^\psi) p = 2\sigma'' + \sigma'^2 + 4R^{-1} R' \sigma' + \lambda - \bar{\lambda} e^{2\sigma}. \quad (26)$$

From (7),  $T_0^0 = \rho$  and  $T = \rho - 3p$ . Hence (20) becomes

$$(\psi_1 + 1)\rho + 3p = 0. \quad (27)$$

This relation is not, however, independent of (25) and (26) and can be deduced from them.

Eliminating  $\psi$  between (25) and (26), we get

$$2\rho\sigma'' + (\rho + 3p)\sigma'^2 + (4\rho + 6p)R^{-1} R' \sigma' + 3(\rho + p)(\lambda - \bar{\lambda} e^{2\sigma}) = 0. \quad (28)$$

This, then, is the only equation which  $\sigma$  must satisfy, and for each solution,  $\psi$  is given by (25) or (26). Hence,

*Every Lemaitre model admits a non-uniform regradaution. Also: for a given Lemaitre model, a regradaution can always be found so that  $\bar{\kappa}$  and  $\bar{\lambda}$  have any chosen values ( $\bar{\kappa} > 0$ ), including  $\bar{\lambda} = 0$  even when  $\lambda \neq 0$ . This proves our earlier statement that the cosmical constant in a Lemaitre universe can always be transformed away. It may not follow that it is always desirable to do so; a static model with non-zero  $\lambda$  may, for example, be preferred to an equivalent non-static model with zero  $\lambda$ .*

12. It is clear from the above equations that any further restriction imposed upon  $\sigma$  and  $\psi$  has the effect of limiting the class of Lemaitre models which admit such a regradaution. For example, McVittie's assumption  $\psi = 0$  implies  $\rho + 3p = 0$ , and so rules out all Lemaitre models which do not satisfy this. It is, however, of interest to consider certain limitations in greater detail, and we shall now consider those Lemaitre models which admit the restricted regradautions

$$\psi = n\sigma + a, \quad (29)$$

where  $n$  and  $a$  are any constants.

\* See H. P. Robertson, 1933, for a detailed account of Lemaitre models. Certain conventions there differ from ours, e.g.  $ds$ , which is a length in Robertson's paper, is a time in ours.

From (27) we have

$$(n+1)\rho + 3p = 0, \quad (30)$$

this, then, being the physical interpretation of (29). We are thus restricted to the class  $\mathcal{N}$  of Lemaitre models in each of which  $\rho$  and  $p$  satisfy a linear relation of the form (30). This relation gives the value of  $n$ , which is therefore known for each member of  $\mathcal{N}$  unless it is an empty model ( $\rho = p = 0$ ). Since the ratio  $\rho/p$  at any event is not affected by a regraduation, every model equivalent to a member of  $\mathcal{N}$  also belongs to  $\mathcal{N}$  and has the same value of  $n$  associated with it. Some well-known members of  $\mathcal{N}$  are the " $p=0$ " Lemaitre models, given by  $n = -1$ , and the "radiation" ( $\rho_0 = 0$ ) models, given by  $n = -2$ .

Substituting from (23) and (24) in (30), we get

$$2RR'' - nR'^2 + \frac{1}{3}\lambda(n-2)R^2 - nk = 0, \quad (31)$$

which is therefore the equation for  $R$  corresponding to (30). Multiplying by  $R^{-n-1}R'$ , this equation can be integrated to give

$$R'^2 = \mu R^n + \frac{1}{3}\lambda R^2 - k, \quad (32)$$

where  $\mu$  is an arbitrary constant. Differentiating and dividing by  $2R'$ , we get

$$R'' = \frac{1}{2}n\mu R^{n-1} + \frac{1}{3}\lambda R. \quad (33)$$

This derivation assumes  $R' \neq 0$ , but (32) and (33) are still equivalent to (31) when  $R' = 0$ ,  $R$  and  $\mu$  then being given by

$$R^2 = \frac{3nk}{\lambda(n-2)}, \quad \mu = -\frac{2k}{n-2}R^{-n}. \quad (34)$$

Substituting from (32) and (33) in (23) and (24), we get

$$\kappa\rho = 3\mu R^{n-2}, \quad \kappa p = -\mu(n+1)R^{n-2}, \quad \kappa\rho_0 = 3\mu(n+2)R^{n-2}. \quad (35)$$

For a sensible model,  $\rho$  and  $\rho_0$  should not be negative, and  $p$ , if negative, should be very small. These conditions are satisfied only when  $\mu \geq 0$  and  $-2 \leq n \leq -1 + \epsilon$ , where  $\epsilon$  is small.

We observe from (35) that  $\mu = 0$  is characteristic of the empty models which belong to  $\mathcal{N}$ . For these,  $n$  is indeterminate, and (32) and (33) are still true but not (34).

When  $n = -1$ , then  $p = 0$  and we have the well-known relation  $\rho \propto R^{-3}$  for such models, giving conservation of mass. This is clearly connected with the fact that the restriction  $\psi = -\sigma + a$  resulted from the definition of mass by means of elementary particles (§ 6).

When  $n = -2$ , then  $\rho_0 = 0$  and  $p \propto R^{-4}$ , which is a well-known relation for radiation models. These models are presumably connected in some reasonable way with the restriction  $\psi = -2\sigma + a$ .

13. In consequence of (29), equation (28) for  $\sigma$  must now be replaced by the first order equation (25). Substituting from (35), we get

$$\sigma'^2 + 2R^{-1}R'\sigma' + \mu R^{n-2}(1 - \kappa^{-1}\bar{\kappa}e^{n\sigma+a}) + \frac{1}{3}\lambda - \frac{1}{3}\bar{\lambda}e^{2\sigma} = 0.$$

Writing

$$\bar{R} = Re^\sigma, \quad \bar{\mu} = \mu\kappa^{-1}\bar{\kappa}e^a, \quad (36)$$

and substituting from (32), this equation becomes

$$R^2\bar{R}'^2 = \bar{\mu}\bar{R}^{n+2} + \frac{1}{3}\bar{\lambda}\bar{R}^4 - k\bar{R}^2. \quad (37)$$

Since only non-uniform regraduations are being considered,  $\sigma$  is not constant, and the possibility  $\bar{R}/R = \text{constant}$  is excluded.

The appropriate space-time after regraduation is  $\bar{V}_4$  with metric  $d\bar{s}^2 = e^{2\sigma}ds^2$ , and this with (22) suggests a change of a basic coordinate from  $t$  to  $\bar{t}$ , where

$$d\bar{t} = e^\sigma dt. \quad (38)$$

We then get

$$d\bar{s}^2 = d\bar{t}^2 - \bar{R}^2 a_{\lambda\mu} dx^\lambda dx^\mu, \quad (39)$$

where  $\bar{R}$  is given by (36). The standard Lemaitre form has thus been recovered for the new model.



From (38), equation (37) can now be written

$$\left(\frac{d\bar{R}}{d\bar{t}}\right)^2 = \bar{\mu}\bar{R}^n + \frac{1}{3}\bar{\lambda}\bar{R}^2 - k. \quad (40)$$

Comparing with (32), we see that  $\bar{\mu}$ , as defined in (36), is the new value of the parameter  $\mu$  after regraduation.

Conversely, if  $M$ ,  $\bar{M}$  are non-static Lemaitre models satisfying (32) and (40) respectively, where  $\mu$  and  $\bar{\mu}$  have the same sign (or vanish together), then they are equivalent. The regraduation from  $M$  to  $\bar{M}$  is given by  $\sigma = \log \bar{R}(\bar{t}) - \log R(t)$  and  $\psi = n\sigma + a$ , where  $t$  and  $\bar{t}$  are related by

$$\frac{d\bar{t}}{\bar{R}(\bar{t})} = \frac{dt}{R(t)}, \quad (41)$$

and

$$a = \log(\kappa\bar{\mu}/\bar{\kappa}\mu). \quad (42)$$

This theorem applies also to the case when one of the models is static ( $R = \text{const.}$ ) except that (32) is replaced by (34) when  $M$  is static, and similarly for  $\bar{M}$ .

The above theorems are remarkably general, and show that in a Lemaitre model of class  $\mathcal{N}$ , the only elements which are invariant under all permissible regraduations are  $k$ ,  $n$ , and the signs of  $\kappa$  and  $\mu$ .

14. Two models of class  $\mathcal{N}$  which are of particular interest are those given by Einstein and de Sitter. In the Einstein model,  $R = R_0$ , a constant, and

$$k = 1, \quad n = -1, \quad \mu = \frac{2}{3}R_0, \quad \lambda = R_0^{-2}. \quad (43)$$

These satisfy (34), and we have from the theorem of § 13:

*Every non-static Lemaitre model of class  $\mathcal{N}$  in which  $k = 1$ ,  $n = -1$ , and  $\mu > 0$  is equivalent to the Einstein model.*

The de Sitter universe is empty, and every representative Lemaitre model therefore has  $\mu = 0$  and  $n$  indeterminate. There are several alternative forms, corresponding to different values of  $k$ , this being due to the fact that the stream-lines are indeterminate. The static form satisfies

$$k = 1, \quad \mu = 0, \quad \lambda = 3R_0^{-2}, \quad R = R_0.$$

Regraduations of a de Sitter model are uninteresting because they merely give other de Sitter models, though they may change the form from static to non-static, or change the value of  $R_0$ .

15. A fact of some interest which arises out of the theorems of § 13 is that we can sometimes regraduate a non-static Lemaitre model so that it becomes static.\* Since a static model is certainly of class  $\mathcal{N}$ ,  $\rho$  and  $p$  being constants, it follows that the only models which can possibly be regraduated to become static are those of class  $\mathcal{N}$ .

Taking  $M$  as a given non-static member of  $\mathcal{N}$ , then  $k$ ,  $n$  and the sign of  $\mu$  are known, and relations of the form (34) must hold for  $\bar{R}$ ,  $\bar{\lambda}$  and  $\bar{\mu}$ . Since  $\bar{\mu}$  must have the same sign as  $\mu$ , we see that  $\mu k(n-2)$  must be negative or zero. Also, from the equations leading to (34), we find that if either  $k$  or  $n-2$  is zero, then the other must be zero.

Conversely, if  $\mu k(n-2) < 0$ , or if  $k = 0$  and  $n = 2$ , then  $M$  is equivalent to a static model in which  $\bar{R}$  can have any assigned value. For equations corresponding to (34) can then be solved for  $\bar{\lambda}$  and  $\bar{\mu}$ , and the theorem of § 13 applies.

16. It has already been mentioned that in the case of a Lemaitre model, a regraduation can always be found so that  $\bar{\lambda} = 0$ . As an example of this and of the calculation described in § 13, let us regraduate an Einstein model so that  $\bar{\lambda} = 0$ .

In this case we see from (43) that  $k = 1$ ,  $n = -1$ , and  $\bar{\mu}$  can have any positive value. It will be found that  $\bar{R}$  varies from 0 to  $\bar{\mu}$ , and as a matter of interest let us choose  $\bar{\mu} = 2R_0$ . Then (40) becomes

$$d\bar{R}/d\bar{t} = \eta(2R_0\bar{R}^{-1} - 1)^{1/2}, \quad \eta = \pm 1.$$

\* This is not equivalent to the result given by McVittie at the end of his paper, because he was concerned not with regraduations but with transformations of basic coordinates.

This shows that  $\bar{R}$  now varies from 0 to  $2R_e$ , and making the substitution

$$\bar{R} = 2R_e \sin^2 \phi/2, \quad (2R_e \bar{R}^{-1} - 1)^{1/2} = \eta \cot \phi/2,$$

we get

$$\bar{t} = R_e(\phi - \sin \phi), \quad (44)$$

where the constant of integration is chosen so that  $\bar{t} = 0$  when  $\bar{R} = 0$ . Eliminating  $\phi$ , we find  $\bar{R}(\bar{t})$ .

We now have all the elements of the new model,  $\bar{\kappa}$  being supposed given. To find the regraduation which produces this model, substitute for  $R$  and  $\bar{R}$  in (41). From (44), this equation becomes  $d\bar{t} = R_e d\phi$ , and taking  $\bar{t} = 0$  in the new model to correspond to  $t = 0$  in the old, we find

$$\bar{t} = t - R_e \sin \frac{t}{R_e}. \quad (45)$$

This, then, is the regraduation of proper-time for all fundamental observers.

Substituting as explained in § 13, we find that the regraduation functions  $\sigma, \psi$  are given by

$$\sigma = 2 \sin^2 \frac{t}{2R_e}, \quad \psi = \frac{3\kappa}{2\bar{\kappa}} \operatorname{cosec}^2 \frac{t}{2R_e}. \quad (46)$$

We thus see that the Einstein model is equivalent to a non-static model in which  $\bar{\lambda} = 0$ . This model oscillates with a constant period  $2\pi R_e$  in both old and new time-scales, and the value of  $\bar{R}$  varies between 0 and  $2R_e$ , so that the model periodically contracts to a point. The new density,  $\bar{\rho}$ , is  $6\bar{\kappa}^{-1}R_e\bar{R}^{-3}$ , and thus varies between  $\frac{3}{2}\bar{\kappa}^{-1}R_e^{-2}$  and  $\infty$ ; the old density is  $\rho = 2\kappa^{-1}R_e^{-2}$ .

17. One other regraduation of special interest is that which gives  $\bar{\kappa} = \kappa$ ,  $\bar{\lambda} = \lambda$ , and  $\bar{\mu} = \mu$ , so that  $a = 0$  from (42). This includes the case considered by McVittie, in which also  $n = 0$ .

It is easily verified that a more general regraduation  $\bar{M} \rightarrow \bar{M}$  in which  $\bar{\lambda}$  has the same sign as  $\lambda$  can always be followed by a uniform regraduation  $\bar{M} \rightarrow \bar{M}$  so that finally  $\bar{\kappa} = \kappa$ , and  $\bar{\lambda} = \lambda$ , (and also  $\bar{t} = t$  if desired). We cannot, however, impose the additional relation  $\bar{\mu} = \mu$ , for it can be verified that under a uniform regraduation,  $\bar{\mu}^2 \bar{\lambda}^{-n} = \mu^2 \lambda^{-n}$ , whence  $\bar{\mu} = \mu$  only when the first regraduation gives  $\bar{\mu} = \mu(\bar{\lambda}/\lambda)^{n/2}$ . Since uniform regraduations are trivial, we deduce that in the general problem of regraduation there is no substantial loss of generality if we assume that  $\bar{t} = t$ ,  $\bar{\kappa} = \kappa$ , and  $\bar{\lambda} = 0$  or  $\lambda$ ; we cannot,\* however, assume that  $\bar{\mu} = \mu$ .

Returning to the regraduation mentioned at the beginning of this section, we now see that this is essentially special except when  $\lambda = 0$  and  $n \neq 0$ .

It will be assumed that  $\bar{M}$  and  $\bar{M}$  are non-static, and comparing (32) and (40) we see at once that the zero of  $\bar{t}$  can be chosen so that  $\bar{R}$  is of the form

$$\bar{R}(\bar{t}) = R(\eta \bar{t}), \quad \eta = \pm 1. \quad (47)$$

The relation between  $t$  and  $\bar{t}$  is found from (41), which integrates to give

$$\bar{t} = H^{-1}(\eta H(t) + \alpha), \quad (48)$$

where  $H' = 1/R$ ,  $H^{-1}(x)$  is the inverse function of  $H(x)$ , and  $\alpha$  is an arbitrary constant. Functions  $\sigma, \psi$  are now given in the usual way,  $\sigma$  being  $\log R(\eta \bar{t}) - \log R(t)$  and therefore not constant unless  $\eta$  and  $\alpha$  have particular values.

When  $\eta = 1$ , we see from (47) that the two models are exactly similar. This means, not that the regraduation is trivial, but that it is equivalent to a correspondence of  $V_4$  with itself. Regarding (48) as defining a correspondence  $P \rightarrow \bar{P}$  in  $V_4$ , then after regraduation, the new description of the physical state at  $P$  is identical with the original description of the state at  $\bar{P}$ .

An illustrative example is the case  $n = 0$ ,  $\lambda > 0$  considered by McVittie. Writing  $\lambda = 3\omega^2$ , then from (33) and (32) we find

$$R(t) = Ae^{\omega t} + Be^{-\omega t}, \quad \mu = \bar{\mu} - 4AB\omega^2,$$

where  $A$  and  $B$  are constants. To obtain (48) generally it is necessary to consider several different cases, and we shall confine ourselves to the one given by McVittie, which is, in our

\* The case when  $\lambda = 0$  is special;  $\bar{\lambda}$  is then arbitrary and we can assume  $\bar{\mu} = \mu$  except when  $n = 0$ .

notation,  $R = Ae^{\omega t}$ . This McVittie describes, rather misleadingly, as of de Sitter type; it is actually a de Sitter model if  $\mu = 0$ , i.e. if  $k = 0$ . We find  $H(t) = -(A\omega)^{-1}e^{-\omega t}$ , and, from (48),

$$\bar{z} = -\omega^{-1} \log (\eta e^{-\omega t} + \beta),$$

where  $\beta$  is a constant replacing  $\alpha$ . We therefore have

$$\begin{aligned} \eta = 1: & \quad \sigma = -\log (1 + \beta e^{\omega t}); \\ \eta = -1: & \quad \sigma = \log (-e^{-2\omega t} + \beta e^{-\omega t}). \end{aligned}$$

When  $\eta = 1$  and  $\beta$  is small,  $\sigma$  is approximately  $-\beta e^{\omega t}$ . This is the approximate result given by McVittie, who considers only infinitesimal regraduations and so does not obtain results corresponding to  $\eta = -1$ , these having no infinitesimal member.

18. This concludes our study of regraduations. We have defined them, and have shown that they are admitted by many systems of General Relativity when that theory is interpreted in a certain reasonable way. In this interpretation, the status of the cosmical constant is changed—it is in fact no longer cosmical in the sense that it is the same for all systems.

Certain restricted regraduations have been considered in detail, and also certain physical systems which admit them, particularly the Lemaitre models. Special regraduations of interest are those which turn a non-static model into a static, those which transform away the cosmical constant, and those which leave constants  $\kappa$ ,  $\lambda$ ,  $\mu$  unaltered in value. Results given by McVittie occur as special cases of these last regraduations.

Finally, it is clear that many problems still remain. There is the study of physical systems with space-time (21) but in which pressure is not isotropic, and there is the singular case  $\psi_1 = -2$  in § 9. Deeper than these is the problem of finding what physical systems admit regraduations for which  $\psi$  is not a function of  $\sigma$ .

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XX.—Evaluation and Application of Certain Ladder-Type Networks. By W. E. Bruges, M.Sc., A.C.G.I., D.I.C., Assoc.M.Inst.C.E. Communicated by Professor M. G. SAY, Ph.D., M.Sc. (With Seven Text-figures.)

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THIS paper has for object the calculation of a ladder network, using trigonometrical functions of real multiples of  $(-1)^{\frac{1}{2}}$  which, in many cases, simplify practical formulæ. The work was prepared particularly for application to transmission lines, conductors in electrical machines, and isolated cylindrical conductors. The effect of the conjunction of two or more dissimilar networks is considered, leading to a method of assessing the impedance of a conductor of any shape embedded in an open slot cut in highly permeable material.

### 1. THE IMPEDANCE OF THE LADDER NETWORK

The circuit in fig. 1 comprises  $(n-1)$  impedances  $\delta z_1$  in series, LN, connected uniformly through  $n$  impedances  $\delta z_2$  to a conductor HK which may sometimes be identified with "earth":  $\delta z_1$  and  $\delta z_2$  are vector quantities of the form  $A+jB$ ,  $A$  and  $B$  being scalar. The vectors  $v_1, v_2, v_3, \dots, v_n$  represent the potential differences between the junctions of the series impedances and the conductor HK. The vectors  $i_1, i_2, i_3, \dots, i_n$  represent the currents in the parallel impedances  $\delta z_2$  in order.  $I_n$  is the vector current flowing into the network at N and out at K, N being at the end of the  $(n-1)$ th series impedance  $\delta z_1$  remotest from L. We then have

$$\delta z_2(i_n - i_{n-1}) = v_n - v_{n-1} = \delta z_1(i_1 + i_2 + i_3 + \dots + i_{n-1}),$$

also

$$v_n = \delta z_2 \cdot i_1 + \delta z_1[i_1 + (i_1 + i_2) + (i_1 + i_2 + i_3) + \dots];$$

and if  $\delta z_1/\delta z_2 = \psi$ , then

$$i_n = v_n/\delta z_2 = i_1(1 + \psi S_{n-1}),$$

where

$$\begin{aligned} i_1 S_n &= i_n + 2i_{n-1} + 3i_{n-2} + \dots + ni_1, \\ i_1 S_{n-1} &= i_{n-1} + 2i_{n-2} + 3i_{n-3} + \dots + (n-1)i_1, \text{ etc.}, \end{aligned}$$

and

$$\begin{aligned} i_{n-1} &= i_1(1 + \psi S_{n-2}), \\ i_{n-2} &= i_1(1 + \psi S_{n-3}), \text{ and so on.} \end{aligned}$$

Further

$$I_n = i_1 + i_2 + i_3 + \dots + i_n = i_1(S_n - S_{n-1})$$

and

$$S_n - S_{n-1} = n + \psi(S_{n-1} + S_{n-2} + \dots + S_1) = U_n \quad (1)$$

with

$$S_1 = 1.$$

Substituting the values so obtained for  $i_n, i_{n-1}, i_{n-2}, \dots$  in the series for  $i_1 S_n$  and dividing by  $i_1$  we have

$$S_n = h_n + \psi(S_{n-1} + 2S_{n-2} + 3S_{n-3} + \dots + S_1),$$

where  $h_n = 1 + 2 + 3 + \dots + n$ ; similarly

$$S_{n-1} = h_{n-1} + \psi(S_{n-2} + 2S_{n-3} + 3S_{n-4} + \dots + S_1),$$

and so on. Repeated substitution of this recurrence relation gives

$$S_n = q_0 + q_1\psi + q_2\psi^2 + \dots + q_{n-1}\psi^{n-1}, \quad (2)$$

where  $q_0 = h_n = 1 + 2 + 3 + \dots + n$ , and  $q_1$  is obtained by multiplying each term of this arithmetical progression by the sum of a similar progression to 1, 2, 3, . . . (n-1) terms, then adding the terms;  $q_2$  is obtained by multiplying each term of  $q_1$  so obtained by the sum of the same arithmetical progression to 1, 2, 3, . . . (n-2) terms, and so on. In practice these

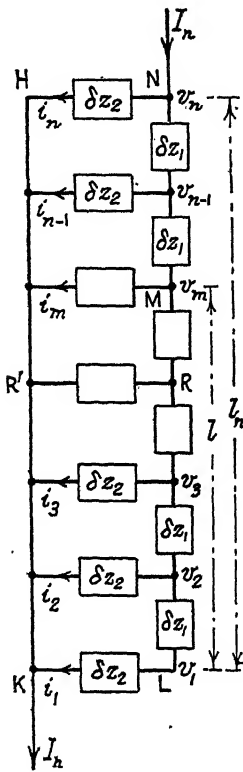


FIG. 1.

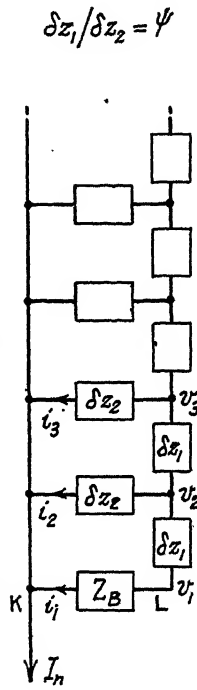


FIG. 2.

Ladder networks.

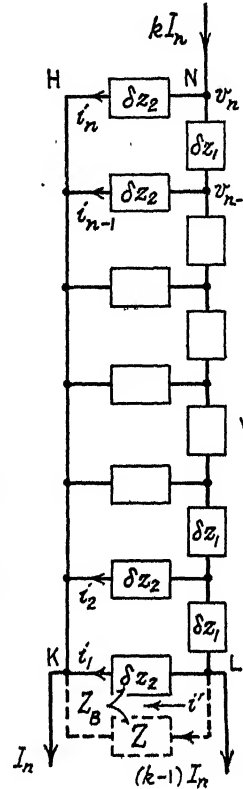


FIG. 3.

terms may be obtained by writing down the progression, first forwards then backwards, moving on the multiplier series a term at a time, multiplying only those terms directly under one another, and adding:

$$q_0 = h_n,$$

$$q_1 = h_{n-1} + 2h_{n-2} + 3h_{n-3} \dots \text{to } (n-1) \text{ terms,}$$

and

$$q_2 = h_{n-2} + (1.2 + 2.1)h_{n-3} + (1.3 + 2.2 + 3.1)h_{n-4} \dots \text{to } (n-2) \text{ terms,}$$

the series in brackets being obtained by multiplying

$$1 + 2 + 3 + 4 + \dots$$

by

$$\dots + 3 + 2 + 1,$$

moving the lower series a term at a time to the right. To obtain  $q_3$  we write down the series so obtained and repeat the process:

$$\dots + 3 + 2 + 1, \\ \dots + 3 + 2 + 1, \\ \dots + 3 + 2 + 1, \\ \dots + 3 + 2 + 1,$$

whence

$$q_3 = h_{n-3} + (1.1.2 + 1.2.1 + 2.1.1)h_{n-4} \\ + (1.1.3 + 1.2.2 + 1.3.1 + 2.1.2 + 2.2.1 + 3.1.1)h_{n-5} \dots \quad (3)$$

to  $(n-3)$  terms, and so on.

The impedance of the network between HK and N is consequently

$$Z_n = \frac{v_n}{I_n} = \frac{i_n \cdot \delta z_2}{i_1(S_n - S_{n-1})} = \frac{i_1 \cdot \delta z_2(1 + \psi S_{n-1})}{S_n - S_{n-1}} = \frac{\delta z_2(1 + \psi S_{n-1})}{U_n} \quad (4)$$

Tables are given of the coefficients of  $\psi$  in  $S_n$  and  $U_n$  for values of  $n$  from 1 to 15.

TABLE I.—TO FIND  $S_n$ 

Coefficient of		$\psi$	$\psi^2$	$\psi^3$	$\psi^4$	$\psi^5$	$\psi^6$	$\psi^7$	$\psi^8$	$\psi^9$	$\psi^{10}$	$\psi^{11}$	$\psi^{12}$	$\psi^{13}$	$\psi^{14}$
$n=$	1														
2	3	1													
3	6	5	1												
4	10	15	7	1											
5	15	35	28	9	1										
6	21	70	84	45	11	1									
7	28	126	210	165	66	13	1								
8	36	210	462	495	286	91	15	1							
9	45	330	924	1,287	1,001	455	120	17	1						
10	55	495	1,716	3,003	3,003	1,820	680	153	19	1					
11	66	715	3,003	6,435	8,008	6,188	3,060	969	190	21	1				
12	78	1,001	5,005	12,870	19,448	18,564	11,628	4,845	1,330	231	23	1			
13	91	1,365	8,008	24,310	43,758	50,388	38,760	20,349	7,315	1,771	276	25	1		
14	105	1,820	12,376	43,758	92,378	125,970	116,280	74,613	33,649	10,626	2,300	325	27	1	
15	120	2,380	18,564	75,582	184,756	293,930	319,770	245,157	134,596	53,130	14,950	2,925	378	29	1
Sign *	+	$+j$	-	$-j$	+	$+j$	-	$-j$	+	$+j$	-	$-j$	+	$+j$	-

TABLE II.—TO FIND  $U_n$ 

Coefficient of		$\psi$	$\psi^2$	$\psi^3$	$\psi^4$	$\psi^5$	$\psi^6$	$\psi^7$	$\psi^8$	$\psi^9$	$\psi^{10}$	$\psi^{11}$	$\psi^{12}$	$\psi^{13}$	$\psi^{14}$
$n=$	1														
2	2	1													
3	3	4	1												
4	4	10	6	1											
5	5	20	21	8	1										
6	6	35	56	36	10	1									
7	7	56	126	120	55	12	1								
8	8	84	252	330	220	78	14	1							
9	9	120	462	792	715	364	105	16	1						
10	10	165	792	1,716	2,002	1,365	560	136	18	1					
11	11	220	1,287	3,432	5,005	4,368	2,380	816	171	20	1				
12	12	286	2,002	6,435	11,440	12,376	8,568	3,876	1,140	210	22	1			
13	13	364	3,003	11,440	24,310	31,824	27,132	15,504	5,985	1,540	253	24	1		
14	14	455	4,368	19,448	48,620	75,582	77,520	54,262	26,334	8,855	2,024	300	26	1	
15	15	560	6,188	31,824	92,378	167,960	203,490	170,544	100,947	42,504	12,650	2,600	351	28	1
Sign *	+	$+j$	-	$-j$	+	$+j$	-	$-j$	+	$+j$	-	$-j$	+	$+j$	-

\* Applicable where  $\psi$  is of the form  $jx/r$ .

To extend the tables downwards for  $\psi^x$ , sum the figures in the column for  $\psi^{x-1}$  and add the bottom existing figure in the  $\psi^x$  column.

In the limit, when  $n$  is large, the process described above for evaluating the coefficients  $q_0, q_1, q_2, \dots$  is equivalent to a successive double integration of these coefficients: thus

$$q_0 = h_n \text{ (with } n \text{ large) approaches } \int_0^n \int_0^n dn \cdot dn,$$

$q_1$  approaches  $\int_0^n \int_0^n q_0 \cdot dn \cdot dn = n^4/4!$ ,  $q_2$  approaches  $\int_0^n \int_0^n q_1 \cdot dn \cdot dn = n^6/6!$ , etc. Hence when  $n$  is large,

$$S_n = \frac{n^2}{2!} + \frac{n^4}{4!} \psi + \frac{n^6}{6!} \psi^2 + \dots,$$

and  $S_n - S_{n-1}$  approaches the differential coefficient of  $S_n$  with respect to  $n$ , i.e.

$S_n - S_{n-1}$  approaches  $\frac{\partial S_n}{\partial n} = n + \frac{n^3}{3!}\psi + \frac{n^5}{5!}\psi^2 + \dots$  (treating  $\psi$  as constant),

and similarly  $U_n - U_{n-1}$  approaches  $1 + \frac{n^2}{2!}\psi + \frac{n^4}{4!}\psi^2 + \dots$

The same results may be obtained by summing the coefficients of  $\psi, \psi^2, \psi^3 \dots$  in turn by the method of differences, and taking the limits when  $n$  is infinite. A rigid proof is possible in this way.

Suppose that the impedance  $NL$  is uniformly distributed over a length  $L_n$ : then the series impedance per unit length is

$$z_1 = n \cdot \delta z_1 / L_n, \quad (5)$$

and the parallel impedance per unit length is

$$z_2 = L_n \cdot \delta z_2 / n. \quad (6)$$

Writing  $\theta = n^2 \psi = L_n^2 \cdot z_1 / z_2$  and  $z_1 z_2 = \delta z_1 \cdot \delta z_2$  we have, for  $n$  large:

$$1 + \psi S_n \text{ approaches } 1 + (\theta/2!) + (\theta^2/4!) + \dots;$$

that is,

$$\lim_{n \rightarrow \infty} (1 + \psi S_n) = \cosh \sqrt{\theta}. \quad (7)$$

Similarly

$$\sqrt{\psi}(S_n - S_{n-1}) \text{ approaches } \sqrt{\theta} + (\theta\sqrt{\theta}/3!) + (\theta^2\sqrt{\theta}/5!) + \dots,$$

that is,

$$\lim_{n \rightarrow \infty} \sqrt{\psi}(S_n - S_{n-1}) = \lim_{n \rightarrow \infty} \sqrt{\psi} \cdot U_n = \sinh \sqrt{\theta}. \quad (8)$$

Also

$$\lim_{n \rightarrow \infty} (U_n - U_{n-1}) = \cosh \sqrt{\theta}, \quad (9)$$

and when  $n$  approaches infinity

$$Z_n = \lim_{n \rightarrow \infty} \frac{\delta z_2 (1 + \psi S_{n-1})}{U_n} = \sqrt{z_1 z_2} \frac{\cosh \sqrt{\theta}}{\sinh \sqrt{\theta}} = \sqrt{z_1 z_2} \coth \sqrt{\theta}. \quad (10)$$

If  $M$  is any point in  $LN$ , where  $LM = l$ , including  $(m-1)$  elements  $\delta z_1$  and  $m$  elements  $\delta z_2$ ,

$$v_m / v_n = i_m / i_n = (1 + \psi S_{m-1}) / (1 + \psi S_{n-1}) = \cosh \sqrt{\theta}_m / \cosh \sqrt{\theta}_n \quad (11)$$

with  $m$  and  $n$  infinite: here  $v_m$  is the potential difference between  $M$  and  $K$ . Thus conditions in any part of the network may be found.

The impedance  $Z_n$  when  $n$  is infinite may also be found by considering conditions at the point  $M$ , where  $v_m = z_2 \cdot dI/dl$ , so that  $z_2/dl$  is an element of parallel impedance corresponding to an element of parallel current  $dI$  ( $dI$  corresponds to the shunt current  $i_m$  in fig. 1). Further,  $dv_m = I z_1 \cdot dl$ , where  $I$  is the current at the point  $M$  in impedances  $\delta z_1$ . Thus

$$d^2 v_m / dl^2 = z_1 \cdot dI/dl = v_m \cdot z_1 / z_2,$$

giving

$$v_m = \frac{1}{2} A e^{l\sqrt{(z_1/z_2)}} + \frac{1}{2} B e^{-l\sqrt{(z_1/z_2)}} \quad (12)$$

in which  $A$  and  $B$  are constants to be determined from limiting conditions. Since, however, the same result should be obtained whether  $l$  is positive or negative, it may be inferred that  $A = B$ ; hence

$$\frac{v_m}{v_n} = \frac{\cosh L_m \sqrt{(z_1/z_2)}}{\cosh L_n \sqrt{(z_1/z_2)}} = \frac{\cosh \sqrt{\theta}_m}{\cosh \sqrt{\theta}_n} \quad (13)$$

and

$$I = \int_0^m dI = \int_0^m (v/z_2) dl = (v_n \sinh \sqrt{\theta}_m) / (\sqrt{(z_1 z_2)} \cosh \sqrt{\theta}_n).$$

Hence

$$Z_n = v_n / I_n = \sqrt{(z_1 z_2)} \coth \sqrt{\theta}_n. \quad (10)$$

It is convenient at this point to notice a case in which the network parameters vary inversely as some function of  $l$ . Suppose that instead of  $z_1$  and  $z_2$  we have  $z_1/\phi(l)$  and  $z_2/\phi(l)$ . Then  $v = z_2 \cdot dI/\phi(l)dl$  and  $dv = I z_1 \cdot d\phi(l)/\phi(l)^2$ , so that

$$\frac{d^2v}{dl^2} = \frac{z_1}{\phi(l)} \cdot \frac{dI}{dl} - I z_1 \frac{\phi'(l)}{[\phi(l)]^2} = \frac{z_1}{z_2} v - \frac{\phi'(l)}{\phi(l)} \cdot \frac{dv}{dl}, \quad (13)$$

and if  $\phi(l) = l$ , for example, we have  $\phi'(l) = 1$ , and

$$\frac{d^2v}{dl^2} + \frac{1}{l} \cdot \frac{dv}{dl} - \frac{z_1}{z_2} v = 0. \quad (13a)$$

After the removal of the constant  $z_1/z_2$  by substitution, this becomes a Bessel equation of zero order.

## 2. THE IMPEDANCE OF THE TERMINATED NETWORK

We now consider the effect of replacing the impedance  $\delta z_2$  across KL by an impedance  $Z_B$  as shown in fig. 2.  $Z_B$  may be taken if needed as representing a second network, continuing the first but having different parameters. We now have

$$\delta z_2(i_n - i_{n-1}) = v_n - v_{n-1} = \delta z_1(i_1 + i_2 + \dots + i_n);$$

also

$$v_n = Z_B i_1 + \delta z_1[i_1 + (i_1 + i_2) + (i_1 + i_2 + i_3) + \dots]$$

and

$$i_n = v_n/\delta z_2 = i_1[(Z_B/\delta z_2) + \psi T_{n-1}],$$

where

$$i_1 T_n = i_n + 2i_{n-1} + 3i_{n-2} + \dots + n i_1$$

as for  $i_1 S_n$ , but differing in terms of  $i_1$ . Again

$$i_{n-1} = i_1[(Z_B/\delta z_2) + \psi T_{n-2}], \text{ and so on;}$$

$$I_n = i_1(T_n - T_{n-1})$$

and

$$T_n - T_{n-1} = (n Z_B/\delta z_2) + \psi(T_{n-1} + T_{n-2} + \dots + T_1), \quad (14)$$

with

$$T_1 = 1.$$

Thus

$$T_n = h_{n-1}(Z_B/\delta z_2) + n + \psi(T_{n-1} + 2T_{n-2} + 3T_{n-3} + \dots + T_1).$$

Similarly

$$T_{n-1} = h_{n-2}(Z_B/\delta z_2) + n + \psi(T_{n-2} + 2T_{n-3} + 3T_{n-4} + \dots + T_1),$$

and so on, and by repeated substitution of the recurrence relation we obtain a series as before of the form

$$T_n = q_0 + q_1 \psi + q_2 \psi^2 + \dots + q_{n-1} \psi^{n-1},$$

differing only from the series previously obtained for  $S_n$  in that we replace  $h_n$  by  $[h_{n-1}(Z_B/\delta z_2) + n]$ ,  $h_{n-1}$  by  $[h_{n-2}(Z_B/\delta z_2) + (n-1)]$ , and so on. Also

$$T_n = (Z_B/\delta z_2) S_{n-1} + (S_n - S_{n-1}), \quad (15)$$

that is,

$$T_n = (Z_B/\delta z_2) S_{n-1} + U_n.$$

The impedance between HK and N is consequently

$$Z_n = \frac{v_n}{I_n} = \frac{\delta z_2[(Z_B/\delta z_2) + \psi T_{n-1}]}{T_n - T_{n-1}}, \quad (16)$$

which, when  $n$  approaches infinity, becomes

$$Z_n = \lim_{n \rightarrow \infty} \frac{Z_B(1 + \psi S_{n-2}) + \delta z_1(S_{n-1} - S_{n-2})}{(Z_B/\delta z_2)(S_{n-1} - S_{n-2}) + (U_n - U_{n-1})} = \frac{Z_B \cosh \sqrt{\theta} + \sqrt{(z_1 z_2)} \sinh \sqrt{\theta}}{(Z_B/\sqrt{z_1 z_2}) \sinh \sqrt{\theta} + \cosh \sqrt{\theta}}. \quad (17)$$



Conditions in the network can be found from

$$\frac{v_m}{v_n} = \frac{Z_B + \delta z_1 \cdot T_{m-1}}{Z_B + \delta z_1 \cdot T_{n-1}} = \frac{Z_B \cosh \sqrt{\theta}_m + \sqrt{(z_1 z_2)} \sinh \sqrt{\theta}_m}{Z_B \cosh \sqrt{\theta}_n + \sqrt{(z_1 z_2)} \sinh \sqrt{\theta}_n} \quad (18)$$

If K and L are joined so that  $Z_B = 0$ , then

$$v_n/I_n = \sqrt{(z_1 z_2)} \tanh \sqrt{\theta}. \quad (19)$$

The impedance between points N and L in fig. 1 could be found by joining the centre points RR'. The impedance between R and either N or L is  $\sqrt{(z_1 z_2)} \tanh \frac{1}{2} l_n \sqrt{(z_1 z_2)}$ , and the total impedance of these two sections in series is twice this value. But the connector RR' may be removed because R and R' are equipotential points under these conditions. The impedance N to L is consequently

$$Z_A = 2\sqrt{(z_1 z_2)} \tanh \frac{1}{2} l_n \sqrt{(z_1 z_2)}. \quad (20)$$

Measured from equipotential points where  $Z_B$  may be considered zero,

$$v_m/v_n = \sinh \sqrt{\theta}_m / \sinh \sqrt{\theta}_n. \quad (21)$$

The impedance  $Z_n$  between K and N, fig. 2, may alternatively be calculated as follows, when  $n$  is infinite: eq. (12) applies, but the constants A and B are different and are no longer equal. When  $l = 0$

$$v_1 = i_1 Z_B = \frac{1}{2}(A + B) \quad \text{and} \quad v_m = i_1 Z_B e^{l\sqrt{(z_1/z_2)}} - B \sinh l\sqrt{(z_1/z_2)}.$$

Again

$$I = \int dI = \int (v/z_2) dl = [i_1 Z_B / \sqrt{(z_1 z_2)}] e^{l\sqrt{(z_1/z_2)}} - [B / \sqrt{(z_1 z_2)}] \cosh l\sqrt{(z_1/z_2)}.$$

For  $l = 0$ ,  $I = i_1$ :

$$i_1 = [i_1 Z_B / \sqrt{(z_1 z_2)}] - B / \sqrt{(z_1 z_2)}, \quad \text{and} \quad B = i_1 [Z_B - \sqrt{(z_1 z_2)}],$$

so that

$$\begin{aligned} v &= i_1 Z_B e^{l\sqrt{(z_1/z_2)}} - i_1 [Z_B - \sqrt{(z_1 z_2)}] \sinh l\sqrt{(z_1/z_2)} \\ &= i_1 [Z_B \cosh l\sqrt{(z_1/z_2)} + \sqrt{(z_1 z_2)} \sinh l\sqrt{(z_1/z_2)}]. \end{aligned}$$

Also

$$I = i_1 \left[ \frac{Z_B}{\sqrt{(z_1 z_2)}} \sinh l\sqrt{\frac{z_1}{z_2}} + \cosh l\sqrt{\frac{z_1}{z_2}} \right].$$

Hence the impedance between L and N is

$$Z_n = \frac{v_n}{I_n} = \frac{Z_B \cosh \sqrt{\theta} + \sqrt{(z_1 z_2)} \sinh \sqrt{\theta}}{[Z_B / \sqrt{(z_1 z_2)}] \sinh \sqrt{\theta} + \cosh \sqrt{\theta}}. \quad (17)$$

If  $Z_B$  represents a network having parameters  $z_1'$ ,  $z_2'$  and  $\theta'$ , such that  $z_1' z_2' = z_1 z_2$ , then

$$Z_n = \sqrt{(z_1 z_2)} \coth (\sqrt{\theta} + \sqrt{\theta'}). \quad (17a)$$

### 3. LADDER NETWORK WITH SUPERIMPOSED LOAD

In fig. 2,  $Z_B$  may be considered as a "load" impedance applied across the termination of the network of fig. 1,  $n$  being large. Alternatively we may place a load impedance  $Z_L$  across the terminals K and L as in fig. 3, when the impedances  $\delta z_2$  and  $Z_L$ , being in parallel, may be manipulated together as  $Z_B$  in fig. 2. Now in certain practical problems  $Z_L$  is unknown, but the load and "leakage" currents are measurable. Consequently it is convenient to eliminate  $Z_L$  and provide expressions in terms of these two currents.

Suppose the load current to be  $(k-1)I_n$ , where  $k$  is any quantity of the form  $A + jB$ .  $I_n$  is the "leakage" current as before, and the total current entering the network at N is now  $kI_n$  (fig. 3). We thus have  $i_1 + i_2 + i_3 + \dots + i_n = I_n$  as before, and  $i' Z_B = i_1 \cdot \delta z_2$  where  $i' = (k-1)I_n + i_1$ , whence

$$Z_B = \frac{i_1 \cdot \delta z_2}{(k-1)I_n + i_1} \quad (22)$$

Further

$$\begin{aligned} I_n &= i'(\Gamma_n - \Gamma_{n-1}) = i'((Z_B/\delta z_2)(S_{n-1} - S_{n-2}) + (U_n - U_{n-1})) \\ &= \lim_{n \rightarrow \infty} (i_1/\sqrt{\psi}) \sinh \sqrt{\theta} + [(k-1)I_n + i_1] \cosh \sqrt{\theta}. \end{aligned}$$

Therefore

$$\begin{aligned} Z_B &= \lim_{n \rightarrow \infty} \left[ \frac{\delta z_2}{\frac{(k-1)[(1/\sqrt{\psi}) \sinh \sqrt{\theta} + \cosh \sqrt{\theta}]}{k - (k-1) \cosh \sqrt{\theta}} + 1} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{\sqrt{(\delta z_1 \cdot \delta z_2)[k - (k-1) \cosh \sqrt{\theta}]}}{(k-1) \sinh \sqrt{\theta} + k\sqrt{\psi}} \right] \\ &= \frac{\sqrt{(z_1 z_2)[k - (k-1) \cosh \sqrt{\theta}]}}{(k-1) \sinh \sqrt{\theta}}, \end{aligned} \quad (23)$$

since  $k\sqrt{\psi} = k\sqrt{\theta}/n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus the impedance of the network between K and N and terms of  $k$  and  $I_n$  is

$$\begin{aligned} Z_n &= \frac{\sqrt{(z_1 z_2)[k - (k-1) \cosh \sqrt{\theta}] \cosh \sqrt{\theta} + \sqrt{(z_1 z_2) \sinh \sqrt{\theta}[(k-1) \sinh \sqrt{\theta}]}}{[k - (k-1) \cosh \sqrt{\theta}] \sinh \sqrt{\theta} + [(k-1) \sinh \sqrt{\theta}] \cosh \sqrt{\theta}} \\ &= \frac{v_n}{kI_n} = \frac{\sqrt{(z_1 z_2)[k \cosh \sqrt{\theta} - (k-1)]}}{k \sinh \sqrt{\theta}}. \end{aligned} \quad (24)$$

As before, if  $v_m$  is the potential difference between K and any point M in LN,

$$\begin{aligned} v_m/v_n &= [Z_B \cosh \sqrt{\theta}_m + \sqrt{(z_1 z_2) \sinh \sqrt{\theta}_m}] / [Z_B \cosh \sqrt{\theta} + \sqrt{(z_1 z_2) \sinh \sqrt{\theta}}] \\ &= \frac{\frac{\sqrt{(z_1 z_2)[k - (k-1) \cosh \sqrt{\theta}]}}{(k-1) \sinh \sqrt{\theta}} \cosh \sqrt{\theta}_m + \sqrt{(z_1 z_2) \sinh \sqrt{\theta}_m}}{\frac{\sqrt{(z_1 z_2)[k - (k-1) \cosh \sqrt{\theta}]}}{(k-1) \sinh \sqrt{\theta}} \cosh \sqrt{\theta} + \sqrt{(z_1 z_2) \sinh \sqrt{\theta}}} \\ &= \frac{k \cosh \sqrt{\theta}_m - (k-1) \cosh (\sqrt{\theta} - \sqrt{\theta}_m)}{k \cosh \sqrt{\theta} - (k-1)}. \end{aligned} \quad (25)$$

The voltampères absorbed by the network can be found by subtracting the load voltampères from the total input at HN, giving

$$I^2 k^2 \frac{\sqrt{(z_1 z_2)[k \cosh \sqrt{\theta} - (k-1)]}}{k \sinh \sqrt{\theta}} - v_1(k-1)I_n. \quad (26)$$

Putting  $\sqrt{\theta}_m = 0$  in eq. (25), we have

$$v_1 = \frac{v_n[k - (k-1) \cosh \sqrt{\theta}]}{k \cosh \sqrt{\theta} - (k-1)} = \sqrt{(z_1 z_2)} \frac{kI_n[k - (k-1) \cosh \sqrt{\theta}]}{k \sinh \sqrt{\theta}}$$

using the value of  $v_n$  obtained from eq. (24). Hence, substituting for  $v_1$  in eq. (26), the active and reactive voltampères absorbed by the network, exclusive of those taken by the load impedance  $Z_L$ , become

$$I_n^2 \sqrt{(z_1 z_2)} [(2k^2 - 2k + 1) \cosh \sqrt{\theta} - 2k(k-1)] / \sinh \sqrt{\theta}. \quad (27)$$

This result enables the "no-load" and "load" voltampères to be compared for various load currents.

#### 4. APPLICATIONS

The application of the ladder network to *transmission lines* is well known. The development above is particularly applicable to cases of discontinuity such as the series connection of overhead and cable lines. Any number of such lengths can be added by use of eq. (17). From eq. (27), line parameters may be chosen such that the magnitude B in  $k = A + jB$  vanishes

for a given frequency, or approximates to zero over a given frequency band, so as to bring the sending- and receiving-end currents into a co-phasal relation.

The networks in figs. 1 and 2 may be used as equivalent circuits representing either an *isolated conductor* or a *conductor in a slot* cut in a mass of highly permeable material (a case covering all rotary electrical machines of the electromagnetic type).

The circuit of fig. 1 applies to the case of a *rectangular conductor in a rectangular slot*, fig. 4, A. Here HK in fig. 1 represents one end and N the other end of the conductor at the points of entry into the slot.  $LN = l_n$  is the depth of the slot occupied by the conductor: elements  $\delta z_1 = j\delta x$  comprise pure inductive reactance, and elements  $\delta z_2 = \delta r$  are purely

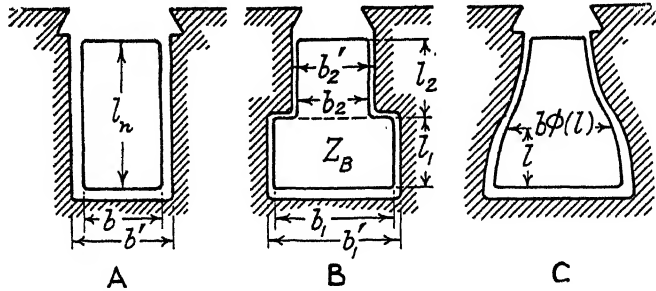


FIG. 4.—Conductors in slots.

resistive. The circuit demonstrates the building up of reactive electro-motive forces and the tendency to force the current to the upper parts of the conductor. For this case

$$z_1 = jx = j4\pi(c/b')2\pi f \cdot 10^{-9} \quad (28)$$

and

$$z_2 = r = \rho c/b, \quad (28a)$$

where  $\rho$  = resistivity of the conductor material,  $b$  = width of conductor,  $b'$  = width of slot,  $l_n$  = depth of slot filled by conductor,  $c$  = length of slot,  $f$  = frequency in cycles per sec. Further

$$\delta x = x l_n / (n - 1), \quad \delta r = r n / l_n, \quad \text{and} \quad \theta = l_n^2 j x / r.$$

The reactance is based on the usual assumptions that (a) the magnetic flux passes across the slot in straight lines normal to the sides of the slot, and (b) the magnetic permeability of the magnetic material is infinite. The impedance of a single conductor in such a slot will thus be  $Z_n = \sqrt{(jxr)} \coth l_n \sqrt{(jx/r)}$ : writing as scalar values  $\alpha = l_n \sqrt{(x/r)}$  and  $\beta = \sqrt{(xr)}$ , then

$$Z_n = \beta \sqrt{j} \cdot \coth \alpha \sqrt{j} \quad (29)$$

$$\begin{aligned} &= \beta \sqrt{j} \frac{\cosh \alpha \sqrt{j} \cdot \sin \alpha \sqrt{j}}{\sinh \alpha \sqrt{j} \cdot \sin \alpha \sqrt{j}} \\ &= \frac{\beta}{\sqrt{2}} \cdot \frac{\sinh \alpha \sqrt{2} + \sin \alpha \sqrt{2} + j(\sinh \alpha \sqrt{2} - \sin \alpha \sqrt{2})}{\cosh \alpha \sqrt{2} - \cos \alpha \sqrt{2}}. \end{aligned} \quad (29a)$$

The real part of this agrees with a well-known result (Field, 1905), but Field does not give the imaginary (reactive) part, which is of great practical importance in the design of certain electrical machines.

The circuit of fig. 2 applies to a *conductor of stepped section* in a slot of similar shape, as shown in fig. 4, B. Although only one step is indicated, any number of steps may be dealt with by application of eq. (17),  $Z_B$  being first found from eq. (29) or (29a) for the lowest or innermost step using  $b_1, b_1', l_1$  in place of  $b, b', l_n$ , fig. 4, B. The composite impedance will be

$$Z_n = \frac{Z_B \cdot \cosh \alpha \sqrt{j} + \beta \sqrt{j} \cdot \sinh \alpha \sqrt{j}}{(Z_B / \beta \sqrt{j}) \sinh \alpha \sqrt{j} + \cosh \alpha \sqrt{j}}. \quad (30)$$

$\alpha$  in this case is found by use of the dimensions of the next step, e.g.  $b_2, b_2', l_2'$ , in fig. 4, B. If there is a further step, the value in eq. (30) becomes a new  $Z_B$ , and the process is repeated.

It will be seen from figs. 1 and 2 that the current density in that part of the conductor nearest the slot-opening will be  $v_n/\rho c$ : hence the current density in any part of such a conductor may be found after evaluating the impedances of the first, first and second, first, second and third . . . steps in the manner described, using eq. (18) and working downwards from the slot-opening. In eq. (18)  $v_m$  and  $v_n$  are proportional to the current-densities at the points M and N. By putting  $\theta_m = 0$  we find the current-density at the bottom of the step nearest the opening, which will be the same as that at the top of the next lower step, and so on.

In the case of fig. 4, C, where a conductor of arbitrary section is embedded in, and conforms proportionally to the shape of, a slot in highly magnetic material, let the width of the conductor be  $b\phi(l)$  at a distance  $l$  from the bottom of the slot, and let that of the slot itself be  $b'\phi(l)$ . Then, from eq. (28) and (28a),  $z_1 = jx/\phi(l)$  and  $z_2 = r/\phi(l)$ . Result (13) then applies, and the solution depends on the nature of  $\phi(l)$ . For example, in the case of a triangular slot with the vertex remote from the slot-opening,  $z_1 = jx/\eta l$  and  $z_2 = r/\lambda l$ , where  $\eta$  and  $\lambda$  are constants: we then have  $\phi'(l)/\phi(l) = 1/l$ , and eq. (13a) applies.

In most cases an approximate solution can more readily be obtained by dividing the conductor into rectangular sections and adding, using eq. (17).

For an isolated cylindrical conductor of unit length, using eq. (28),  $c = 1$ ,  $b' = b = 2\pi l$ , where  $l$  is the radius, and eq. (13a) is applicable. We consider the cylinder as divided into elemental concentric cylinders having sections  $2\pi l \cdot dl$ ; and further in the case of a hollow tube we may consider a finite number of sections and so obtain the approximate impedances and current-densities, from eqs. (17) and (18). This may be quicker than the more exact method using Bessel functions, especially if the hollow cylinder comprises a number of thin cylinders of materials differing in magnetic and conductional properties.

Fig. 5 is drawn to represent conditions in a rectangular slot in magnetic material where a number of insulated rectangular conductors, not transposed within the slot, each carry the same

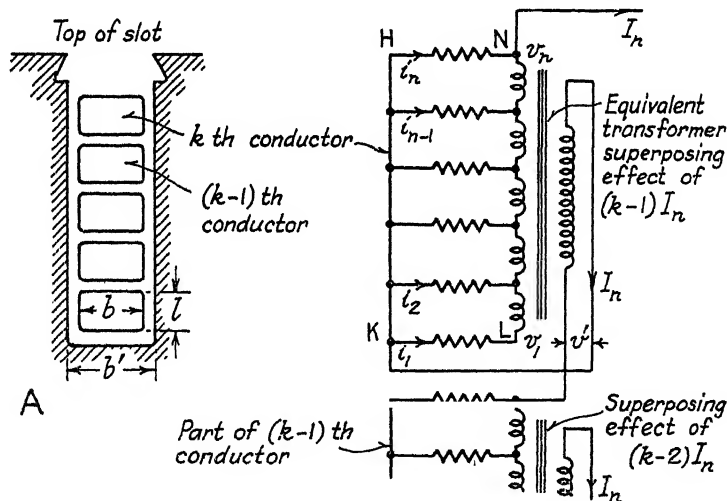


FIG. 5.—Slot with several layers of conductors.

load current. The  $k$ th conductor (counting upwards from the bottom of the slot) is shown with part of the next, the  $(k-1)$ th conductor below it. By mutual induction an electro-motive force will be developed in any conductor by the currents in all those beneath it. The effect of the currents  $I_n$  in the  $(k-1)$  lower conductors is superposed on the reactances  $\delta x$  of the equivalent circuit for the  $k$ th conductor. The formulæ already obtained are now applicable. Here  $\theta = l^2 jx/r$ , where  $l$  is the depth of a single conductor:  $x$ ,  $r$ ,  $a$  and  $\beta$  are found as before from eqs. (28) and (28a). If  $v_n$  is the potential difference between K and N, and  $v'$  as shown in fig. 5 represents the electro-motive force induced between the  $(k-1)$ th and  $k$ th conductors, then from eq. (27) the voltamperes absorbed are

$$I_n(v_n + v') = I_n^2 \beta \sqrt{j} \frac{(2k^2 - 2k + 1) \cosh a\sqrt{j} - 2k(k-1)}{\sinh a\sqrt{j}} \quad (31)$$

$$= \frac{I_n^2 \beta}{\sqrt{2}} \left[ \frac{4k(k-1)(\cosh \alpha/\sqrt{2} - \cos \alpha/\sqrt{2})[\sinh \alpha/\sqrt{2} - \sin \alpha/\sqrt{2} + j(\sinh \alpha/\sqrt{2} + \sin \alpha/\sqrt{2})] + \sinh \alpha/\sqrt{2} + \sin \alpha/\sqrt{2} + j(\sinh \alpha/\sqrt{2} - \sin \alpha/\sqrt{2})}{\cosh \alpha/\sqrt{2} - \cos \alpha/\sqrt{2}} \right] \quad (31a)$$

The real part of this agrees with that obtained by Field, who does not give the imaginary part.

The current-density in that part of the  $k$ th conductor nearest to the slot opening is  $v_n/\rho c$ , obtainable from eq. (24) for any given current  $I_n$ ; and that in any other part of the conductor can be inferred from eq. (25). The conductors may be in series or in parallel. If the latter they will normally be transposed so that there will be the same current in each.

Often a mean result for all the conductors in a slot is required rather than for the conductors

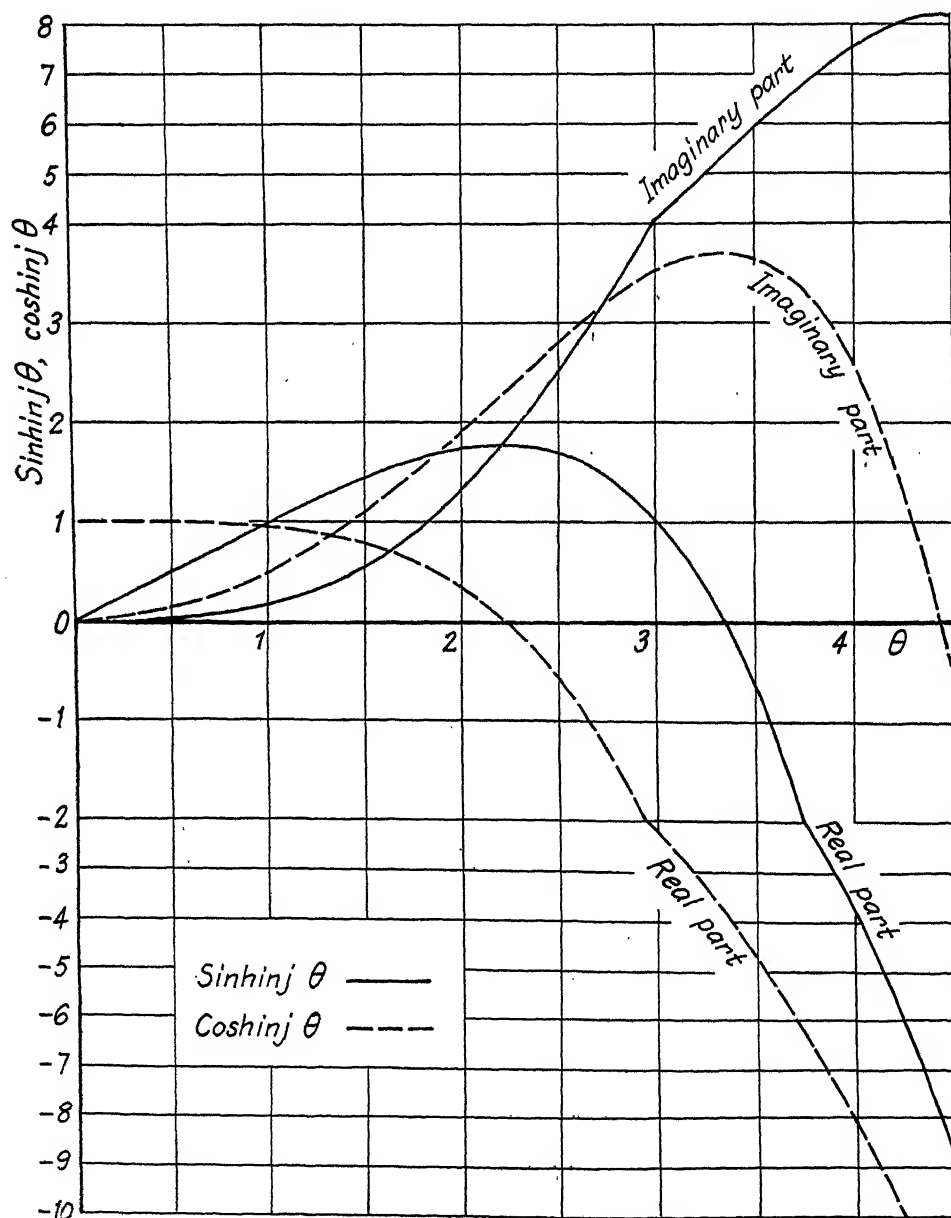


FIG. 6.— $\sinh j\theta$  and  $\cosh j\theta$ .

To find  $\sin j\theta$  change sign of imaginary part of  $\sinh j\theta$ : e.g.  $\sin j(1) = 0.992 - j0.166$ .  
To find  $\cos j\theta$  change sign of imaginary part of  $\cosh j\theta$ : e.g.  $\cos j(1) = 0.958 - j0.498$ .

individually. A mean value can be had by making  $k=1, 2, 3, \dots k$  in succession in eq. (31), adding the separate results and dividing by  $k$ . The mean impedance per conductor is then

$$\frac{v_n + v'}{I_n} = \beta \sqrt{j} \frac{(2k^2 + 1) \cosh \alpha \sqrt{j} - (2k^2 - 2)}{3 \sinh \alpha \sqrt{j}}.$$

It is to be observed that the impedance so calculated does not include reactance resulting from magnetic leakage flux across the slot-opening above the conductors, nor from overhang leakage. These effects are obtained and added in the manner well known to designers. If the conductors, not being transposed, comprise a number of insulated strips joined together at the ends of the slot, or elsewhere, an allowance must be made when calculating  $r$  in eq. (28a).

## 5. TRIGONOMETRICAL AND HYPERBOLIC FUNCTIONS OF REAL MULTIPLES OF $(-1)^{\frac{1}{2}}$

In many practical cases, such as that above, a ladder network of the form shown in fig. 1 may have all the series impedances  $\delta z_1$  pure resistance and all the parallel impedances  $\delta z_2$  pure reactance; or *vice versa*. It is evident from eqs. (29), (29a), (31), and (31a) that solutions of such networks can be more conveniently in the form of hyperbolic or trigonometrical functions of real multiples of  $(-1)^{\frac{1}{2}}$  than as ordinary functions of this type. Tables of these functions are not readily available in a convenient form for general use, *vide Funktionentafeln*, Jahnke and Emde, 1938; or *Complex Functions and Atlas*, Kennelly, 1900. The curves in figs. 6

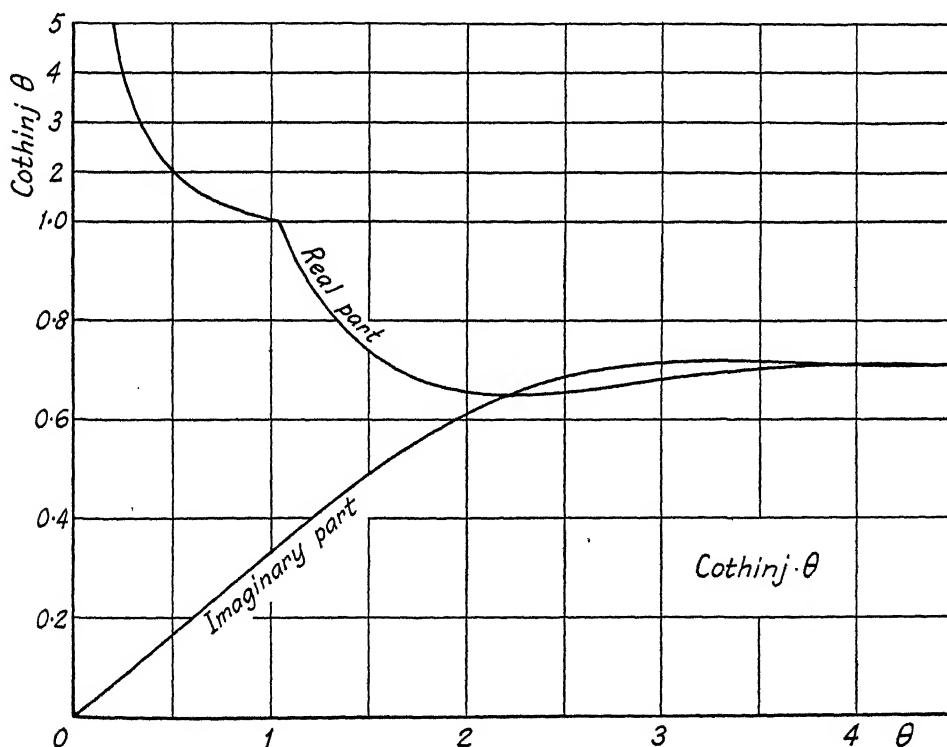


FIG. 7.—Cothinj  $\theta$ .

To find cotinj  $\theta$ , change sign of imaginary part: e.g. cotinj (1) = 1.022 - j0.331.

and 7 are therefore given, from which, for many practical purposes, sufficiently close values may be obtained.

In order to simplify nomenclature it is suggested that the functions following might be written in the form shown immediately under each:—

$(1/\sqrt{j}) \sin \theta \sqrt{j}$	$(1/\sqrt{j}) \sinh \theta \sqrt{j}$	$\cos \theta \sqrt{j}$	$\cosh \theta \sqrt{j}$	$(1/\sqrt{j}) \tanh \theta \sqrt{j}$	$\sqrt{j} \coth \theta \sqrt{j}$
sininj $\theta$	sinhij $\theta$	cosinj $\theta$	coshinj $\theta$	tanhinj $\theta$	cothinj $\theta$

and similarly for other functions. We then have

$$\begin{aligned} 2\sqrt{2} \sininj \theta \cdot \coshinj \theta &= \sinh \theta\sqrt{2} + \sin \theta\sqrt{2} + j (\sinh \theta\sqrt{2} - \sin \theta\sqrt{2}), \\ 2\sqrt{2} \sinhinj \theta \cdot \cosinj \theta &= \sinh \theta\sqrt{2} + \sin \theta\sqrt{2} - j (\sinh \theta\sqrt{2} - \sin \theta\sqrt{2}), \\ 2 \sininj \theta \cdot \sinhinj \theta &= \cosh \theta\sqrt{2} - \cos \theta\sqrt{2}, \\ 2 \cosinj \theta \cdot \coshinj \theta &= \cosh \theta\sqrt{2} + \cos \theta\sqrt{2}, \end{aligned}$$

and a number of other formulæ analogous to the ordinary trigonometric relations.

The author records his thanks to Mr H. E. Clapham for the original suggestion that the ladder network could be applied to the case of a slot-conductor; to Professor M. G. Say for help in compiling the paper; and to a referee for several suggestions for shortening the proofs of certain expressions.

*(Issued separately March 25, 1946)*

XXI.—Tables of Chebyshev Polynomials. By C. W. Jones, M.Sc., and J. C. P. Miller, Ph.D., of the University of Liverpool, and J. F. C. Conn, D.Sc., and R. C. Pankhurst, Ph.D., of the National Physical Laboratory. Communicated by Dr A. C. AITKEN, F.R.S.

(MS. received October 23, 1944. Read May 7, 1945)

1. *Introduction.*—The object of this paper is twofold: firstly, to present a table of the Chebyshev polynomials  $C_n(x) = 2 \cos(n \cos^{-1} \frac{1}{2}x)$  for  $n = 1(1)12$  and  $x = 0(0.02)2$ , values being exact or to 10 decimals; secondly, to provide a working list of coefficients and formulæ relating to these and allied functions.

Valuable accounts of applications and properties will be found in Van der Pol and Weijers, 1933, in Lanczos, 1938, and in Szegő, 1939. Further applications are indicated in the following paper, by J. C. P. Miller, which also suggests methods of reducing the inconvenience caused by the present lack of tables of the allied polynomials  $S_n(x)$ . It is hoped that suitable tables will be prepared later.

Manuscript tables of  $C_n(x)$  and  $S_n(x)$  at interval 0.001 in  $x$  to 12 decimals have been prepared by the *New York Mathematical Tables Project*, under the supervision of Dr A. N. Lowan (see *Mathematical Tables and other Aids to Computation*, 1, No. 4, p. 125, October 1943), but considerable delay in publication seems probable. Because of this delay and because the present tables seem likely to be of particular convenience to computers, expected to be in the majority, who need only tabular values, and for whom a fairly wide interval will suffice, it has seemed desirable to proceed with publication.

2.1. *Definitions and Notations.*—We define the Chebyshev Polynomials \* as

$$C_n(x) = 2 \cos n\theta \quad \text{where} \quad x = C_1(x) = 2 \cos \theta. \quad (2.11)$$

The more usual definition, for which it is also convenient to have a notation, is

$$T_n(\mu) = \cos n\theta \quad \text{where} \quad \mu = T_1(\mu) = \cos \theta, \quad (2.12)$$

so that we have

$$x = 2\mu \quad C_n(x) = 2T_n(\mu) \quad (2.13)$$

and

$$C_n(x) = 2 \cos(n \cos^{-1} \frac{1}{2}x) \quad T_n(\mu) = \cos(n \cos^{-1} \mu). \quad (2.14)$$

The revised definition gives simpler numerical coefficients; it has been found more convenient in theoretical investigations, and equally good for numerical applications.

For allied functions we use the notations

$$\left. \begin{aligned} S_{n-1}(x) = U_{n-1}(\mu) &= \frac{\sin n\theta}{\sin \theta} & \sigma_n(x) &= 2 \sin n\theta = 2v_n(\mu), \\ \sigma_n(x) &= \sigma_1(x)S_{n-1}(x) & v_n(\mu) &= v_1(\mu)U_{n-1}(\mu). \end{aligned} \right\} \quad (2.15)$$

It should be noted that  $S_{n-1}(x)$  and  $U_{n-1}(\mu)$  are polynomials of degree  $n-1$  in  $x$  and  $\mu$  respectively, but that  $\sigma_n(x)$  and  $v_n(\mu)$  are not polynomials.

The range of  $C_n(x)$ , including  $x = C_1(x)$ , is from  $-2$  to  $+2$ , if  $\theta$  is to be real, whilst that of  $T_n(\mu)$ , including  $\mu = T_1(\mu)$ , is from  $-1$  to  $+1$ . It is convenient on occasion to use instead the range 0 to 1 for the independent variable, which we shall then denote by  $X$ , such that

\* To obviate confusion we note that this term has also been applied to several other sets of polynomials; for instance, it has been used as synonymous with the term *orthogonal polynomials* (see Szegő, 1939, p. 25, fn.), although this use now seems largely to have died out. Other more particular polynomials known by the name of Chebyshev are (i) the orthogonal polynomials associated with sums over a set of equidistant abscissæ, namely  $n! \Delta_n^N(x) \binom{x-N}{n}$ , see Szegő, 1939, p. 32; (ii) the polynomials whose zeros give the abscissæ associated with Chebyshev's formulæ for numerical quadrature.



$\frac{1}{2}x = \mu = 2X - 1$ , as in Lanczos, 1938, p. 140, although the advantage of symmetry about the origin is then lost.

For the general range,  $a \leq z \leq b$  (Lanczos, 1938, p. 137), the definitions are

$$C_n = 2T_n = 2 \cos n\theta \quad S_n = U_n = \frac{\sin(n+1)\theta}{\sin \theta},$$

where

$$\theta = \cos^{-1} \frac{2z - b - a}{b - a}.$$

We do not consider this range further apart from a mention in § 3.5.

2.2. *Allied Functions*.—If we write  $\phi = \frac{1}{2}\pi - \theta$ , so that  $2 \cos \theta = 2 \sin \phi = x$ , we have

$$\left. \begin{aligned} 2 \sin(2m+1)\phi &= (-1)^m \cdot 2 \cos(2m+1)\theta = (-1)^m C_{2m+1}(x), \\ 2 \sin 2m\phi &= (-1)^{m+1} \cdot 2 \sin 2m\theta = (-1)^{m+1} \sigma_1(x) S_{2m-1}(x) = (-1)^{m+1} \sigma_{2m}(x), \\ 2 \cos(2m+1)\phi &= (-1)^m \cdot 2 \sin(2m+1)\theta = (-1)^m \sigma_1(x) S_{2m}(x) = (-1)^m \sigma_{2m+1}(x), \\ 2 \cos 2m\phi &= (-1)^m \cdot 2 \cos 2m\theta = (-1)^m C_{2m}(x). \end{aligned} \right\} \quad (2.21)$$

Van der Pol and Weijers, 1933, p. 81, give the same relations in terms of  $T_n(\mu)$  and  $u_n(\mu)$ .

Other polynomials of interest (Szegő, 1939, p. 3, etc.) are

$$\left. \begin{aligned} \frac{\cos(n+\frac{1}{2})\theta}{\cos \frac{1}{2}\theta} &= S_n(x) - S_{n-1}(x) = U_n(\mu) - U_{n-1}(\mu) = (-1)^n U_{2n}(\sqrt{\tfrac{1}{2} - \tfrac{1}{2}\mu}), \\ \frac{\sin(n+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta} &= S_n(x) + S_{n-1}(x) = U_n(\mu) + U_{n-1}(\mu) = U_{2n}(\sqrt{X}). \end{aligned} \right\} \quad (2.22)$$

3. *Formulae*.—Most of these are readily derived from well-known properties of the circular functions.

### 3.1. *Recurrence Relations*.

$$y_{n+1} - xy_n + y_{n-1} = 0 \quad \text{Satisfied by } C_n, T_n, S_n, U_n, \sigma_n, v_n. \quad (3.11)$$

$2\mu$  may replace  $x$ .

$$y_{n+m} - C_m y_n + y_{n-m} = 0 \quad \text{Satisfied by } C_n, T_n, S_n, U_n, \sigma_n, v_n. \quad (3.12)$$

$2T_m$  may replace  $C_m$ .

3.2. *Explicit Forms*.—These are readily derived from (3.11) starting with the values for  $n=0$  and  $n=1$ . One set of coefficients suffices for expansions in powers both of  $X$  and of  $\mu$ , for

$$T_n(\mu) = \cos n\theta = \cos 2n(\tfrac{1}{2}\theta) = T_{2n}(\cos \tfrac{1}{2}\theta) = T_{2n}(\sqrt{X}) \quad (3.21)$$

since

$$X = \tfrac{1}{2}(1 + \mu) = \cos^2 \tfrac{1}{2}\theta. \quad (3.22)$$

Thus, for example,

$$T_2(\mu) = 2\mu^2 - 1 = 8X^2 - 8X + 1 \quad T_4(\mu) = 8\mu^4 - 8\mu^2 + 1.$$

Likewise

$$v_n(\mu) = v_1(\mu) U_{n-1}(\mu) = v_{2n}(\sqrt{X}) = \frac{v_1(\mu)}{2\sqrt{X}} U_{2n-1}(\sqrt{X}) \quad (3.23)$$

and

$$U_{n-1}(\mu) = \frac{1}{2\sqrt{X}} U_{2n-1}(\sqrt{X}). \quad (3.24)$$

We write

$$\left. \begin{aligned} C_n(x) &= c_{n,0}x^n - c_{n,2}x^{n-2} + c_{n,4}x^{n-4} - \dots & S_n(x) &= s_{n,0}x^n - s_{n,2}x^{n-2} + s_{n,4}x^{n-4} - \dots \\ T_n(\mu) &= t_{n,0}\mu^n - t_{n,2}\mu^{n-2} + t_{n,4}\mu^{n-4} - \dots & U_n(\mu) &= u_{n,0}\mu^n - u_{n,2}\mu^{n-2} + u_{n,4}\mu^{n-4} - \dots \end{aligned} \right\} \quad (3.25)$$

where

$$\left. \begin{aligned} c_{n,2p} &= \frac{n \cdot (n-p-1)!}{p!(n-2p)!} & s_{n,2p} &= \frac{(n-p)!}{p!(n-2p)!} \\ t_{n,2p} &= 2^{n-2p-1} c_{n,2p} & u_{n,2p} &= 2^{n-2p} s_{n,2p} \end{aligned} \right\} \quad n \geq 2p \geq 0. \quad (3.26)$$

and all coefficients are zero if  $2p > n$ . Values of some of the coefficients are tabulated below. Then, as we have seen,

$$\left. \begin{aligned} T_n(2X-1) &= t_{2n,0}X^n - t_{2n,2}X^{n-1} + t_{2n,4}X^{n-2} - \dots \\ U_n(2X-1) &= \frac{1}{2}u_{2n+1,0}X^n - \frac{1}{2}u_{2n+1,2}X^{n-1} + \frac{1}{2}u_{2n+1,4}X^{n-2} - \dots \end{aligned} \right\} \quad (3.27)$$

Also

$$\left. \begin{aligned} \sigma_0(x) &= v_0(\mu) = 0 & \sigma_1(x) &= \sqrt{4-x^2} & v_1(\mu) &= \sqrt{1-\mu^2} \\ \sigma_n(x) &= \sigma_1(x)S_{n-1}(x) & v_n(\mu) &= v_1(\mu)U_{n-1}(\mu). \end{aligned} \right\} \quad (3.28)$$

		$c_{n,2p}$											
$n \setminus 2p$	0	2	4	6	8	10	12	14	16	18	20		
0	2												
1	1												
2	1	2											
3	1	3											
4	1	4	2										
5	1	5	5										
6	1	6	9	2									
7	1	7	14	7									
8	1	8	20	16	2								
9	1	9	27	30	9								
10	1	10	35	50	25	2							
11	1	11	44	77	55	11							
12	1	12	54	112	105	36	2						
13	1	13	65	156	182	91	13						
14	1	14	77	210	294	196	49	2					
15	1	15	90	275	450	378	140	15					
16	1	16	104	352	660	672	336	64	2				
17	1	17	119	442	935	1122	714	204	17				
18	1	18	135	546	1287	1782	1386	540	81	2			
19	1	19	152	665	1729	2717	2508	1254	285	19			
20	1	20	170	800	2275	4004	4290	2640	825	100	2		

		$s_{n,2p}$											
$n \setminus 2p$	0	2	4	6	8	10	12	14	16	18	20		
0	1												
1	1												
2	1	1											
3	1	2											
4	1	3	1										
5	1	4	3										
6	1	5	6	1									
7	1	6	10	4									
8	1	7	15	10	1								
9	1	8	21	20	5								
10	1	9	28	35	15	1							
11	1	10	36	56	35	6							
12	1	11	45	84	70	21	1						
13	1	12	55	120	126	56	7						
14	1	13	66	165	210	126	28	1					
15	1	14	78	220	330	252	84	8					
16	1	15	91	286	495	462	210	36	1				
17	1	16	105	364	715	792	462	120	9				
18	1	17	120	455	1001	1287	924	330	45	1			
19	1	18	136	560	1365	2002	1716	792	165	10			
20	1	19	153	680	1820	3003	3003	1716	495	55	1		

		$t_{n,2p}$									
$n \setminus 2p$	0	2	4	6	8	10	12	14	16	18	20
0	1										
1	1										
2	2	1									
3	4	3									
4	8	8	1								
5	16	20	5								
6	32	48	18	1							
7	64	112	56	7							
8	128	256	160	32	1						
9	256	576	432	120	9						
10	512	1280	1120	400	50	1					
11	1024	2816	2816	1232	220	11					
12	2048	6144	6912	3584	840	72	1				
13	4096	13312	16640	9984	2912	364	13				
14	8192	28672	39424	26880	9408	1568	98	1			
15	16384	61440	92160	70400	28800	6048	560	15			
16	32768	131072	212992	180224	84480	21504	2688	128	1		
17	65536	278528	487424	452608	239360	71808	11424	816	17		
18	131072	589824	1105920	118208	658944	228096	44352	4320	162	1	
19	262144	1245184	2490368	2723840	1770496	695552	160512	20064	1140	19	
20	524288	2621440	5570560	6553600	4659200	2050048	549120	84480	6600	200	1

		$u_{n,2p}$									
$n \setminus 2p$	0	2	4	6	8	10	12	14	16	18	20
0	1										
1	2										
2	4	1									
3	8	4									
4	16	12	1								
5	32	32	6								
6	64	80	24	1							
7	128	192	80	8							
8	256	448	240	40	1						
9	512	1024	672	160	10						
10	1024	2304	1792	560	60	1					
11	2048	5120	4608	1792	280	12					
12	4096	11264	11520	5376	1120	84	1				
13	8192	24576	28160	15360	4032	448	14				
14	16384	53248	67584	42240	13440	2016	112	1			
15	32768	114688	159744	112640	42240	8064	672	16			
16	65536	245760	372736	292864	126720	29568	3360	144	1		
17	131072	524288	860160	745472	366080	101376	14784	960	18		
18	262144	1114112	1966080	1863680	1025024	329472	59136	5280	180	1	
19	524288	2359296	4456448	4587520	2795520	1025024	219648	25344	1320	20	
20	1048576	4980736	10027008	11141120	7454720	3075072	768768	109824	7920	220	1

3.3. *Inverse Relations.*—It is sometimes useful to express a given power series as the sum of a series of Chebyshev polynomials; the following inverse relations may then be used (cf. Lanczos, 1938, p. 147).

$$\left. \begin{aligned}
 x^{2m} &= C_{2m}(x) + 2mC_{2m-2}(x) + \binom{2m}{2}C_{2m-4}(x) + \dots + \binom{2m}{p}C_{2m-2p}(x) + \dots \\
 &\quad + \frac{1}{2}\binom{2m}{m}C_0(x) \\
 x^{2m+1} &= C_{2m+1}(x) + (2m+1)C_{2m-1}(x) + \dots + \binom{2m+1}{p}C_{2m-2p+1}(x) + \dots \\
 &\quad + \binom{2m+1}{m}C_1(x)
 \end{aligned} \right\} \quad (3.31)$$

and

$$\left. \begin{aligned} 2^{2m-1}\mu^{2m} &= T_{2m}(\mu) + 2mT_{2m-2}(\mu) + \binom{2m}{2}T_{2m-4}(\mu) + \dots + \binom{2m}{p}T_{2m-2p}(\mu) + \dots \\ &\quad + \frac{1}{2}\binom{2m}{m}T_0(\mu) \\ 2^{2m}\mu^{2m+1} &= T_{2m+1}(\mu) + (2m+1)T_{2m-1}(\mu) + \dots + \binom{2m+1}{p}T_{2m-2p+1}(\mu) + \dots \\ &\quad + \binom{2m+1}{m}T_1(\mu) \end{aligned} \right\} \quad (3.32)$$

Likewise, dropping the argument  $(2X-1)$  in  $T_r(2X-1)$ , we have

$$2^{2m-1}X^m = T_m + 2mT_{m-1} + \binom{2m}{2}T_{m-2} + \dots + \binom{2m}{p}T_{m-p} + \dots + \frac{1}{2}\binom{2m}{m}T_0 \quad (3.33)$$

The coefficients of the successive terms of the expansions of equations (3.31) are tabulated below.

	$C_0$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	$C_{11}$	$C_{12}$
1	$\frac{1}{2}$												
$x$		1											
$x^2$	1		1										
$x^3$		3		1									
$x^4$	3		4		1								
$x^5$		10		5		1							
$x^6$	10		15		6		1						
$x^7$		35		21		7		1					
$x^8$	35		56		28		8		1				
$x^9$		126		84		36		9		1			
$x^{10}$	126		210		120		45		10		1		
$x^{11}$		462		330		165		55		11		1	
$x^{12}$	462		792		495		220		66		12		1

This table may be easily extended by noting that each entry may be obtained from the adjacent items in the preceding line.

### 3.4. Differential Equations.

$$(4-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0 \quad \text{Solution: } y = aC_n(x) + b\sigma_n(x) \quad (3.41)$$

$$(1-\mu^2)\frac{d^2y}{d\mu^2} - \mu\frac{dy}{d\mu} + n^2y = 0 \quad \text{Solution: } y = \alpha T_n(\mu) + \beta v_n(\mu) \quad (3.42)$$

$$(4-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + (n^2-1)y = 0 \quad \text{Solution: } y = a'S_{n-1}(x) + b'\frac{C_n(x)}{\sqrt{4-x^2}} \quad (3.43)$$

$$(1-\mu^2)\frac{d^2y}{d\mu^2} - 3\mu\frac{dy}{d\mu} + (n^2-1)y = 0 \quad \text{Solution: } y = \alpha'U_{n-1}(\mu) + \beta'\frac{T_n(\mu)}{\sqrt{1-\mu^2}} \quad (3.44)$$

### 3.5. Orthogonality.

$$\int_{-2}^2 C_m(x)C_n(x)\frac{dx}{\sqrt{4-x^2}} = 0 \quad \int_{-2}^2 S_m(x)S_n(x)\sqrt{4-x^2}dx = 0 \quad m \neq n \quad (3.51)$$

$$\int_{-2}^2 C_n^2(x)\frac{dx}{\sqrt{4-x^2}} = 2\pi \quad \int_{-2}^2 S_n^2(x)\sqrt{4-x^2}dx = 2\pi \quad n \geq 1 \quad (3.52)$$

$$\int_{-1}^1 T_m(\mu)T_n(\mu)\frac{d\mu}{\sqrt{1-\mu^2}} = 0 \quad \int_{-1}^1 U_m(\mu)U_n(\mu)\sqrt{1-\mu^2}d\mu = 0 \quad m \neq n \quad (3.53)$$

$$\int_{-1}^1 T_n^2(\mu) \frac{d\mu}{\sqrt{1-\mu^2}} = \frac{1}{2}\pi \quad \int_{-1}^1 U_n^2(\mu) \sqrt{1-\mu^2} d\mu = \frac{1}{2}\pi \quad n \geq 1 \quad (3.54)$$

$$\int_0^1 T_m T_n \frac{dX}{\sqrt{X-X^2}} = 0 \quad \int_0^1 U_m U_n \sqrt{X-X^2} dX = 0 \quad m \neq n \quad (3.55)$$

$$\int_0^1 T_n^2 \frac{dX}{\sqrt{X-X^2}} = \frac{1}{2}\pi \quad \int_0^1 U_n^2 \sqrt{X-X^2} dX = \frac{1}{8}\pi \quad n \geq 1 \quad (3.56)$$

$$\int_a^b T_m T_n \frac{dz}{\sqrt{(z-a)(b-z)}} = 0 \quad \int_a^b U_m U_n \sqrt{(z-a)(b-z)} dz = 0 \quad m \neq n \quad (3.57)$$

$$\int_a^b T_n^2 \frac{dz}{\sqrt{(z-a)(b-z)}} = \frac{1}{2}\pi \quad \int_a^b U_n^2 \sqrt{(z-a)(b-z)} dz = \frac{1}{8}\pi(b-a)^2 \quad n \geq 1 \quad (3.58)$$

For brevity, the appropriate argument  $\mu = 2X - 1 = (2z - b - a)/(b - a)$  has been omitted from  $T_p, U_p$  in (3.55)–(3.58).

### 3.6. Generating Functions.

$$\sum_{n=0}^{\infty} C_n(x) t^n = \frac{2 - xt}{1 - xt + t^2} \quad \sum_{n=0}^{\infty} S_n(x) t^n = \frac{1}{1 - xt + t^2} \quad (3.61)$$

$$\sum_{n=0}^{\infty} T_n(\mu) t^n = \frac{1 - \mu t}{1 - 2\mu t + t^2} \quad \sum_{n=0}^{\infty} U_n(\mu) t^n = \frac{1}{1 - 2\mu t + t^2} \quad (3.62)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} &= 2e^{\frac{1}{2}xt} \cos\left(\frac{1}{2}t\sqrt{4-x^2}\right) \\ \sum_{n=0}^{\infty} \sqrt{4-x^2} S_{n-1}(x) \frac{t^n}{n!} &= 2e^{\frac{1}{2}xt} \sin\left(\frac{1}{2}t\sqrt{4-x^2}\right) \end{aligned} \right\} \quad (3.63)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} T_n(\mu) \frac{t^n}{n!} &= e^{\mu t} \cos(t\sqrt{1-\mu^2}) \\ \sum_{n=0}^{\infty} \sqrt{1-\mu^2} U_{n-1}(\mu) \frac{t^n}{n!} &= e^{\mu t} \sin(t\sqrt{1-\mu^2}) \end{aligned} \right\} \quad (3.64)$$

The connection with Jacobi polynomials (Szegő, 1939, pp. 59, 68) gives a third set of generating functions:

$$\left. \begin{aligned} \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} C_n(x) t^n &= \left\{ \frac{2 - tx + 2\sqrt{1-tx+t^2}}{1 - tx + t^2} \right\}^{\frac{1}{2}} \\ \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \cdot 6 \dots (2n+2)} S_n(x) t^n &= \frac{1}{t\sqrt{4-x^2}} \left\{ \frac{-2 + tx + 2\sqrt{1-tx+t^2}}{1 - tx + t^2} \right\}^{\frac{1}{2}} \\ &= \{(1-tx+t^2)(2-tx+2\sqrt{1-tx+t^2})\}^{-\frac{1}{2}} \end{aligned} \right\} \quad (3.65)$$

with similar expressions involving  $T_n, U_n$  and  $\mu$ .

3.7. *Further Relationships*.—Since the tables give only  $C_n(x)$ , the following relations may be found useful. We have (omitting the argument  $(x)$ )

$$C_n = S_n - S_{n-2} \quad dC_n/dx = nS_{n-1} \quad (3.71)$$

whence

$$\left. \begin{aligned} S_{2n-1} &= C_{2n-1} + C_{2n-3} + \dots + C_1 \\ S_{2n} &= C_{2n} + C_{2n-2} + \dots + C_2 + 1 \end{aligned} \right\} \quad (3.72)$$

and  $dC_n/dx$  follows. Also

$$\frac{dS_{n-1}}{dx} = \frac{1}{n} \frac{d^2 C_n}{dx^2} = (xS_{n-1} - nC_n)/(4-x^2). \quad (3.73)$$

### 4. Calculation of the Tables.

4.1. The functions  $C_n(x)$  were computed by C. W. J. and J. C. P. M. to 14 decimals, using (3.11), for each particular value of  $x$ , and checked by (3.12) with  $m=n$ , i.e.  $C_{2n} = C_n^2 - 2$ . Typed tables, suitable for printer's copy, were then prepared giving  $C_n(x)$  for  $n=1(1)12$ ,  $x=0(0.02)2$ ; exact values are given for  $n \leq 7$  and 10-decimal values for  $n \geq 8$ .

All typed values, together with the extra MS. digits, have been checked by formation of selected differences. For  $n \leq 7$  every value was used in the formation of at least one value of  $\delta^{10}$  which was required to be exactly zero; every tenth such difference was evaluated. For  $n \geq 8$  (where exact values were not calculated by C. W. J. and J. C. P. M.), selected values of  $\delta^{10}$  were similarly evaluated, and the results recorded; further, the four final digits (11th to 14th decimals) were used to give other values of  $\delta^{10}$ , one half-way between each pair of values previously obtained. For  $n \leq 9$ , these differences should all be close to zero, and for  $n \geq 10$  they should be close to values calculated in advance; agreement was satisfactory in all cases.

4.2. Values of the polynomials were obtained independently by J. F. C. C. and R. C. P., all being exact up to and including  $C_{10}$ . For each of the lower degree polynomials a complete difference table was built up, starting from constant  $\delta^n$ ; for the higher degree polynomials to  $n = 10$ , (3.12) was used, with various values of  $m$ , and the results checked by formation of selected values of  $\delta^n$ , as explained above. In each case  $T_n$  was found and the results doubled to give  $C_n$ . Finally all entries for  $x = 0.012$  were checked by direct evaluation of  $C_n$ . For  $n = 11$  and 12, values were found to 12 decimals only, again by using (3.12) to give  $T_{11}$  and  $T_{12}$ , and then doubling the results.

4.3. The two independent sets of values were compared with each other as an additional check (although the only check on the 12-decimal values of  $C_{11}$  and  $C_{12}$ ); no discrepancies were found.

Typed copies of the tables, giving exact values of  $C_n(x)$  for  $n \leq 10$  have been presented to the Royal Society of Edinburgh and to the British Association Committee on the Calculation of Mathematical Tables.

5. *Values for Non-Tabular Arguments.*—No provision has been made for interpolation. This is because it seems likely that tabular arguments will usually suffice. Again, if this is not the case, it seems almost certain that values of  $C_n(x)$ , when required, will be needed for several different values of  $n$ , for each individual value of  $x$  used, *i.e.* that interpolation at the same argument in several columns would be necessary. In these circumstances, it is better to use (3.11), starting with  $C_0 = 2$ ,  $C_1 = x$ , than to interpolate, checking finally by means of  $C_{2n} = C_n^2 - 2$ . An alternative check is to obtain the final value by interpolation from the present tables; formulæ of the Lagrange type seem most appropriate for such infrequent calculations. Yet another possibility is to evaluate  $\theta = \cos^{-1} \frac{1}{2}x = \tan^{-1} (x/\sqrt{4-x^2})$  and from it to obtain  $2 \cos n\theta$  for the required value of  $n$ .

Isolated values may also be found in the same way, if needed, but it seems useful to indicate how the recurrence formulæ may be combined to cut out intermediate steps. Thus

$$C_2 = x^2 - 2, \quad C_4 = C_2^2 - 2, \quad C_8 = C_4^2 - 2$$

gives  $C_8$  in three stages;

$$C_2 = x^2 - 2, \quad C_3 = xC_2 - C_1, \quad C_6 = C_3^2 - 2, \quad C_9 = C_3(C_6 - 1)$$

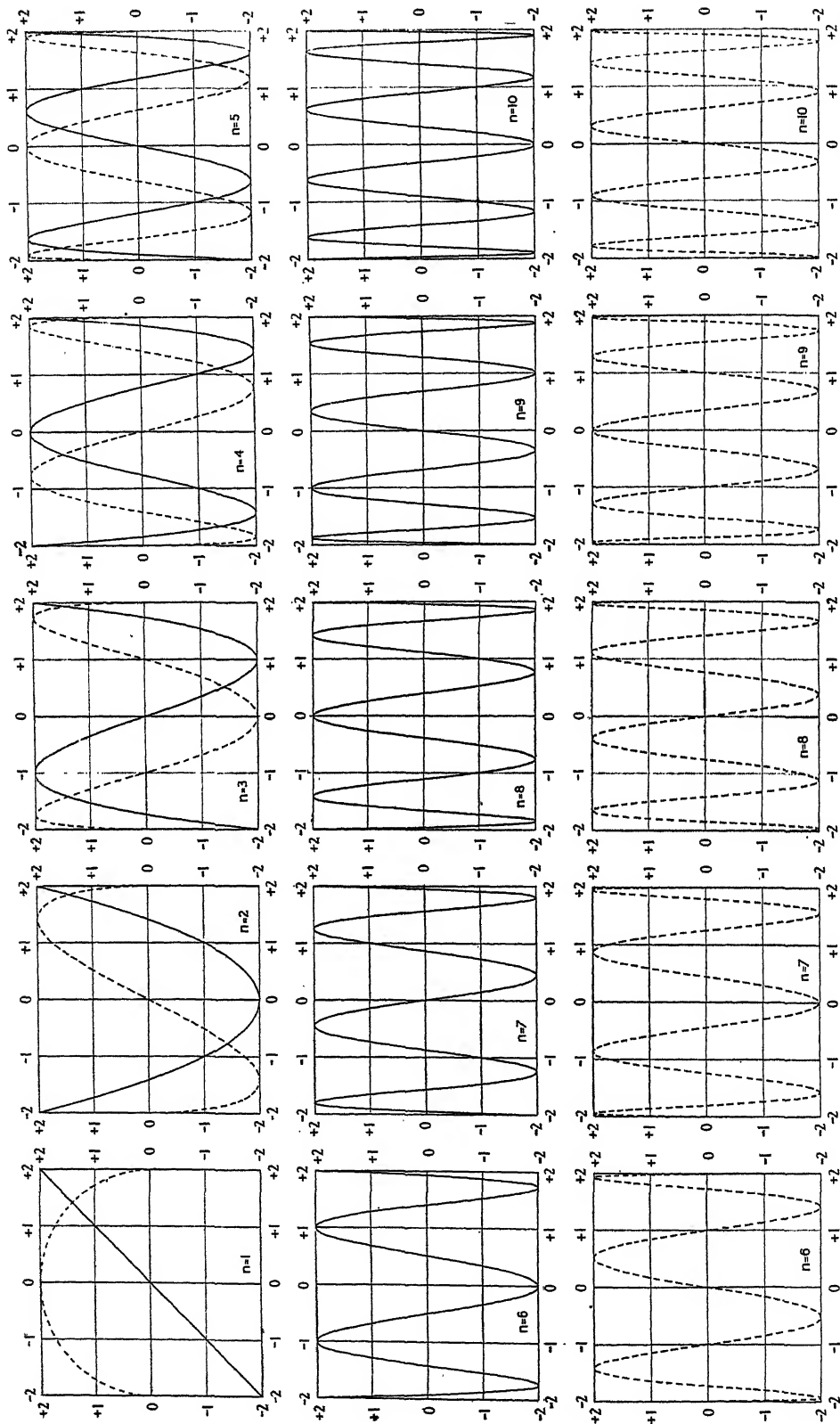
gives  $C_9$  in 4 stages. The processes are not ideal, but may often be less troublesome than interpolation.

In all calculations of this type, mental interpolation in the tables gives a useful check against gross errors, such as errors of sign.

6. *Acknowledgment.*—The writers wish to express their thanks to Dr L. J. Comrie for help in the design and preparation of the diagrams and for verifying the last 11 digits of the exact values of  $C_7$  to  $C_{10}$  by differencing them on a National accounting machine.

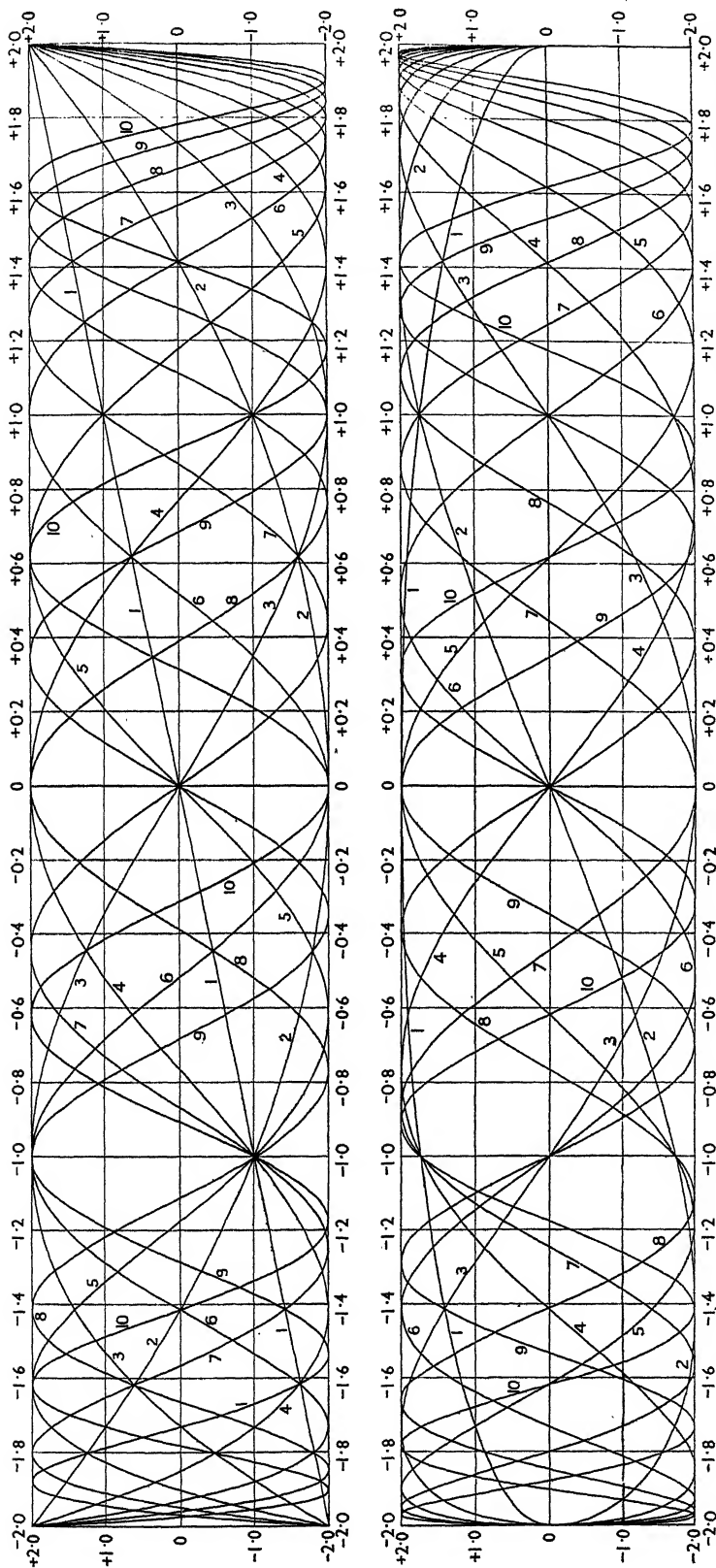
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By courtesy of Scientific Computing Service Ltd.

The figures show, for individual values of  $n$ ,  $C_n(x) = 2T_n(\mu) = 2 \cos n\theta$  (full lines) and  $\sigma_n(x) = 2V_n(\mu) = 2 \sin n\theta$  (broken lines) with argument  $x = 2\mu = 2 \cos \theta$ ,  $0 \leq \theta \leq \pi$ . By this definition the range in  $x$ , and in  $C_n(x)$  and  $\sigma_n(x)$ , is limited to the interval  $(-2, +2)$  covered by the diagrams. Extensions outside this range may be made by means of similar definitions in terms of hyperbolic functions; these lead to the *same* polynomials  $C_n(x)$  and  $S_{n-1}(x) = \sigma_n(x)/\sigma_1(x)$  as with  $|x| \leq 2$ , but with  $\sigma_1(x) = \sqrt{x^2 - 4}$  in place of  $\sigma_1(x) = \sqrt{4 - x^2}$ . Note that, for any given  $n$ , each of  $C_n(x)$  and  $\sigma_n(x)$  has a stationary value  $\pm 2$  whenever the other has a zero.



By courtesy of Scientific Computing Service Ltd.

The upper figure shows  $C_n(x) = 2T_n(\mu) = 2 \cos n\theta$  for  $n = 1, 2, 3, \dots$  to 10 on a single diagram; the lower figure shows  $\sigma_n(x) = 2v_n(\mu) = 2 \sin n\theta$ , also for  $n = 1, 2, 3, \dots$  to 10. In each case the argument is  $x = 2\mu = 2 \cos \theta$ ,  $0 \leq \theta \leq \pi$ , and ranges from  $-2$  to  $+2$ , as in the diagrams for individual values of  $n$  on the opposite page. The numbers on the diagrams show the value of  $n$  to be associated with each curve.

The increase in scale of the abscissa  $x$ , relative to that for the ordinate, makes it easier to note the following points of interest: Every intersection of two curves has an abscissa  $x = 2 \cos \alpha$  for which  $\alpha/n$  is a rational fraction. This follows from the relations

$$2 \cos p\theta - 2 \cos q\theta = 4 \sin \frac{1}{2}(p+q)\theta \sin \frac{1}{2}(p-q)\theta, \quad 2 \sin p\theta - 2 \sin q\theta = 4 \cos \frac{1}{2}(p+q)\theta \sin \frac{1}{2}(p-q)\theta.$$

Since the ordinates are  $2 \cos n\alpha$  or  $2 \sin n\alpha$ , with integral  $n$ , these also are of the form  $2 \cos \beta$  (or  $2 \sin \beta$ ), with  $\beta/\pi$  a rational fraction. In consequence of this, whenever  $\alpha$  is a particular fraction of  $\pi$ , all curves must pass through a limited number of definite points on the corresponding ordinate. Thus, when  $\alpha = \frac{1}{2}\pi$ ,  $x = 2 \cos \frac{1}{2}\pi = 0.618$ , and the possible values of the ordinate are: in the upper diagram,  $2, 2 \cos \frac{2}{3}\pi = 0.618$  and  $2 \cos \frac{4}{3}\pi = -1.618$ ; in the lower diagram,  $0, \pm 2 \sin \frac{1}{2}\pi = \pm 1.902$  and  $\pm 2 \sin \frac{3}{2}\pi = \pm 1.176$ .

The abscissae of all maxima, minima and zeros are also of the same form  $x = 2 \cos \alpha$ ,  $\alpha/\pi$  rational, but it is of interest to note further that, for example,  $C_{2n}(x) = C_n^2(x) - 2$  has maxima when  $C_n(x) = \pm 2$  and is a maximum or minimum, and has minima when  $C_n(x) = 0$ . On the other hand  $\sigma_{2n}(x) = C_n(x)\sigma_n(x)$  has zeros when  $C_n(x) = 0$  and when  $\sigma_n(x) = 0$ , which is also when  $C_n(x) = \pm 2$ . Further relations, such as  $C_{3n}(x) = C_n^3(x) - 3C_n(x)$ , may be interpreted on the diagrams in a similar way.



## CHEBYSHEV POLYNOMIALS

$x$	$\mu$	$X$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$1-X$
0.00	0.00	0.500	0.00	-2.0000	-0.00000 0	+2.00000 000	+0.00000 00000	0.500
0.02	0.01	.505	.02	1.9996	.05999 2	1.99840 016	.09996 00032	.495
0.04	0.02	.510	.04	.9984	.11993 6	.99360 256	.19968 01024	.490
0.06	0.03	.515	.06	.9964	.17978 4	.98561 296	.29892 07776	.485
0.08	0.04	.520	.08	.9936	.23948 8	.97444 096	.39744 32768	.480
0.10	0.05	0.525	0.10	-1.9900	-0.29900 0	+1.96010 000	+0.49501 00000	0.475
0.12	0.06	.530	.12	.9856	.35827 2	.94260 736	.59138 48832	.470
0.14	0.07	.535	.14	.9804	.41725 6	.92198 416	.68633 37824	.465
0.16	0.08	.540	.16	.9744	.47590 4	.89825 536	.77962 48576	.460
0.18	0.09	.545	.18	.9676	.53416 8	.87144 976	.87102 89568	.455
0.20	0.10	0.550	0.20	-1.9600	-0.59200 0	+1.84160 000	+0.96032 00000	0.450
0.22	0.11	.555	.22	.9516	.64935 2	.80874 256	.104727 53632	.445
0.24	0.12	.560	.24	.9424	.70617 6	.77291 776	.13167 62624	.440
0.26	0.13	.565	.26	.9324	.76242 4	.73416 976	.21330 81376	.435
0.28	0.14	.570	.28	.9216	.81804 8	.69254 656	.29196 10368	.430
0.30	0.15	0.575	0.30	-1.9100	-0.87300 0	+1.64810 000	+1.36743 00000	0.425
0.32	0.16	.580	.32	.8976	.092723 2	.60088 576	.43951 54432	.420
0.34	0.17	.585	.34	.8844	.098069 6	.55096 336	.50802 35424	.415
0.36	0.18	.590	.36	.8704	.103334 4	.49839 616	.57276 66176	.410
0.38	0.19	.595	.38	.8556	.108512 8	.44325 136	.63356 35168	.405
0.40	0.20	0.600	0.40	-1.8400	-1.13600 0	+1.38560 000	+1.69024 00000	0.400
0.42	0.21	.605	.42	.8236	.18591 2	.60088 576	.74262 91232	.395
0.44	0.22	.610	.44	.8064	.23481 6	.26308 096	.79057 16224	.390
0.46	0.23	.615	.46	.7884	.28266 4	.19837 456	.83391 62976	.385
0.48	0.24	.620	.48	.7696	.32940 8	.13148 416	.87252 03968	.380
0.50	0.25	0.625	0.50	-1.7500	-1.37500 0	+1.06250 000	+1.90625 00000	0.375
0.52	0.26	.630	.52	.7296	.41939 2	.099151 616	.93498 04032	.370
0.54	0.27	.635	.54	.7084	.46253 6	.91863 056	.95859 65024	.365
0.56	0.28	.640	.56	.6864	.50438 4	.84394 496	.97699 31776	.360
0.58	0.29	.645	.58	.6636	.54488 8	.76756 496	.99007 56768	.355
0.60	0.30	0.650	0.60	-1.6400	-1.58400 0	+0.68960 000	+1.99776 00000	0.350
0.62	0.31	.655	.62	.6156	.62167 2	.61016 336	.99997 32832	.345
0.64	0.32	.660	.64	.5904	.65785 6	.52937 216	.99665 41824	.340
0.66	0.33	.665	.66	.5644	.69250 4	.44734 736	.98775 32576	.335
0.68	0.34	.670	.68	.5376	.72556 8	.36421 376	.97323 33568	.330
0.70	0.35	0.675	0.70	-1.5100	-1.75700 0	+0.28010 000	+1.95307 00000	0.325
0.72	0.36	.680	.72	.4816	.78675 2	.19513 856	.92725 17632	.320
0.74	0.37	.685	.74	.4524	.81477 6	.10946 576	.89578 06624	.315
0.76	0.38	.690	.76	.4224	.84102 4	0.02322 176	.85867 25376	.310
0.78	0.39	.695	.78	.3916	.86544 8	0.06344 944	.81595 74368	.305
0.80	0.40	0.700	0.80	-1.3600	-1.88800 0	-0.15040 000	+1.76768 00000	0.300
0.82	0.41	.705	.82	.3276	.90863 2	.23747 824	.71389 98432	.295
0.84	0.42	.710	.84	.2944	.92729 6	.32452 864	.65469 19424	.290
0.86	0.43	.715	.86	.2604	.94394 4	.41139 184	.59014 70176	.285
0.88	0.44	.720	.88	.2256	.95852 8	.49790 464	.52037 19168	.280
0.90	0.45	0.725	0.90	-1.1900	-1.97100 0	-0.58390 000	+1.44549 00000	0.275
0.92	0.46	.730	.92	.1536	.98131 2	.66920 704	.36564 15232	.270
0.94	0.47	.735	.94	.1164	.98941 6	.75365 104	.28098 40224	.265
0.96	0.48	.740	.96	.0784	.99526 4	.83705 344	.19169 26976	.260
0.98	0.49	.745	0.98	.0396	1.99880 8	0.91923 184	.09796 07968	.255
1.00	0.50	0.750	1.00	-1.0000	-2.00000 0	-1.00000 000	+1.00000 00000	0.250
$-x$	$-\mu$	$1-X$	$-C_1$	$+C_2$	$-C_3$	$+C_4$	$-C_5$	$X$

$$\frac{1}{2}x = \mu = 2X - 1 = \cos \theta \quad C_0 = 2 = 2T_0 \quad C_n = 2 \cos n\theta = 2T_n \quad X = \cos^2 \frac{1}{2}\theta \quad 1 - X = \sin^2 \frac{1}{2}\theta$$

## CHEBYSHEV POLYNOMIALS

$x$	$\mu$	$X$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$1 - X$
1.00	0.50	0.750	1.00	-1.0000	-2.00000 0	-1.00000 000	+1.00000 00000	0.250
.02	.51	.755	.02	0.9596	1.99879 2	.07916 784	0.89804 08032	.245
.04	.52	.760	.04	.9184	.99513 6	.15654 144	.79233 29024	.240
.06	.53	.765	.06	.8764	.98898 4	.23192 304	.68314 55776	.235
.08	.54	.770	.08	.8336	.98028 8	.30511 104	.57076 80768	.230
1.10	0.55	0.775	1.10	-0.7900	-1.96900 0	-1.37590 000	+0.45551 00000	0.225
.12	.56	.780	.12	.7456	.95507 2	.44408 064	.33770 16832	.220
.14	.57	.785	.14	.7004	.93845 6	.50943 984	.21769 45824	.215
.16	.58	.790	.16	.6544	.91910 4	.57176 064	+ .09586 16576	.210
.18	.59	.795	.18	.6076	.89696 8	.63082 224	- .02740 22432	.205
1.20	0.60	0.800	1.20	-0.5600	-1.87200 0	-1.68640 000	-0.15168 00000	0.200
.22	.61	.805	.22	.5116	.84415 2	.73826 544	.27653 18368	.195
.24	.62	.810	.24	.4624	.81337 6	.78618 624	.40149 49376	.190
.26	.63	.815	.26	.4124	.77962 4	.82992 624	.52608 30624	.185
.28	.64	.820	.28	.3616	.74284 8	.86924 544	.64978 61632	.180
1.30	0.65	0.825	1.30	-0.3100	-1.70300 0	-1.90390 000	-0.77207 00000	0.175
.32	.66	.830	.32	.2576	.66003 2	.93364 224	0.89237 57568	.170
.34	.67	.835	.34	.2044	.61389 6	.95822 064	1.01011 96576	.165
.36	.68	.840	.36	.1504	.56454 4	.97737 984	.12469 25824	.160
.38	.69	.845	.38	.0956	.51192 8	.99086 064	.23545 96832	.155
1.40	0.70	0.850	1.40	-0.0400	-1.45600 0	-1.99840 000	-1.34176 00000	0.150
.42	.71	.855	.42	+ .0164	.39671 2	.99973 104	.44290 60768	.145
.44	.72	.860	.44	.0736	.33401 6	.99458 304	.53818 35776	.140
.46	.73	.865	.46	.1316	.26786 4	.98268 144	.62685 09024	.135
.48	.74	.870	.48	.1904	.19820 8	.96374 784	.70813 88032	.130
1.50	0.75	0.875	1.50	+0.2500	-1.12500 0	-1.93750 000	-1.78125 00000	0.125
.52	.76	.880	.52	.3104	1.04819 2	.90365 184	.84535 87968	.120
.54	.77	.885	.54	.3716	0.96773 6	.86191 344	.89961 06976	.115
.56	.78	.890	.56	.4336	.88358 4	.81199 104	.94312 20224	.110
.58	.79	.895	.58	.4964	.79568 8	.75358 704	.97497 95232	.105
1.60	0.80	0.900	1.60	+0.5600	-0.70400 0	-1.68640 000	-1.99424 00000	0.100
.62	.81	.905	.62	.6244	.60847 2	.61012 464	.99992 99168	.095
.64	.82	.910	.64	.6896	.50905 6	.52445 184	.99104 50176	.090
.66	.83	.915	.66	.7556	.40570 4	.42906 864	.96654 99424	.085
.68	.84	.920	.68	.8224	.29836 8	.23265 824	.92537 78432	.080
1.70	0.85	0.925	1.70	+0.8900	-0.18700 0	-1.20790 000	-1.86643 00000	0.075
.72	.86	.930	.72	0.9584	- .07155 2	1.08146 944	.78857 54368	.070
.74	.87	.935	.74	1.0276	+ .04802 4	0.94403 824	.69065 05376	.065
.76	.88	.940	.76	.0976	.17177 6	.79527 424	.57145 86624	.060
.78	.89	.945	.78	.1684	.29975 2	.63484 144	.42976 97632	.055
1.80	0.90	0.950	1.80	+1.2400	+0.43200 0	-0.46240 000	-1.26432 00000	0.050
.82	.91	.955	.82	.3124	.56856 8	.27760 624	1.07381 13568	.045
.84	.92	.960	.84	.3856	.70950 4	- .08011 264	0.85691 12576	.040
.86	.93	.965	.86	.4596	0.85485 6	+ .13043 216	.61225 21824	.035
.88	.94	.970	.88	.5344	1.00467 2	.35438 336	.33843 12832	.030
1.90	0.95	0.975	1.90	+1.6100	+1.15900 0	+0.59210 000	-0.03401 00000	0.025
.92	.96	.980	.92	.6864	.31788 8	0.84394 496	+0.30248 63232	.020
.94	.97	.985	.94	.7636	.48138 4	1.11028 496	0.67256 88224	.015
.96	.98	.990	.96	.8416	.64953 6	1.39149 056	1.07778 54976	.010
.98	.99	.995	.98	1.9204	1.82239 2	1.68793 616	1.51972 15968	.005
2.00	1.00	1.000	2.00	+2.0000	+2.00000 0	+2.00000 0000	+2.00000 00000	0.000
$-x$	$-\mu$	$1 - X$	$-C_1$	$+C_2$	$-C_3$	$+C_4$	$-C_5$	$X$

$$\frac{1}{2}x = \mu = 2X - 1 = \cos \theta \quad C_0 = 2 = 2T_0 \quad C_n = 2 \cos n\theta = 2T_n \quad X = \cos^2 \frac{1}{2}\theta \quad 1 - X = \sin^2 \frac{1}{2}\theta$$

## CHEBYSHEV POLYNOMIALS

$x$	$\mu$	$X$	$C_6$	$C_7$	$C_8$	$1 - X$
0.00	0.00	0.500	-2.00000 00000 00	-0.00000 00000 0000	+2.00000 00000	0.500
.02	.01	.505	1.99640 09599 36	.13988 80223 9872	1.99360 31995	.495
.04	.02	.510	.98561 53559 04	.27910 47166 3616	.97445 11672	.490
.06	.03	.515	.96767 77133 44	.41698 14404 0064	.94265 88269	.485
.08	.04	.520	.94264 54978 56	.55285 49166 2848	.89841 71045	.480
0.10	0.05	0.525	-1.91059 90000 00	-0.68606 99000 0000	+1.84199 20100	0.475
.12	.06	.530	.87164 11740 16	.081598 18240 8192	.77372 33551	.470
.14	.07	.535	.82589 74304 64	.094195 94226 6496	.69402 31113	.465
.16	.08	.540	.77351 53827 84	1.06338 73188 4544	.60337 34118	.460
.18	.09	.545	.71466 45477 76	1.17966 85753 9968	.50232 42042	.455
0.20	0.10	0.550	-1.64953 60000 00	-1.29022 72000 0000	+1.39149 05600	0.450
.22	.11	.555	.57834 19800 96	.39451 05988 2112	.27154 96484	.445
.24	.12	.560	.50131 54570 24	.49199 19720 8576	.14323 73837	.440
.26	.13	.565	.41870 96442 24	.58217 26450 9824	1.00734 47565	.435
.28	.14	.570	.33079 74696 96	.66458 43283 1488	.086471 38578	.430
0.30	0.15	0.575	-1.23787 10000 00	-1.73879 13000 0000	+0.71623 36100	0.425
.32	.16	.580	.14024 08181 76	.80439 25050 1632	.56283 52166	.420
.34	.17	.585	1.03823 53555 84	.86102 35632 9856	.40548 73441	.415
.36	.18	.590	.093220 01776 64	.90835 86815 5904	.24519 10523	.410
.38	.19	.595	.082249 72236 16	.94611 24617 7408	+ .08297 44881	.405
0.40	0.20	0.600	-0.70950 40000 00	-1.97404 16000 0000	-0.08011 26400	0.400
.42	.21	.605	.59361 27282 56	.99194 64690 6752	.24300 47888	.395
.44	.22	.610	.47522 94461 44	.99967 25787 0336	.40462 64885	.390
.46	.23	.615	.35477 30631 04	.99711 19066 2784	.56389 84139	.385
.48	.24	.620	.23267 43695 36	.98420 40941 7728	.71974 35957	.380
0.50	0.25	0.625	-0.10937 50000 00	-1.96093 75000 0000	-0.87109 37500	0.375
.52	.26	.630	+ .01467 36496 64	.92735 01053 7472	1.01689 57045	.370
.54	.27	.635	.13901 15512 96	.88353 02647 0016	.15611 78942	.365
.56	.28	.640	.26317 12194 56	.82961 72947 0464	.28775 69045	.360
.58	.29	.645	.38667 89325 44	.76580 18959 2448	.41084 40322	.355
0.60	0.30	0.650	+0.50905 60000 00	-1.69232 64000 0000	-1.52445 18400	0.350
.62	.31	.655	.62982 00755 84	.60948 48363 3792	.62770 06741	.345
.64	.32	.660	.74848 65167 36	.51762 28116 8896	.71976 51162	.340
.66	.33	.665	.86456 97900 16	.41713 71961 8944	.79988 03395	.335
.68	.34	.670	.097758 49226 24	.30847 56094 1568	.86734 83370	.330
0.70	0.35	0.675	+1.08704 90000 00	-1.19213 57000 0000	-1.92154 39900	0.325
.72	.36	.680	.19248 27095 04	1.06866 42123 5712	.96192 09424	.320
.74	.37	.685	.29341 19301 76	.093865 58340 6976	.98801 72474	.315
.76	.38	.690	.38936 93685 76	.80275 18174 8224	.99946 07499	.310
.78	.39	.695	.47989 62407 04	.66163 83690 5088	.99597 41686	.305
0.80	0.40	0.700	+1.56454 40000 00	-0.51604 48000 0000	-1.97737 98400	0.300
.82	.41	.705	.64287 61114 24	.36674 14318 3232	.94360 40855	.295
.84	.42	.710	.71446 98716 16	.21453 72502 4256	.89468 11618	.290
.86	.43	.715	.77891 82751 36	-.06027 73009 8304	.83075 67540	.285
.88	.44	.720	.83583 19267 84	+ .09516 01787 6992	.75209 09695	.280
0.90	0.45	0.725	+1.88484 10000 00	+0.25086 69000 0000	-1.65906 07900	0.275
.92	.46	.730	.92559 72413 44	.40590 79388 3648	.55216 19376	.270
.94	.47	.735	.95777 60210 56	.55932 54373 9264	.43201 01099	.265
.96	.48	.740	.98107 84296 96	.71014 25949 0816	.29934 15386	.260
.98	.49	.745	1.99523 34208 64	.085736 79556 4672	.15501 28243	.255
1.00	0.50	0.750	+2.00000 00000 00	+1.00000 00000 0000	-1.00000 00000	0.250
$-x$	$-\mu$	$1 - X$	$+C_6$	$-C_7$	$+C_8$	$X$

$$\frac{1}{2}x = \mu = 2X - 1 = \cos \theta \quad C_n = 2 \cos n\theta = 2T_n \quad X = \cos^2 \frac{1}{2}\theta \quad 1 - X = \sin^2 \frac{1}{2}\theta$$

## CHEBYSHEV POLYNOMIALS

$x$	$\mu$	$X$	$C_6$	$C_7$	$C_8$	$1 - X$
1.00	0.50	0.750	+ 2.00000 00000 00	+ 1.00000 00000 0000	- 1.00000 00000	0.250
.02	.51	.755	1.99516 94592 64	.13703 20452 4928	0.83539 67731	.245
.04	.52	.760	.98056 76584 96	.26745 74624 3584	.66241 18976	.240
.06	.53	.765	.95605 73522 56	.39027 52157 9136	.48236 56235	.235
.08	.54	.770	.92154 05629 44	.50449 57311 7952	.29668 51733	.230
1.10	0.55	0.775	+ 1.87696 10000 00	+ 1.60914 71000 0000	- 1.10689 91900	0.225
.12	.56	.780	.82230 65251 84	.70328 16250 0608	+ .08536 88948	.220
.14	.57	.785	.75761 16639 36	.78598 27144 8704	.27840 86306	.215
.16	.58	.790	.68296 01628 16	.85637 21312 6656	.47043 15095	.210
.18	.59	.795	.59848 75930 24	.91361 76029 6832	.65958 11785	.205
1.20	0.60	0.800	+ 1.50438 40000 00	+ 1.95694 08000 0000	+ 0.84394 49600	0.200
.22	.61	.805	.40089 65991 04	.98562 56877 0688	1.02156 67399	.195
.24	.62	.810	.28833 25173 76	.99902 72591 4624	.19046 12840	.190
.26	.63	.815	.16706 15813 76	.99658 06549 3376	.34863 00438	.185
.28	.64	.820	1.03751 91511 04	.97781 06766 1312	.49407 85150	.180
1.30	0.65	0.825	+ 0.90020 90000 00	+ 1.94234 17000 0000	+ 1.62483 52100	0.175
.32	.66	.830	.75570 62410 24	.88990 79949 5168	.73897 23123	.170
.34	.67	.835	.60466 02988 16	.82036 44580 1344	.83462 80749	.165
.36	.68	.840	.44779 79279 36	.73369 77643 9296	.91003 10316	.160
.38	.69	.845	.28592 62771 84	.63003 79457 1392	.96352 60879	.155
1.40	0.70	0.850	+ 0.11993 60000 00	+ 1.50967 04000 0000	+ 1.99360 25600	0.150
.42	.71	.855	- .04919 55890 56	.37304 83403 4048	.99892 42323	.145
.44	.72	.860	.22040 13117 44	.22080 56886 8864	.97836 15035	.140
.46	.73	.865	.39252 08775 04	1.05377 04212 4416	.93102 56925	.135
.48	.74	.870	.56429 75887 36	0.87297 83718 7072	.85630 55791	.130
1.50	0.75	0.875	- 0.73437 50000 00	+ 0.67968 75000 0000	+ 1.75390 62500	0.125
.52	.76	.880	0.90129 35311 36	.47539 26294 7328	.62389 03279	.120
.54	.77	.885	1.06348 70343 04	.26184 06647 7184	.46672 16581	.115
.56	.78	.890	.21927 93149 44	+ .04104 62910 8736	.28331 15290	.110
.58	.79	.895	.36688 06066 56	- .18469 18353 1648	1.07506 75069	.105
1.60	0.80	0.900	- 1.50438 40000 00	- 0.41277 44000 0000	+ 0.84394 49600	0.100
.62	.81	.905	.62976 18252 16	0.64028 42400 4992	.59250 13563	.095
.64	.82	.910	.74086 19888 64	0.86396 86441 3696	.32395 34125	.090
.66	.83	.915	.83540 42643 84	1.08022 11364 7744	+ .04223 71778	.085
.68	.84	.920	.91097 65365 76	1.28506 27382 4768	- .24792 88637	.080
1.70	0.85	0.925	- 1.96503 10000 00	- 1.47412 27000 0000	- 0.54097 75900	0.075
.72	.86	.930	.99488 03112 96	.64261 86986 2912	0.83042 38503	.070
.74	.87	.935	.99769 36954 24	.78533 64924 3776	1.10879 18014	.065
.76	.88	.940	.97049 30058 24	.89660 90278 5024	.36753 88832	.060
.78	.89	.945	.91014 87384 96	.97029 49913 2288	.59697 63461	.055
1.80	0.90	0.950	- 1.81337 60000 00	- 1.99975 68000 0000	- 1.78618 62400	0.050
.82	.91	.955	.67673 04293 76	.97783 80246 6432	.92293 47755	.045
.84	.92	.960	.49660 40739 84	.89684 02385 3056	.99358 19649	.040
.86	.93	.965	1.26922 12192 64	.74849 92854 3104	.98298 74516	.035
.88	.94	.970	0.99063 41724 16	.52396 09609 4208	.87441 24342	.030
1.90	0.95	0.975	- 0.65671 90000 00	- 1.21375 61000 0000	- 1.64941 75900	0.025
.92	.96	.980	- 0.26317 12194 56	0.80777 50645 5552	1.28775 69045	.020
.94	.97	.985	+ 0.19449 85554 56	- 0.29524 16248 1536	0.76726 73076	.015
.96	.98	.990	1.72096 90152 96	+ 0.33531 37723 8016	- 0.06375 40214	.010
.98	.99	.995	1.32111 26016 64	1.09608 13544 9472	+ 0.84912 84802	.005
2.00	1.00	1.000	+ 2.00000 00000 00	+ 2.00000 00000 0000	+ 2.00000 00000	0.000
$-x$	$-\mu$	$1 - X$	$+C_6$	$-C_7$	$+C_8$	$X$

$$\frac{1}{2}x = \mu = 2X - 1 = \cos \theta \quad C_n = 2 \cos n\theta = 2T_n \quad X = \cos^2 \frac{1}{2}\theta \quad 1 - X = \sin^2 \frac{1}{2}\theta$$

## CHEBYSHEV POLYNOMIALS

$x$	$\mu$	$X$	$C_0$	$C_{10}$	$C_{11}$	$C_{12}$	$1 - X$
0.00	0.00	0.500	+0.00000 00000	-2.00000 00000	-0.00000 00000	+2.00000 00000	0.500
	.01	.505	.17976 00864	1.99000 79978	.21956 02463	1.98561 67928	.495
.02	.02	.510	.35808 27633	.96012 78567	.43648 78776	.94266 83416	.490
.04	.03	.515	.53354 09700	.91064 63687	.64817 97521	.87175 55836	.485
.06	.04	.520	.70472 82850	.84203 88417	.085209 13923	.77387 15303	.480
.08							
0.10	0.05	0.525	+0.87026 91010	-1.75496 50999	-1.04576 56110	+1.65038 85388	0.475
	.06	.530	1.02882 86267	.65026 39199	.22686 02971	.50304 06843	.470
.12	.07	.535	.17912 26582	.52894 59391	.39317 50897	.33390 14266	.465
.14	.08	.540	.31992 70647	.39218 50814	.54267 66778	1.14535 88130	.460
.16	.09	.545	.45008 69322	.24130 85564	.67352 24723	.094007 45114	.455
.18							
0.20	0.10	0.550	+1.56852 53120	-1.07778 54976	-1.78408 24115	+0.72096 90153	0.450
	.11	.555	.67425 15215	.090321 43136	.87295 86705	.49116 34061	.445
.22	.12	.560	.76636 89442	.71930 88371	.93900 30651	.25394 81015	.440
.24	.13	.565	.84408 22818	.52788 33632	.98133 19562	+ .01273 70546	.435
.26	.14	.570	.90670 42085	.33083 66794	.99933 84787	- .22897 80947	.430
.28							
0.30	0.15	0.575	+1.95366 13830	-0.13013 51951	-1.99270 19415	-0.46767 53874	0.425
	.16	.580	.98449 97743	+ .07220 47112	.96139 42667	.069985 08766	.420
.32	.17	.585	.99888 92603	.27413 50044	.90568 33588	.092206 73464	.415
.34	.18	.590	.99662 74604	.47359 48334	.82613 33204	1.13100 28288	.410
.36	.19	.595	.97764 27673	.66852 97034	.72360 14572	1.32349 83171	.405
.38							
0.40	0.20	0.600	+1.94199 65440	+0.85691 12576	-1.59923 20410	-1.49660 40740	0.400
	.21	.605	.88988 44578	1.03675 62610	.45444 68282	.64762 39289	.395
.42	.22	.610	.82163 69238	.20614 67349	.29093 23604	.77415 69735	.390
.44	.23	.615	.73771 86362	.36324 89866	1.11062 41024	.87413 60737	.385
.46	.24	.620	.63872 71683	.50633 26364	.091568 75028	.94586 26378	.380
.48							
0.50	0.25	0.625	+1.52539 06250	+1.63378 90625	-0.70849 60938	-1.98803 71094	0.375
	.26	.630	.39856 43391	.74414 91608	.49160 67755	.99978 46840	.370
.52	.27	.635	.25922 66018	.83610 02592	.26773 24618	.98067 57886	.365
.54	.28	.640	1.10847 34282	.90850 20243	- .03971 22946	.93074 09093	.360
.56	.29	.645	.094751 23573	.96040 11994	+ .18952 03384	.85047 94031	.355
.58							
0.60	0.30	0.650	+0.77765 25960	+1.99104 50176	+0.41697 17146	-1.74086 19889	0.350
	.31	.655	.60031 04184	.99989 31335	.063962 33244	.60332 66724	.345
.62	.32	.660	.41697 31373	.98662 79241	.085446 87341	.43976 79343	.340
.64	.33	.665	.22921 61721	.95116 30131	1.05855 14165	.25251 90782	.335
.66	.34	.670	+ .03867 87402	.89364 98804	1.24900 31784	1.04432 77191	.330
.68							
0.70	0.35	0.675	-0.15294 50930	+1.81448 24249	+1.42308 27904	-0.81832 44716	0.325
	.36	.680	.34391 88662	.71429 93588	.57821 44045	.57798 49875	.320
.72	.37	.685	.53247 69290	.59398 43199	.71202 53257	.32708 55789	.315
.74	.38	.690	.71683 83524	.45466 36020	.82238 26900	- .06965 27577	.310
.76	.39	.695	.089522 14824	.29770 14123	.90742 85840	+ .19009 28832	.305
.78							
0.80	0.40	0.700	-1.06585 90720	+1.12469 25824	+1.96561 31379	+0.44779 79279	0.300
	.41	.705	.22701 39183	.093745 26725	.99572 51098	.069904 19175	.295
.82	.42	.710	.37699 49257	.73800 54242	.99691 94821	.093940 69407	.290
.84	.43	.715	.51417 35074	.52856 75376	.96874 15898	1.16455 02296	.285
.86	.44	.720	.63700 30219	.31153 07654	.91114 73054	1.37027 88634	.280
.88							
0.90	.045	0.725	-1.74402 16110	+0.08944 13401	+1.82451 88171	+1.55262 55953	0.275
	.46	.730	.83389 69214	- .13502 32301	.70967 55497	.70792 47359	.270
.92	.47	.735	.90541 49407	.35907 99344	.56787 98024	.83288 69486	.265
.94	.48	.740	.95751 04719	.57986 85145	.40083 66980	.92467 17446	.260
.96	.49	.745	1.98928 05235	.079448 20887	.21068 80766	1.98095 64037	.255
.98							
1.00	0.50	0.750	-2.00000 00000	-1.00000 00000	+1.00000 00000	+2.00000 00000	0.250
-x	-μ	1-X	-C <sub>0</sub>	+C <sub>10</sub>	-C <sub>11</sub>	+C <sub>12</sub>	X

$$\frac{1}{2}x = \mu = 2X - 1 = \cos \theta \quad C_n = 2 \cos n\theta = 2T_n \quad X = \cos^2 \frac{1}{2}\theta \quad 1 - X = \sin^2 \frac{1}{2}\theta$$

## CHEBYSHEV POLYNOMIALS

$x$	$\mu$	$X$	$C_9$	$C_{10}$	$C_{11}$	$C_{12}$	$1 - X$
1.00	0.50	0.750	-2.00000 00000	-1.00000 00000	+1.00000 00000	+2.00000 00000	0.250
.02	.51	.755	1.98913 67538	.19352 27158	0.77174 35837	1.98070 11712	.245
.04	.52	.760	.95636 58359	.37220 85718	.52926 89213	.92264 82499	.240
.06	.53	.765	.90158 27767	.53331 21198	.27627 19297	.82616 03653	.235
.08	.54	.770	.82491 57183	.67422 38025	+ .01675 40116	.69231 81350	.230
1.10	0.55	0.775	-1.72673 62090	-1.79251 06399	-0.24502 54949	+1.52298 25955	0.225
.12	.56	.780	.60766 84628	.88595 75732	.50460 40191	1.32080 10717	.220
.14	.57	.785	.46859 68756	.95260 90688	.75737 74628	1.08919 87612	.215
.16	.58	.790	.31067 15803	.99081 05426	0.99866 86491	0.83235 49096	.210
.18	.59	.795	1.13531 18124	.99924 91171	1.22380 21458	0.55516 25851	.205
1.20	0.60	0.800	-0.94420 68480	-1.97699 31776	-1.42818 49651	+0.26317 12195	0.200
.22	.61	.805	.73931 42650	.92353 01432	.60739 25097	-.03748 87186	.195
.24	.62	.810	.52285 52670	.83880 18151	.75725 89837	.34019 93247	.190
.26	.63	.815	.29730 67997	.72323 66115	.87397 13307	.63796 72653	.185
.28	.64	.820	-.06539 01775	.57777 79421	.95416 55884	0.92355 40111	.180
1.30	0.65	0.825	+0.16994 40730	-1.40390 79151	-1.99502 43626	-1.18962 37563	0.175
.32	.66	.830	.40553 54573	1.20366 55087	.99437 39288	.42890 80773	.170
.34	.67	.835	.63803 71624	0.97965 82773	0.97767 92540	.95142 32618	.165
.36	.68	.840	0.86394 44386	.73506 65951	.86363 50080	.79947 70157	.160
.38	.69	.845	1.07962 80556	.47363 93712	.73325 03878	.91824 61640	.155
1.40	0.70	0.850	+1.28137 31840	-0.19968 01024	-1.56092 53274	-1.98561 53559	0.150
.42	.71	.855	.46542 40696	+ .08197 79465	1.34901 53856	.99757 97940	.145
.44	.72	.860	.62803 71624	.36600 87184	1.10098 23218	.95142 32618	.140
.46	.73	.865	.76552 70898	.64664 38586	0.82142 70562	.84592 73607	.135
.48	.74	.870	.87435 38852	0.91773 81710	.051610 13921	.68156 82313	.130
1.50	0.75	0.875	+1.95117 18750	+1.17285 15625	-0.19189 45312	-1.46069 33594	0.125
.52	.76	.880	.99292 06690	.40534 90889	+0.14320 99462	1.18766 99707	.120
.54	.77	.885	.99691 06886	.60852 08024	.048021 13471	0.86899 53279	.115
.56	.78	.890	.96091 96942	.77572 31939	0.80920 84883	0.51335 79521	.110
.58	.79	.895	.88329 84961	.90054 41171	1.11956 12088	-0.13163 74071	.105
1.60	0.80	0.900	+1.76308 63360	+1.97699 31776	+1.40010 27482	+0.26317 12195	0.100
.62	.81	.905	.60013 64373	.99971 96721	.63940 94315	0.65612 36069	.095
.64	.82	.910	.39525 22406	.96426 02621	.82613 45893	1.03060 04643	.090
.66	.83	.915	1.15033 48517	.86731 86760	.94941 41504	.36870 88137	.085
.68	.84	.920	0.86854 22473	.70707 98391	.99935 18824	.65183 13233	.080
1.70	0.85	0.925	+0.55446 07970	+1.48356 09449	+1.96759 28093	+1.86134 68310	0.075
.72	.86	.930	+0.21428 96760	1.19900 20931	1.84799 39241	1.97954 74564	.070
.74	.87	.935	-0.14396 12420	0.85829 92403	1.63740 19201	1.99078 01007	.065
.76	.88	.940	0.51025 94066	0.46948 23276	1.33654 83032	1.88284 26860	.060
.78	.89	.945	0.87232 29047	+0.04424 15758	0.95107 29095	1.64866 82032	.055
1.80	0.90	0.950	-1.21537 84320	-0.40149 49376	+0.49268 75443	+1.28833 25174	0.050
.82	.91	.955	1.52190 32668	0.84692 91700	-0.01950 56945	0.81142 49328	.045
.84	.92	.960	1.77135 05769	1.26570 30966	0.55754 31208	+0.23982 37543	.040
.86	.93	.965	1.93985 73746	1.62514 72651	1.08291 65386	-0.38907 74966	.035
.88	.94	.970	1.99993 44153	1.88546 42666	1.54473 84058	1.01864 39364	.030
1.90	0.95	0.975	-1.92013 73210	-1.99884 33199	-1.87766 49868	-1.56872 01550	0.025
.92	.96	.980	1.66471 81921	1.90850 20243	1.99960 56945	1.93074 09093	.020
.94	.97	.985	1.19325 69519	1.54765 11791	1.80918 63356	1.96217 03119	.015
.96	.98	.990	-0.46027 16544	-0.83837 84212	-1.18295 00511	1.48020 36790	.010
.98	0.99	0.995	+0.58519 30364	+0.30955 37318	+0.02772 33526	-0.25466 14937	.005
2.00	1.00	1.000	+2.00000 00000	+2.00000 00000	+2.00000 00000	+2.00000 00000	0.000
$-x$	$-\mu$	$1 - X$	$-C_9$	$+C_{10}$	$-C_{11}$	$+C_{12}$	$X$

$$\frac{1}{2}x = \mu = 2X - 1 = \cos \theta \quad C_n = 2 \cos n\theta = 2T_n \quad X = \cos^2 \frac{1}{2}\theta \quad 1 - X = \sin^2 \frac{1}{2}\theta$$

## AUXILIARY TABLES

$x$	$\mu$	$\sqrt{4-x^2}$	$\sqrt{2+x}$	$\sqrt{2-x}$	$\cos^{-1} \frac{1}{2}x$	
$2 \cos \theta$	$\cos \theta$	$2 \sin \theta$	$2 \cos \frac{1}{2}\theta$	$2 \sin \frac{1}{2}\theta$	$\theta$	
					radians	°
0.00	0.00	2.00000 00000	1.41421 35624	1.41421 35624	1.57079 63268	90.00000 000
.02	.01	1.99989 99975	.42126 70404	.40712 47279	.56079 61601	89.42703 266
.04	.02	.99959 99600	.42828 56857	.40000 00000	.55079 49932	88.85400 800
.06	.03	.99909 97974	.43527 00094	.39283 88277	.54079 18250	88.28086 868
.08	.04	.99839 93595	.44222 05102	.38564 06461	.53078 56524	87.70755 722
0.10	0.05	1.99749 84355	1.44913 76746	1.37840 48752	1.52077 54700	87.13401 602
.12	.06	.99639 67542	.45602 19779	.37113 09201	.51076 02683	86.56018 723
.14	.07	.99509 39827	.46287 38838	.36381 81697	.50073 90337	85.98601 278
.16	.08	.99358 97271	.46969 38457	.35646 59966	.49071 07468	85.41143 426
.18	.09	.99188 35307	.47648 23060	.34907 37563	.48067 43818	84.83639 291
0.20	0.10	1.98997 48742	1.48323 96974	1.34164 07865	1.47062 89056	84.26082 952
.22	.11	.98786 31744	.48996 64426	.33416 64064	.46057 32768	83.68468 443
.24	.12	.98554 77834	.49666 29547	.32664 99161	.45050 64444	83.10789 742
.26	.13	.98302 79877	.50332 96378	.31909 05958	.44042 73471	82.53040 768
.28	.14	.98030 30071	.50996 68871	.31148 77049	.43033 49121	81.95215 375
0.30	0.15	1.97737 19933	1.51657 50888	1.30384 04810	1.42022 80540	81.37307 344
.32	.16	.97423 40287	.52315 46212	.29614 81397	.41010 56738	80.79310 378
.34	.17	.97088 81247	.52970 58541	.28840 98727	.39996 66577	80.21218 094
.36	.18	.96733 32204	.53622 91496	.28062 48475	.38980 98755	79.63024 019
.38	.19	.96356 81806	.54272 48621	.27279 22061	.37963 41803	79.04721 580
0.40	0.20	1.95959 17942	1.54919 33385	1.26491 10641	1.36943 84060	78.46304 097
.42	.21	.95540 27718	.55563 49186	.25698 05090	.35922 13670	77.87764 776
.44	.22	.95099 97437	.56204 99352	.24899 95997	.34898 18563	77.29096 701
.46	.23	.94638 12576	.56843 87141	.24096 73646	.33871 86439	76.70292 825
.48	.24	.94154 57759	.57480 15748	.23288 28006	.32843 04758	76.11345 964
0.50	0.25	1.93649 16731	1.58113 88301	1.22474 48714	1.31811 60717	75.52248 781
.52	.26	.93121 72327	.58745 07866	.21655 25061	.30777 41239	74.92993 786
.54	.27	.92572 06443	.59373 77451	.20830 45974	.29740 32953	74.33573 315
.56	.28	.92000 00000	.60000 00000	.20000 00000	.28700 22176	73.73979 529
.58	.29	.91405 32908	.60623 78404	.19163 75288	.27656 94890	73.14204 398
0.60	0.30	1.90787 84028	1.61245 15497	1.18321 59566	1.26610 36728	72.54239 688
.62	.31	.90147 31131	.61864 14056	.17473 40124	.25560 32944	71.94076 951
.64	.32	.89483 50852	.62480 76809	.16619 03790	.24506 68395	71.33707 512
.66	.33	.88796 18640	.63095 06430	.15758 36903	.23449 27516	70.73122 451
.68	.34	.88085 08713	.63707 05544	.14891 25293	.22387 94293	70.12312 593
0.70	0.35	1.87349 93995	1.64316 76725	1.14017 54251	1.21322 52231	69.51268 489
.72	.36	.86590 46064	.64924 22502	.13137 08499	.20252 84334	68.89980 398
.74	.37	.85806 35081	.65529 45357	.12249 72160	.19178 73061	68.28438 272
.76	.38	.84997 29728	.66132 47726	.11355 28726	.18100 00303	67.66631 734
.78	.39	.84162 97131	.66733 32001	.10453 61017	.17016 47341	67.04550 060
0.80	0.40	1.83303 02780	1.67332 00531	1.09544 51150	1.15927 94807	66.42182 152
.82	.41	.82417 10446	.67928 55624	.08627 80491	.14834 22646	65.79516 520
.84	.42	.81504 82087	.68522 99546	.07703 29614	.13735 10067	65.16541 251
.86	.43	.80565 77749	.69115 34525	.06770 78252	.12630 35499	64.53243 986
.88	.44	.79599 55457	.69705 62748	.05830 05244	.11519 76534	63.89611 886
0.90	0.45	1.78605 71099	1.70293 86366	1.04880 88482	1.10403 09877	63.25631 605
.92	.46	.77583 78304	.70880 07491	.03923 04845	.09280 11283	62.61289 250
.94	.47	.76533 28298	.71464 28199	.02956 30141	.08150 55488	61.96570 347
.96	.48	.75453 69760	.72046 50534	.01980 39027	.07014 16144	61.31459 799
.98	.49	.74344 48658	.72626 76502	.00995 04938	.05870 65739	60.65941 842
1.00	0.50	1.73205 08076	1.73205 08076	1.00000 00000	1.04719 75512	60.00000 000

## AUXILIARY TABLES

$x$	$\mu$	$\sqrt{4-x^2}$	$\sqrt{2+x}$	$\sqrt{2-x}$	$\cos^{-1} \frac{1}{2}x$	
$2 \cos \theta$	$\cos \theta$	$2 \sin \theta$	$2 \cos \frac{1}{2}\theta$	$2 \sin \frac{1}{2}\theta$	$\theta$	
					radians	°
1.00	0.50	1.73205 08076	1.73205 08076	1.00000 00000	1.04719 75512	60.00000 000
.02	.51	.72034 88018	.73781 47197	0.98994 94937	.03561 15365	59.33617 026
.04	.52	.70833 25203	.74355 95774	.97979 58971	.02394 53761	58.66774 850
.06	.53	.69599 52830	.74928 55685	.96953 59715	.01219 57615	57.99454 517
.08	.54	.68333 00330	.75499 28775	.95916 63047	1.00035 92174	57.31636 115
1.10	0.55	1.67032 93088	1.76068 16862	0.94868 32981	0.98843 20889	56.63298 703
.12	.56	.65698 52142	.76635 21733	.93808 31520	.97641 05268	55.94420 226
.14	.57	.64328 93841	.77200 45147	.92736 18495	.96429 04716	55.24977 425
.16	.58	.62923 29484	.77763 88835	.91651 51390	.95206 76361	54.54945 736
.18	.59	.61480 64900	.78325 54500	.90553 85138	.93973 74860	53.84299 180
1.20	0.60	1.60000 00000	1.78885 43820	0.89442 71910	0.92729 52180	53.13010 235
.22	.61	.58480 28269	.79443 58445	.88317 60866	.91473 57359	52.41049 704
.24	.62	.56920 36197	.80000 00000	.87177 97887	.90205 36236	51.68386 553
.26	.63	.55319 02652	.80554 70085	.86023 25267	.88924 31152	50.94987 746
.28	.64	.53674 98170	.81107 70276	.84852 81374	.87629 80612	50.20818 050
1.30	0.65	1.51986 84154	1.81659 02125	0.83666 00265	0.86321 18901	49.45839 813
.32	.66	.50253 11977	.82208 67158	.82462 11251	.84997 75659	48.70012 721
.34	.67	.48472 21962	.82756 66882	.81240 38405	.83658 75393	47.93293 520
.36	.68	.46642 42224	.83303 02780	.80000 00000	.82303 36921	47.15635 696
.38	.69	.44761 87343	.83847 76311	.78740 07874	.80930 72740	46.36989 113
1.40	0.70	1.42828 56857	1.84390 88915	0.77459 66692	0.79539 88302	45.57299 600
.42	.71	.40840 33513	.84932 42009	.76157 73106	.78129 81174	44.76508 467
.44	.72	.38794 81258	.85472 36991	.74833 14774	.76699 40079	43.94551 956
.46	.73	.36689 42900	.86010 75238	.73484 69228	.75247 43762	43.11360 595
.48	.74	.34521 37377	.86547 58106	.72111 02551	.73772 59685	42.26858 443
1.50	0.75	1.32287 56555	1.87082 86934	0.70710 67812	0.72273 42478	41.40962 211
.52	.76	.29984 61447	.87616 63039	.69282 03230	.70748 32118	40.53580 211
.54	.77	.27608 77713	.88148 87722	.67823 29983	.69195 51751	39.64611 115
.56	.78	.25155 90278	.88679 62264	.66332 49581	.67613 05096	38.73942 460
.58	.79	.22621 36845	.89208 87928	.64807 40698	.65998 73294	37.81448 851
1.60	0.80	1.20000 00000	1.89736 65961	0.63245 55320	0.64350 11088	36.86989 765
.62	.81	.17285 97529	.90262 97590	.61644 14003	.62664 42116	35.90406 858
.64	.82	.14472 70417	.90787 84028	.60000 00000	.60938 53080	34.91520 625
.66	.83	.11552 67814	.91311 26470	.58309 51895	.59168 86424	33.90126 200
.68	.84	.08517 27973	.91833 26093	.56568 54249	.57351 31044	32.85988 038
1.70	0.85	1.05356 53753	1.92353 84062	0.54772 25575	0.55481 10330	31.78833 062
.72	.86	.102058 80658	.92873 01522	.52915 02622	.53552 66543	30.68341 711
.74	.87	.098610 34428	.93390 79606	.50990 19514	.51559 40062	29.54136 050
.76	.88	.94994 73670	.93907 19430	.48989 79486	.49493 41263	28.35763 658
.78	.89	.91192 10492	.94422 22095	.46904 15760	.47345 11573	27.12675 312
1.80	0.90	0.87177 97887	1.94935 88690	0.44721 35955	0.45102 68118	25.84193 276
.82	.91	.82921 64977	.95448 20286	.42426 40687	.42751 22649	24.49454 847
.84	.92	.78383 67177	.95959 17942	.40000 00000	.40271 58416	23.07391 807
.86	.93	.73511 90380	.96468 82704	.37416 57387	.37638 34823	21.56518 502
.88	.94	.68234 88844	.96977 15604	.34641 01615	.34816 60213	19.94844 359
1.90	0.95	0.62449 97998	1.97484 17658	0.31622 77660	0.31756 04293	18.19487 234
.92	.96	.56000 00000	.97989 89873	.28284 27125	.28379 41092	16.26020 471
.94	.97	.48620 98312	.98494 33241	.24494 89743	.24556 55175	14.06986 775
.96	.98	.39799 49748	.98997 48742	.20000 00000	.20033 48423	11.47834 095
.98	.99	.28213 47196	1.99499 37343	.14142 13562	.14153 94733	8.10961 446
2.00	1.00	0.00000 00000	2.00000 00000	0.00000 00000	0.00000 00000	0.00000 000



XXII.—Two Numerical Applications of Chebyshev Polynomials. By J. C. P. Miller,  
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### I. INTRODUCTION

IN this paper two computational processes are outlined in which the table of Chebyshev Polynomials  $C_n(x) = 2 \cos(n \cos^{-1} \frac{1}{2}x)$  given in the preceding paper may be used with effect; these processes are (a) interpolation and (b) Fourier synthesis. A brief outline is also given of the idea behind the process of "Economization of Power Series" developed in Lanczos, 1938; this is related to (a). Finally the application of (b) to the calculation of Mathieu functions is considered.

Ten decimal tables are given of the auxiliary functions needed for Fourier synthesis. These functions are  $\sqrt{4-x^2} = 2 \sin \theta$ ,  $\sqrt{2+x} = 2 \cos \frac{1}{2}\theta$ ,  $\sqrt{2-x} = 2 \sin \frac{1}{2}\theta$ ;  $\cos^{-1} \frac{1}{2}x = \theta$  is given also, to 10 decimals of a radian and to 8 decimals of a degree.

Throughout this paper the notation used is that of the preceding paper (in particular § 2).

### 2. ECONOMISATION OF POWER SERIES

The reader is referred to the valuable and very readable paper of Lanczos, 1938, for a full discussion of this process, which is closely related to the interpolation process described in § 3; only a brief indication of the underlying principle can be given here. Lanczos's process replaces a power series (possibly asymptotic in character) by a derived series in which Chebyshev polynomials are used in place of powers, and which in general requires fewer, often many fewer, terms to give results with a specified accuracy over a given range of the argument. Instead of using this series in conjunction with tables of Chebyshev Polynomials, Lanczos then curtails it at the appropriate term, say the  $m$ th, and reconverts to a power series, which includes powers up to the  $m$ th only; the coefficients are modified and depend, of course, on  $m$ . Nevertheless, within the specified accuracy and range of argument, the modified series is equivalent to the original power series carried perhaps to many more terms.

Thus, for example, in terms of  $T_n(\mu) = \cos(n \cos^{-1} \mu)$ , Lanczos (p. 156) finds:

$$\begin{aligned} 1/(1+\mu) &= 1 - \mu + \mu^2 - \mu^3 + \dots \\ &= \sqrt{2}\{\frac{1}{2} - pT_1(\mu) + p^2T_2(\mu) - p^3T_3(\mu) + \dots\} \end{aligned}$$

in which  $p = 3 - 2\sqrt{2} = 0.17157$ . Retaining terms to  $T_6(\mu)$  in the latter series,

$$1/(1+\mu) \div 0.9999925 - 0.9992202\mu + 0.9863809\mu^2 - 0.9070856\mu^3 + 0.6753632\mu^4 - 0.3293104\mu^5 + 0.0738853\mu^6$$

in the range  $0 \leq \mu \leq 1$ , with an error which nowhere exceeds  $6 \times 10^{-6}$  within this range. For comparison, the following table gives  $N$ , the number of terms of the original power series needed for similar accuracy in the range  $-M \leq \mu \leq +M$ .

M	0.1	0.5	0.8	0.9
N	6	17	53	114

### 3. INTERPOLATION BY CHEBYSHEV POLYNOMIALS

3.1. It is a well-known property of the Chebyshev Polynomials that, of all polynomials

$$p_n(x) = x^n + a_1x^{n-1} + \dots + a_n$$

with leading coefficient unity,  $C_n(x)$  is the one for which the greatest departure from zero in

the interval  $-2 \leq x \leq +2$  is as small as possible. This property can be used with effect for interpolation. The power series

$$f(T) = f(a+t) = A_0 + A_1 \cdot \frac{4t}{h} + A_2 \cdot \left(\frac{4t}{h}\right)^2 + \dots + A_p \cdot \left(\frac{4t}{h}\right)^p + \dots \quad (3.11)$$

may be replaced over the interval  $-2 \leq 4t/h \leq +2$  (that is, over  $-\frac{1}{2}h \leq t \leq +\frac{1}{2}h$ , or  $a - \frac{1}{2}h \leq T \leq a + \frac{1}{2}h$ ) by the series

$$f(T) = f(a+t) = a_0 + a_1 C_1(4t/h) + a_2 C_2(4t/h) + \dots + a_p C_p(4t/h) + \dots \quad (3.12)$$

We shall see below (3.23, 3.24) that in general (*i.e.* if  $h$  is not too great)  $A_p$  and  $a_p$  are quantities of similar magnitude. Thus, in the interval  $a - \frac{1}{2}h \leq T \leq a + \frac{1}{2}h$ , the series (3.12) is more rapidly convergent than (3.11) when  $1 < |4t/h| \leq 2$ . It is true that (3.11) is more rapidly convergent when  $|4t/h| < 1$ , but to retain the degree of convergence of (3.12) when  $|4t/h| \leq 2$ , the power series could be used only over the range  $|4t/h| \leq 1$ ; to cover a large range of  $T$  with a given maximum number of terms, it would thus be necessary to provide series of type (3.11) for twice as many values of  $x$ , *i.e.* at half the interval  $h$  in  $x$ , as are needed for series of type (3.12).

3.2. In terms of derivatives, the coefficients  $a_p$  may be determined as follows, using the calculus of operators. Writing  $t = \frac{1}{2}h \cos \theta$ , and assuming

$$f(a+t) = a_0 + \sum_{p=1}^{\infty} a_p C_p\left(\frac{4t}{h}\right) = a_0 + 2 \sum_{p=1}^{\infty} a_p \cos p\theta, \quad (3.21)$$

it follows (*cf.* Watson, 1922, p. 181) that

$$\begin{aligned} a_p &= \frac{1}{\pi} \int_0^\pi f(a + \tfrac{1}{2}h \cos \theta) \cos p\theta \, d\theta \\ &= \frac{1}{\pi} \int_0^\pi e^{\frac{1}{2}h \cos \theta D} \cos p\theta \, d\theta \cdot f(a) \\ &= I_p(\tfrac{1}{2}hD) \cdot f(a), \end{aligned} \quad (3.22)$$

in which  $D^n f(a)$  is written for  $\left\{ \left( \frac{d}{dt} \right)^n f(a+t) \right\}_{t=0}$ . Thus

$$a_p = \frac{(\frac{1}{2}h)^p}{p!} \left\{ f^{(p)}(a) + \frac{(\frac{1}{2}h)^2}{1!(p+1)} f^{(p+2)}(a) + \frac{(\frac{1}{2}h)^4}{2!(p+1)(p+2)} f^{(p+4)}(a) + \dots \right\}. \quad (3.23)$$

Since

$$A_p = \frac{(\frac{1}{2}h)^p}{p!} f^{(p)}(a), \quad (3.24)$$

it follows that, if  $h$  is not too large,  $a_p \doteq A_p$ , as stated above.

3.3. The coefficients  $a_p$  may also be expressed in terms of central differences of  $f(a+\theta h)^*$  for  $\theta = 0, \pm 1, \pm 2, \dots$ . This may be done in two ways: (*a*) by substitution for the  $f^{(p)}(a)$  of equivalent expressions in terms of central differences, see, for example, Comrie, 1936, p. 802, or, if more coefficients are needed, see Oppolzer, 1880, pp. 21, 23; (*b*) by writing  $\theta = \frac{1}{2}x$  in Stirling's interpolation formula

$$f_\theta = f(a + \theta h) = f_0 + \theta \mu \delta f_0 + \frac{\theta^2}{2!} \delta^2 f_0 + \frac{\theta(\theta^2 - 1)}{3!} \mu \delta^3 f_0 + \frac{\theta^2(\theta^2 - 1)}{4!} \delta^4 f_0 + \dots, \quad (3.31)$$

or by writing  $\theta - \frac{1}{2} = \psi = \frac{1}{2}x$  in Bessel's interpolation formula

$$f_\theta = f(a + \theta h) = \mu f_{\frac{1}{2}} + \psi \delta f_{\frac{1}{2}} + \frac{\left(\psi^2 - \frac{1}{2^2}\right)}{2!} \mu \delta^2 f_{\frac{1}{2}} + \frac{\psi\left(\psi^2 - \frac{1}{2^2}\right)}{3!} \delta^3 f_{\frac{1}{2}} + \frac{\left(\psi^2 - \frac{1}{2^2}\right)\left(\psi^2 - \frac{3^2}{2^2}\right)}{4!} \mu \delta^4 f_{\frac{1}{2}} + \dots \quad (3.32)$$

and then substituting for powers of  $x$  their expressions in terms of the polynomials  $C_n(x)$ ; these expressions are given in section 3.3 of the preceding paper.

\* Note the use of the customary  $\theta$  for a fraction or multiple of the tabular interval  $h$ ; no confusion with  $\theta = \cos^{-1} \frac{1}{2}x$  should arise.

The coefficients  $\alpha_p$  are given below in two forms, (i) suitable for the interval  $(a - \frac{1}{2}h, a + \frac{1}{2}h)$  of the argument  $a + \theta h$ , that is for  $-\frac{1}{2} \leq \theta \leq +\frac{1}{2}$ : coefficients derived from Stirling's formula (3.31) and checked by process (a); (ii) suitable for the tabular interval  $(a, a + h)$ , that is for  $0 \leq \theta \leq 1$  so that  $\theta = X$  of the preceding paper: coefficients derived from Bessel's formula (3.32) and checked by process (a), but with substitution for  $f^{(p)}(a + \frac{1}{2}h)$  instead of  $f^{(p)}(a)$ .

$$f(a + \theta h) = \alpha_0 + \alpha_1 C_1(4\theta) + \alpha_2 C_2(4\theta) + \dots + \alpha_p C_p(4\theta) + \dots \quad -\frac{1}{2} \leq \theta \leq +\frac{1}{2}$$

$$\begin{aligned} \frac{1}{2}\alpha_0 &= \frac{1}{2}f_0 + \frac{1}{2^4 \cdot 2!}\delta^2 - \frac{13}{2^8 \cdot 4!}\delta^4 + \frac{794}{2^{12} \cdot 6!}\delta^6 - \frac{1}{2^{16} \cdot 8!}12029\delta^8 + \frac{283}{2^{20} \cdot 10!}54782\delta^{10} - \frac{1}{2^{24} \cdot 12!}1259867730\delta^{12} + \dots \\ \alpha_1 &= \frac{1}{2^2 \cdot 1!}\mu\delta - \frac{13}{2^6 \cdot 3!}\mu\delta^3 + \frac{794}{2^{10} \cdot 5!}\mu\delta^5 - \frac{1}{2^{14} \cdot 7!}12029\mu\delta^7 + \frac{283}{2^{18} \cdot 9!}54782\mu\delta^9 - \frac{1}{2^{22} \cdot 11!}1259867730\mu\delta^{11} + \dots \\ \alpha_2 &= \frac{1}{2^4 \cdot 2!}\delta^2 - \frac{12}{2^8 \cdot 4!}\delta^4 + \frac{719}{2^{12} \cdot 6!}\delta^6 - \frac{1}{2^{16} \cdot 8!}00584\delta^8 + \frac{253}{2^{20} \cdot 10!}35506\delta^{10} - \frac{1}{2^{24} \cdot 12!}0029774536\delta^{12} + \dots \\ \alpha_3 &= \frac{1}{2^6 \cdot 3!}\mu\delta^3 - \frac{75}{2^{10} \cdot 5!}\mu\delta^5 + \frac{11445}{2^{14} \cdot 7!}\mu\delta^7 - \frac{30}{2^{18} \cdot 9!}19276\mu\delta^9 + \frac{12300}{2^{22} \cdot 11!}93194\mu\delta^{11} - \dots \\ \alpha_4 &= \frac{1}{2^8 \cdot 4!}\delta^4 - \frac{74}{2^{12} \cdot 6!}\delta^6 + \frac{11228}{2^{16} \cdot 8!}\delta^8 - \frac{29}{2^{20} \cdot 10!}52712\delta^{10} + \frac{12005}{2^{24} \cdot 12!}80975\delta^{12} - \dots \\ \alpha_5 &= \frac{1}{2^{10} \cdot 5!}\mu\delta^5 - \frac{217}{2^{14} \cdot 7!}\mu\delta^7 + \frac{66564}{2^{18} \cdot 9!}\mu\delta^9 - \frac{295}{2^{22} \cdot 11!}12219\mu\delta^{11} + \dots \\ \alpha_6 &= \frac{1}{2^{12} \cdot 6!}\delta^6 - \frac{216}{2^{16} \cdot 8!}\delta^8 + \frac{66093}{2^{20} \cdot 10!}\delta^{10} - \frac{292}{2^{24} \cdot 12!}58196\delta^{12} + \dots \\ \alpha_7 &= \frac{1}{2^{14} \cdot 7!}\mu\delta^7 - \frac{471}{2^{18} \cdot 9!}\mu\delta^9 + \frac{2}{2^{22} \cdot 11!}54023\mu\delta^{11} - \dots \quad \alpha_8 = \frac{1}{2^{18} \cdot 9!}\mu\delta^9 - \frac{869}{2^{22} \cdot 11!}\mu\delta^{11} + \dots \\ \alpha_8 &= \frac{1}{2^{16} \cdot 8!}\delta^8 - \frac{470}{2^{20} \cdot 10!}\delta^{10} + \frac{2}{2^{24} \cdot 12!}53154\delta^{12} - \dots \quad \alpha_{10} = \frac{1}{2^{20} \cdot 10!}\delta^{10} - \frac{868}{2^{24} \cdot 12!}\delta^{12} + \dots \end{aligned}$$

$$f(a + \theta h) = \alpha_0 + \alpha_1 C_1(4\theta - 2) + \alpha_2 C_2(4\theta - 2) + \dots + \alpha_p C_p(4\theta - 2) + \dots \quad 0 \leq \theta \leq 1$$

$$\begin{aligned} \frac{1}{2}\alpha_0 &= \frac{1}{2}\mu f_{\frac{1}{2}} - \frac{1}{2^4 \cdot 2!}\mu\delta^2 + \frac{35}{2^8 \cdot 4!}\mu\delta^4 - \frac{3466}{2^{12} \cdot 6!}\mu\delta^6 + \frac{6}{2^{16} \cdot 8!}76003\mu\delta^8 - \frac{2183}{2^{20} \cdot 10!}78238\mu\delta^{10} + \frac{10}{2^{24} \cdot 12!}5486789902\mu\delta^{12} - \dots \\ \alpha_1 &= \frac{1}{2^2 \cdot 1!}\delta - \frac{1}{2^6 \cdot 3!}\delta^3 + \frac{34}{2^{10} \cdot 5!}\delta^5 - \frac{3333}{2^{14} \cdot 7!}\delta^7 + \frac{6}{2^{18} \cdot 9!}46734\delta^9 - \frac{2082}{2^{22} \cdot 11!}77290\delta^{11} + \dots \\ \alpha_2 &= \frac{1}{2^4 \cdot 2!}\mu\delta^2 - \frac{36}{2^8 \cdot 4!}\mu\delta^4 + \frac{3599}{2^{12} \cdot 6!}\mu\delta^6 - \frac{7}{2^{16} \cdot 8!}05272\mu\delta^8 + \frac{2284}{2^{20} \cdot 10!}79186\mu\delta^{10} - \frac{11}{2^{24} \cdot 12!}0573959640\mu\delta^{12} + \dots \\ \alpha_3 &= \frac{1}{2^6 \cdot 3!}\delta^3 - \frac{35}{2^{10} \cdot 5!}\delta^5 + \frac{3465}{2^{14} \cdot 7!}\delta^7 - \frac{6}{2^{18} \cdot 9!}75676\delta^9 + \frac{2182}{2^{22} \cdot 11!}43674\delta^{11} - \dots \\ \alpha_4 &= \frac{1}{2^8 \cdot 4!}\mu\delta^4 - \frac{134}{2^{12} \cdot 6!}\mu\delta^6 + \frac{29596}{2^{16} \cdot 8!}\mu\delta^8 - \frac{102}{2^{20} \cdot 10!}35512\mu\delta^{10} + \frac{51621}{2^{24} \cdot 12!}31183\mu\delta^{12} - \dots \\ \alpha_5 &= \frac{1}{2^{10} \cdot 5!}\delta^5 - \frac{133}{2^{14} \cdot 7!}\delta^7 + \frac{29268}{2^{18} \cdot 9!}\delta^9 - \frac{101}{2^{22} \cdot 11!}00299\delta^{11} + \dots \\ \alpha_6 &= \frac{1}{2^{12} \cdot 6!}\mu\delta^6 - \frac{328}{2^{16} \cdot 8!}\mu\delta^8 + \frac{1}{2^{20} \cdot 10!}35213\mu\delta^{10} - \frac{754}{2^{24} \cdot 12!}08828\mu\delta^{12} + \dots \\ \alpha_7 &= \frac{1}{2^{14} \cdot 7!}\delta^7 - \frac{327}{2^{18} \cdot 9!}\delta^9 + \frac{1}{2^{22} \cdot 11!}34563\delta^{11} - \dots \quad \alpha_8 = \frac{1}{2^{18} \cdot 9!}\delta^9 - \frac{649}{2^{22} \cdot 11!}\delta^{11} + \dots \\ \alpha_8 &= \frac{1}{2^{16} \cdot 8!}\mu\delta^8 - \frac{650}{2^{20} \cdot 10!}\mu\delta^{10} + \frac{4}{2^{24} \cdot 12!}48514\mu\delta^{12} - \dots \quad \alpha_{10} = \frac{1}{2^{20} \cdot 10!}\mu\delta^{10} - \frac{1132}{2^{24} \cdot 12!}\mu\delta^{12} + \dots \end{aligned}$$

## DENOMINATORS

$n$	$4^n \cdot n!$	$n$	$4^n \cdot n!$
1	$4 = 2^2$	7	$825 \ 75360 = 2^{18} \cdot 3^2 \cdot 5 \cdot 7$
2	$32 = 2^5$	8	$26424 \ 11520 = 2^{23} \cdot 3^2 \cdot 5 \cdot 7$
3	$384 = 2^7 \cdot 3$	9	$9 \ 51268 \ 14720 = 2^{28} \cdot 3^4 \cdot 5 \cdot 7$
4	$6144 = 2^{11} \cdot 3$	10	$380 \ 50725 \ 88800 = 2^{28} \cdot 3^4 \cdot 5^2 \cdot 7$
5	$1 \ 22880 = 2^{13} \cdot 3 \cdot 5$	11	$16742 \ 31939 \ 07200 = 2^{30} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$
6	$29 \ 49120 = 2^{16} \cdot 3^2 \cdot 5$	12	$8 \ 03631 \ 33075 \ 45600 = 2^{34} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$

A study of the coefficients makes it clear that, even if high differences are needed for interpolation, a smaller number of  $\alpha_n$  will usually suffice. If much interpolation within each interval of tabulation is likely, it may, therefore, be better first to evaluate the first few  $\alpha_n$ —as far as needed—and to use these in conjunction with a table of Chebyshev Polynomials.

3.4. *Numerical Illustration*.—As an example take

$$f(a+t) = \sqrt{2} \cos\left(\frac{1}{4}\pi + t\right), \quad \text{with } h = \frac{1}{2}\pi.$$

The formula (3.22) then gives the following convenient expansion of  $f(T)$  for

$$0 \leq T = \frac{1}{4}\pi + t \leq \frac{1}{2}\pi,$$

that is for  $-\frac{1}{2} \leq \theta = 2t/\pi \leq +\frac{1}{2}$ ,

$$\sqrt{2} \cos\left(\frac{1}{4}\pi + \frac{1}{2}\pi\theta\right) = J_0\left(\frac{1}{4}\pi\right) - J_1\left(\frac{1}{4}\pi\right)C_1(4\theta) - J_2\left(\frac{1}{4}\pi\right)C_2(4\theta) + J_3\left(\frac{1}{4}\pi\right)C_3(4\theta) + \dots$$

The power series in  $\theta$  is exhibited alongside for comparison.

$\sqrt{2} \cos\left(\frac{1}{4}\pi + \frac{1}{2}\pi\theta\right) = 0.85163 \ 19137 \ 0$	$= 1$
$- 36318 \ 78383 \ 5 \ C_1$	$- 0.78539 \ 81634 \ 0 \ (2\theta)$
$- 7321 \ 83222 \ 0 \ C_2$	$- 30842 \ 51375 \ 3 \ (2\theta)^2$
$+ 971 \ 00145 \ 3 \ C_3$	$+ 8074 \ 55121 \ 9 \ (2\theta)^3$
$+ 96 \ 07246 \ 6 \ C_4$	$+ 1585 \ 43442 \ 4 \ (2\theta)^4$
$- 7 \ 58464 \ 6 \ C_5$	$- 249 \ 03945 \ 7 \ (2\theta)^5$
$- 49824 \ 8 \ C_6$	$- 32 \ 59918 \ 9 \ (2\theta)^6$
$+ 2802 \ 9 \ C_7$	$+ 3 \ 65762 \ 0 \ (2\theta)^7$
$+ 137 \ 9 \ C_8$	$+ 35908 \ 6 \ (2\theta)^8$
$- 6 \ 0 \ C_9$	$- 3133 \ 6 \ (2\theta)^9$
$- 2 \ C_{10}$	$- 246 \ 1 \ (2\theta)^{10}$
	$+ 17 \ 6 \ (2\theta)^{11}$
	$+ 1 \ 1 \ (2\theta)^{12}$
	$- 1 \ (2\theta)^{13}$

When  $|\theta| \leq \frac{1}{2}$ ,  $|C_n(4\theta)| \leq 2$  while  $|2\theta|^n \leq 1$ ; the power series has more terms for  $\frac{1}{2} < |2\theta| \leq 1$ , that is over half the total range.

The interval chosen,  $h = \frac{1}{2}\pi$ , would be too large for the convenient tabulation of  $f(T)$ , but the example illustrates the principle that a table of the coefficients  $\alpha_p$  at a given interval,  $h$ , of the argument would be smaller than a table of function and derivatives giving equal accuracy. As shown by the example in section 2, the saving may be even greater when the power series has radius of convergence finite, or zero (as with asymptotic series).

#### 4. FOURIER OR HARMONIC SYNTHESIS

4.1. A table of Chebyshev Polynomials may also be useful in conjunction with Fourier expansions, such as

$$f(x) = f(2 \cos \theta) = \alpha_0 + 2\alpha_1 \cos \theta + 2\alpha_2 \cos 2\theta + \dots \quad (4.11)$$

$$+ 2\beta_1 \sin \theta + 2\beta_2 \sin 2\theta + \dots$$

If values of  $f(x)$  are needed for which  $x = 2 \cos \theta$  is a tabular argument—that is, if  $100x$  is an integer as with the tables given in the preceding paper—the expansion may be written

$$f(x) = \alpha_0 + \alpha_1 C_1(x) + \alpha_2 C_2(x) + \alpha_3 C_3(x) + \dots$$

$$+ \beta_1 S_1(x) + \beta_2 S_2(x) + \beta_3 S_3(x) + \dots \quad (4.12)$$

$$= \alpha_0 + \alpha_1 C_1(x) + \alpha_2 C_2(x) + \alpha_3 C_3(x) + \dots$$

$$+ \sqrt{4-x^2} \{ \beta_1 + \beta_2 S_1(x) + \beta_3 S_2(x) + \dots \}. \quad (4.13)$$

Tables of  $\sigma_p(x)$  are not available, nor are tables of  $S_p(x)$ ; it is hoped later to supplement the table of  $C_p(x)$  with a similar table giving  $S_p(x)$  for  $p = 2(1)11$  or  $12$ .

The need for tables of  $S_p(x)$  may be circumvented, however, by means of the formulæ (3.72) of the preceding paper.

$$S_p = C_p + C_{p-2} + C_{p-4} + \dots \quad (4.14)$$

with  $C_1$  or  $1 = \frac{1}{2}C_0$  as the final term. This gives

$$\begin{aligned} f(x) &= \alpha_0 + \alpha_1 C_1(x) + \alpha_2 C_2(x) + \alpha_3 C_3(x) + \dots \\ &\quad + \sqrt{4-x^2} \{ \beta_1 + \beta_2 C_1(x) + \beta_3 C_2(x) + 1 + \beta_4 C_3(x) + C_1(x) + \dots \} \\ &= \alpha_0 + \alpha_1 C_1(x) + \alpha_2 C_2(x) + \alpha_3 C_3(x) + \dots \\ &\quad + \sqrt{4-x^2} \{ \gamma_1 + \gamma_2 C_1(x) + \gamma_3 C_2(x) + \gamma_4 C_3(x) + \dots \} \end{aligned} \quad (4.15)$$

in which

$$\left. \begin{aligned} \gamma_1 &= \beta_1 + \beta_3 + \beta_5 + \dots & \gamma_2 &= \beta_2 + \beta_4 + \beta_6 + \dots \\ \gamma_3 &= \gamma_1 - \beta_1 = \beta_3 + \beta_5 + \dots & \gamma_4 &= \gamma_2 - \beta_2 = \beta_4 + \beta_6 + \dots \\ \gamma_5 &= \gamma_3 - \beta_3 = \beta_5 + \beta_7 + \dots & \gamma_6 &= \gamma_4 - \beta_4 = \beta_6 + \beta_8 + \dots \end{aligned} \right\} \quad (4.16)$$

A table of  $\sqrt{4-x^2}$  is given to help in the application of (4.15), see p. 202.

It may be noted that for work with a few figures only, up to perhaps 5 or 6, it is better to use  $\beta_1 \sigma_1(x) + \beta_2 \sigma_2(x) + \dots$  for computation, but for many-figure work, in which  $x$  is an exact 2-decimal value, it is of some advantage to multiply  $\beta_1, \beta_2, \beta_3, \beta_4, \dots$  by the 0-, 2-, 4-, 6-, ... decimal values  $1, S_2, S_3, S_4, \dots$  instead of by the many-figure  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots$ ; the single final multiplication by  $\sqrt{4-x^2}$  then applies to the sum of the whole series, and not to  $\beta_1$  alone.

4.2. If  $100 \sin \theta$  is an integer, it is best to substitute  $\theta = \frac{1}{2}\pi - \phi$  immediately, so that the expansion, say,

$$\left. \begin{aligned} f(2 \sin \theta) &= \alpha_0 + 2\alpha_1 \cos \theta + 2\alpha_2 \cos 2\theta + 2\alpha_3 \cos 3\theta + \dots \\ &\quad + 2b_1 \sin \theta + 2b_2 \sin 2\theta + 2b_3 \sin 3\theta + \dots \end{aligned} \right\} \quad (4.21)$$

becomes

$$\left. \begin{aligned} f(2 \cos \phi) &= \alpha_0 + 2b_1 \cos \phi - 2\alpha_2 \cos 2\phi - 2b_3 \cos 3\phi + \dots \\ &\quad + 2\alpha_1 \sin \phi + 2b_2 \sin 2\phi - 2\alpha_3 \sin 3\phi - \dots \end{aligned} \right\} \quad (4.22)$$

in which  $100 \cos \phi$  is integral, and can be dealt with as in section 4.1, compare formulæ (12) on p. 81 of Van der Pol and Weljers, 1933.

4.3. Processes similar to that described in section 4.1 may also be applied to the series

$$g(x) = g(2 \cos \theta) = \kappa_1 \cos \frac{1}{2}\theta + \kappa_3 \cos \frac{3}{2}\theta + \kappa_5 \cos \frac{5}{2}\theta + \dots \quad (4.31)$$

$$h(x) = h(2 \cos \theta) = \mu_1 \sin \frac{1}{2}\theta + \mu_3 \sin \frac{3}{2}\theta + \mu_5 \sin \frac{5}{2}\theta + \dots \quad (4.32)$$

in which terms  $\kappa_{2p} \cos p\theta$  or  $\mu_{2p} \sin p\theta$ , with  $p$  integral, have been omitted from consideration; if they do occur, the process of section 4.1 is immediately applicable. Series of this type arise as part of normal Fourier series in which the variable is  $\psi = \frac{1}{2}\theta$ , but for which  $x = 2 \cos 2\psi$  is taken as variable rather than  $2 \cos \psi$ .

Using (2.22) of the preceding paper, (4.31) becomes

$$\left. \begin{aligned} g(x) &= \cos \frac{1}{2}\theta \sum_{p=0}^{\infty} \kappa_{2p+1} \{ S_p(x) - S_{p-1}(x) \} \\ &= \frac{1}{2} \sqrt{2+x} \{ \kappa_1 + \kappa_3 (C_1 - 1) + \kappa_5 (C_2 - C_1 + 1) + \kappa_7 (C_3 - C_2 + C_1 - 1) + \dots \} \\ &= \frac{1}{2} \sqrt{2+x} \{ \lambda_1 + \lambda_3 C_1 + \lambda_5 C_2 + \lambda_7 C_3 + \dots \} \end{aligned} \right\} \quad (4.33)$$

in which  $x = 2 \cos \theta$  and

$$\left. \begin{aligned} \lambda_1 &= \kappa_1 - \kappa_3 + \kappa_5 - \kappa_7 + \kappa_9 - \dots \\ \lambda_3 &= \kappa_1 - \lambda_1 = \kappa_3 - \kappa_5 + \kappa_7 - \dots \\ \lambda_5 &= \kappa_3 - \lambda_3 = \kappa_5 - \kappa_7 + \kappa_9 - \dots \end{aligned} \right\} \quad (4.34)$$

Likewise (4.32) gives

$$\left. \begin{aligned} h(x) &= \sin \frac{1}{2} \theta \sum_{p=0}^{\infty} \mu_{2p+1} \{S_p(x) + S_{p-1}(x)\} \\ &= \frac{1}{2} \sqrt{2-x} \{ \mu_1 + \mu_3(C_1 + 1) + \mu_5(C_2 + C_1 + 1) + \mu_7(C_3 + C_2 + C_1 + 1) + \dots \} \\ &= \frac{1}{2} \sqrt{2-x} (\nu_1 + \nu_3 C_1 + \nu_5 C_2 + \nu_7 C_3 + \dots) \end{aligned} \right\} \quad (4.35)$$

in which, again,  $x = 2 \cos \theta$  and

$$\left. \begin{aligned} \nu_1 &= \mu_1 + \mu_3 + \mu_5 + \mu_7 + \mu_9 + \dots \\ \nu_3 &= \nu_1 - \mu_1 = \mu_3 + \mu_5 + \mu_7 + \dots \\ \nu_5 &= \nu_3 - \mu_3 = \mu_5 + \mu_7 + \mu_9 + \dots \end{aligned} \right\} \quad (4.36)$$

Tables of  $\sqrt{2+x}$  and  $\sqrt{2-x}$  are given to help in the application of (4.33) and (4.35), see p. 202.

## 5. APPLICATION TO MATHIEU FUNCTIONS

5.1. As an application of these processes of Fourier Synthesis, consider the calculation of Mathieu Functions. These satisfy the differential equation

$$\frac{d^2 y}{dt^2} + (a - 2q \cos 2t)y = 0. \quad (5.11)$$

Periodic solutions exist for particular characteristic values of  $a$ , depending on  $q$ ; these solutions may be expressed in the form (see Ince, 1932, p. 356).

$$\left. \begin{aligned} ce_{2n}(t, q) &= \sum A_{2p} \cos 2pt \\ se_{2n+1}(t, q) &= \sum B_{2p+1} \sin (2p+1)t \\ ce_{2n+1}(t, q) &= \sum A_{2p+1} \cos (2p+1)t \\ se_{2n+2}(t, q) &= \sum B_{2p+2} \sin (2p+2)t \end{aligned} \right\} \quad (5.12)$$

all summations being for  $p=0, 1, 2, \dots \rightarrow \infty$ , the A's and B's being constants depending on  $n$  and  $q$ .

If in (5.11) we substitute  $\mu = \cos 2t$ , the equation becomes

$$(1 - \mu^2) \frac{d^2 y}{d\mu^2} - \mu \frac{dy}{d\mu} + \frac{1}{4}(a - 2q\mu)y = 0. \quad (5.13)$$

This equation, and the rather similar one in which  $X = \frac{1}{2}(\mu + 1) = \cos^2 t$  is taken as independent variable, are not uncommon as variants of Mathieu's equation (see, for example, Stratton, etc., 1941, p. 260≡2). In terms of the variable  $x = 2\mu = 2 \cos 2t$ , (5.13) may be re-written as

$$(4 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + \frac{1}{4}(a - qx)y = 0. \quad (5.14)$$

The expansions (5.12) may then be written in a form suitable for giving solutions of (5.14), or of (5.13),

$$ce_{2n}(t, q) = \sum A_{2p} \cos p\theta = \frac{1}{2} \sum A_{2p} C_p(x) \quad (5.151)$$

$$\begin{aligned} se_{2n+1}(t, q) &= \sum B_{2p+1} \sin (p + \frac{1}{2})\theta = \sum B_{2p+1} \sin \frac{1}{2} \theta \{S_p(x) + S_{p-1}(x)\} \\ &= \frac{1}{2} \sqrt{2-x} \sum (B_{2p+1} + B_{2p+3} + B_{2p+5} + \dots) C'_p(x) \end{aligned} \quad (5.152)$$

$$\begin{aligned} ce_{2n+1}(t, q) &= \sum A_{2p+1} \cos (p + \frac{1}{2})\theta = \sum A_{2p+1} \cos \frac{1}{2} \theta \{S_p(x) - S_{p-1}(x)\} \\ &= \frac{1}{2} \sqrt{2+x} \sum (A_{2p+1} - A_{2p+3} + A_{2p+5} - \dots) C'_p(x) \end{aligned} \quad (5.153)$$

$$\begin{aligned} se_{2n+2}(t, q) &= \sum B_{2p+2} \sin (p+1)\theta = \sum B_{2p+2} \sin \theta S_p(x) \\ &= \frac{1}{2} \sqrt{4-x^2} \sum (B_{2p+2} + B_{2p+6} + B_{2p+10} + \dots) C'_p(x) \end{aligned} \quad (5.154)$$

where, in each case,  $C'_p(x) = C_p(x)$ ,  $p \geq 1$ ;  $C'_0(x) = \frac{1}{2}C_0(x) = 1$ .

5.2. *Numerical Illustrations.*—Application of (5.152) and (5.153) to the evaluation of  $se_3(\frac{1}{8}\pi, 10)$  and  $ce_3(\frac{1}{8}\pi, 10)$ . In examining these numerical examples it should be noted that the coefficients  $B'_{2p+1}$  and  $A'_{2p+1}$  apply to all values of the argument and not only to the particular values,  $\frac{1}{8}\pi$  or  $\frac{3}{8}\pi$ , chosen here; again, the individual items in the columns of products need not be recorded if a calculating machine is available.

$se_3(\frac{1}{8}\pi, 10) = \frac{1}{2}\sqrt{2-x} \Sigma B'_{2p+1} C'_p(x)$ , where  $B'_{2p+1} = B_{2p+1} + B_{2p+3} + B_{2p+5} + \dots$

$$\theta = 2t = \frac{1}{8}\pi \quad x = 2 \cos 2t = 1 \quad 2 \sin t = \sqrt{2-x} = 1.$$

$p$	$B_{2p+1}^*$	$B'_{2p+1}$	$C'_p(1)$	Products
0	+0.43239 50	+0.77165 62	$\frac{1}{2}C_0 = +1$	+0.77165 62
1	+0.73446 92	+0.33926 12	$C_1 = +1$	+0.33926 12
2	-0.50686 51	-0.39520 80	$C_2 = -1$	+0.39520 80
3	+0.12790 76	+0.11165 71	$C_3 = -2$	-0.22331 42
4	- 1773 44	- 1625 05	$C_4 = -1$	+ 1625 05
5	+ 157 79	+ 148 39	$C_5 = +1$	+ 148 39
6	- 9 83	- 9 40	$C_6 = +2$	- 18 80
7	+ 45	+ 43	$C_7 = +1$	+ 43
8	- 2	- 2	$C_8 = -1$	+ 2
				+1.30036 21

Thus  $se_3(\frac{1}{8}\pi, 10) = +0.65018 10^+$ . Ince, 1932, p. 398, gives +0.65018.

$ce_3(\frac{1}{8}\pi, 10) = \frac{1}{2}\sqrt{2+x} \Sigma A'_{2p+1} C'_p(x)$ , where  $A'_{2p+1} = A_{2p+1} - A_{2p+3} + A_{2p+5} - + \dots$

$$\theta = 2t = \frac{3}{8}\pi \quad x = 2 \cos 2t = -1 \quad 2 \sin t = \sqrt{2+x} = 1.$$

$p$	$A_{2p+1}^*$	$A'_{2p+1}$	$C'_p(-1)$	Products
0	+0.75526 89	-0.31466 20	$\frac{1}{2}C_0 = +1$	-0.31466 20
1	+0.34008 13	+1.06993 09	$C_1 = -1$	-1.06993 09
2	-0.53412 14	-0.72984 96	$C_2 = -1$	+0.72984 96
3	+0.16718 53	+0.19572 82	$C_3 = +2$	+0.39145 64
4	- 2590 28	- 2854 29	$C_4 = -1$	+ 2854 29
5	+ 247 06	+ 264 01	$C_5 = -1$	- 264 01
6	- 16 15	- 16 95	$C_6 = +2$	- 33 90
7	+ 77	+ 80	$C_7 = -1$	- 80
8	- 3	- 3	$C_8 = -1$	+ 3
				-0.23773 08

Thus  $ce_3(\frac{1}{8}\pi, 10) = -0.11886 54$ . Ince, 1932, p. 403, gives -0.11887.

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\* The values in these columns are from Ince, 1932, pp. 369 and 370.

XXIII.—The Number of the Elements. By N. Feather, Ph.D., Cavendish Laboratory, Cambridge. Communicated by Sir EDMUND WHITTAKER, F.R.S. (With Three Text-figures.)

(*Ritchie Lecture delivered March 1, 1945*)

(MS. received July 3, 1945. Read November 5, 1945)

IT is natural—perhaps almost inevitable—that I should begin with a quotation from *The Sceptical Chymist*, for it is to Robert Boyle, more than to any one man, that the change from ancient to modern ideas in relation to my main topic is chiefly due. *The Sceptical Chymist* was published in 1661, and by that time the systems of the Greeks and the theories of the alchemists had alike long outlived any usefulness which, scientifically, either had ever possessed. Fire, earth, air and water—or mercury, sulphur, salt and earth (it matters little which list we choose) were inadequate as prototypes in a world in which the scientific spirit was already newly alive. So at least it seemed to Robert Boyle, for in the spacious language of his day he wrote:

“Notwithstanding the subtle reasonings I have met with in the books of the Peripatetiks, and the pretty experiments which have been shew’d me in the Laboratories of Chymists, I am of so diffident and dull a Nature as to think that if neither of them can bring more cogent arguments to evince the truth of their assertion than are wont to be brought; a Man may rationally enough retain some doubts concerning the very number of those materiall Ingredients of mixt bodies, which some would have us call Elements and others Principles.”

Yet it was more than a hundred years later before a list of the chemical elements, in any way recognisable as the forerunner of the lists which we employ to-day, was drawn up on the basis of exact experiment. It was not that the rules which Boyle had laid down for deciding for or against the elementary nature of substances had been entirely disregarded by those who followed, but that the Newtonian concept of mass had not, during the period in question, been accepted and understood by the majority of scientists. So it was that the phlogiston theory of Stahl, which was put forward during the last years of the seventeenth century, was not wholly unscientific, and so it was that this theory formed the basis of speculation and experiment during the greater part of the century which followed. But Newtonian mass was slowly taking its place as the basic concept of physical theory, and very gradually the gravimetric analyses of Lavoisier (1743–94), Richter (1762–1807) and others were forcing upon the attention of chemists the prime importance of the notion of “combining weight” in the further development of their science.

John Dalton—starting, it must be admitted, from purely physical considerations—translated the idea of combining weight into that of “atomic weight”, and from that date (1805) the reality of atoms was no longer merely a matter of philosophical speculation, but a theory was in existence, in which inconsistencies were trivial in relation to the range of success as a whole, whereby the relative masses of the various types of atoms were given as the result of experiment. From that date, too, the philosophical inquiry concerning the number of the elements came to be interpreted almost entirely along atomic lines. Whitehead (*Science and the Modern World*, 1926) has said:

“In the eighteenth century every well-educated man read Lucretius, and entertained ideas about atoms. But John Dalton made them efficient in the stream of science; and in this function of efficiency atomicity was a new idea.”

I will not digress to enlarge upon this theme, but it is pertinent to wonder whether, in view of his own scant early education, John Dalton himself was ever well educated in this respect, whether he ever read Lucretius in the original text.



I cannot afford, either, to digress on the more exact chemistry of Berzelius or Cannizzaro, or the theorising of Ampère and Avogadro, on the basis of which Dalton's original atomic weights were amended and his ideas purified and extended in scope. I accept the chemical atom as a scientific reality, and pass on to the systematisers. Of them there are many: Döbereiner earliest of all (1817), then, when more exact data were available on which to build, Lothar Meyer (1860), de Chancourtois (1862), Newlands (1863) and Mendeléeff (1869). Not all were hailed as prophets, but each had a contribution to make, and the last quarter of the century saw the Periodic Table of the Russian accepted as fundamental in any discussion of the number and interrelations of the elementary varieties of matter of which there was then actual knowledge. More than this, its intelligent use led to the discovery of new elements and to a revision of assigned atomic weights in other cases, but with its acceptance the contribution of classical chemistry to our subject was complete, and the problem of the number of the elements passed to the physicist. Before the periodic table could be further elucidated it was necessary that the nuclear model of the atom should emerge. This development did not occur until 1911.

It is a commonplace to say that the nuclear model would never have emerged, with that claim to immediate acceptance which it possessed at the time of Rutherford's formulation, but for the discoveries of X-rays and radioactivity in 1895 and 1896, and the isolation of the negative electron in the following year; nor is it cause for surprise that more than a dozen years should have elapsed between these momentous experimental discoveries and the full realisation of their implications in the form of a simplifying theory. Rather is it surprising that the advance should have progressed so far in so short a time; there was nothing very easy in the type of experiment required, and the incidence of outstanding genius cannot be guaranteed. Indeed, there were hints along the way which were missed—the work of Barkla (Barkla, 1906) and Kaye (Kaye, 1909) on the characteristic X-rays established a connection between position in Mendeléeff's table and the exact physical measure of the penetrating power of the radiation from any element which was very obviously fundamental, though no one knew exactly how to explain it—but when the unexpected occurred in the experiments of Geiger and Marsden (Geiger and Marsden, 1909) on  $\alpha$ -particle scattering Rutherford was not to miss the implication. It required nearly two years' thought, but when the nuclear model was finally advanced (Rutherford, 1911) there was nothing for the physicist to do but to accept it.

Acceptance by Bohr, as is well known, resulted in the first quantum theory of the atom as a whole (Bohr, 1913), acceptance by Moseley gave direction to his repetition of the work of Barkla and Kaye using an experimental technique of greater precision and power (Moseley, 1913, 1914), acceptance of the nuclear model in the general field of radioactive research fixed upon the atom nucleus as the system involved in spontaneous radioactive disintegration, and the displacement rule of Russell, Fajans and Soddy placed the shorter-lived radioelements securely in the periodic table between uranium (atomic number 92) and thallium (atomic number 81). It was here that the first hint of a new possibility arose: it was frequently necessary to accommodate distinct radioactive types in the same cell of the table, and it was necessary, on the incontrovertible evidence of the chemist, to bracket, with the inactive elements thallium, lead and bismuth, active species of short lifetime. In the writings of Soddy at this period (1913–17) the idea of isotopy, which had its origin in these observations, was expanded and elaborated with great foresight and understanding (*cf.* Soddy, 1917). When the war of 1914–18 was over, Aston took up the matter on the basis of an earlier observation of Thomson on the "parabolas" of neon (Thomson, 1913), and a more powerful instrument was built involving a novel principle with which an astonishingly fruitful start was made on the mass analysis of the elements in general (Aston, 1919). Two parameters, then, forced themselves on the attention of physicists for the specifying of a nucleus: its positive charge, expressed as a multiple ( $Z$ ) of the charge on the electron, and its mass number ( $A$ ), the nearest integer to the number expressing the mass of the corresponding neutral atom in terms of chemical oxygen as 16. Departures of exact masses from integral values on this scale proved to be small throughout the whole system of elements, and the elements themselves were defined and distinguished, of course, by the  $Z$  values, the "atomic numbers" of van den Broek and Moseley.

Parallel with the purely experimental attack on the nuclear side, experiment and detailed theory were keeping pace in investigations on the electronic, or extra-nuclear side of the picture. With these developments I cannot deal; I can only mention the names of pioneers, whose work has long since been greatly extended and, in form at least, largely superseded, but to whom the credit belongs of first giving a coherent "explanation" of the periodic table in its original exclusively chemical aspect. Bohr, Main-Smith and Stoner developed their ideas on this subject in the early nineteen-twenties, and it can now be said that as a result of their work, and that of those who have followed them, the general interpretation of the many regularities with which the table abounds is solidly based in an acceptable framework of physical theory.

I return, then, to the periodic table as viewed from the standpoint of the nuclear physicist, and I would say that by 1932 the survey of "existing" atom types was fairly complete. Not only for the well-known elements, but also for such recently discovered substances as hafnium ( $Z=72$ ), discovered by Coster and Hevesy (1923), and rhenium ( $Z=75$ ), identified by Noddack (1925), both by the method of Moseley, had satisfactory isotopic analyses been carried out. Though the survey of types was well-nigh complete, precision in the determination of exact masses still left much to be desired; in spite of this, however, a new accession of data was available for the systematiser to work upon. Let us see, therefore, what he made of it. First, there are stable species for all values of  $Z$  from 1 to 83, inclusive, with the exception of 43 and 61. Secondly, amongst the stable species all values of  $A$  are represented except  $A=5$  and  $A=8$ , in the range from 1 to 209, and many values of  $A$  are represented twice. Third, when  $Z$  is odd  $A$  is odd (for  $Z > 7$ ), and for these odd-numbered elements the number of stable isotopes is never greater than two. Lastly, when  $A$  has the same value for two stable nuclei (isobars) it is much more likely that  $Z$  values differ by two units than by one unit: "neighbouring" isobars are very rare. Obviously, significant regularities are involved here, though we shall not be able to make much progress in their interpretation without hypotheses concerning the structure of the nucleus itself. To this I shall return presently; meanwhile let us take a last look at the extra-nuclear structure of the atom, to dispose of the suggestion that one clue to an understanding of the number of the elements is to be found there. This is the contention of those who have believed—or who have professed to believe—that we can understand in a general way why atomic numbers higher than that of uranium ( $Z=92$ ) are not represented on the earth, without any knowledge of nuclear properties at all.

At bottom the argument of these theorists—Sommerfeld, Narliker and Flint (1932) have all made suggestions of the type I have in mind—is bound up with the non-dimensional character of the quantity  $\hbar c/2\pi e^2$ , the number 137. For one reason or another they say that  $Z$  cannot be greater than this number, or some specified fraction of it (Flint says not greater than  $137/2^{\frac{1}{2}}$ , *i.e.* not greater than 97). Of the reasons advanced the one most intelligible, physically, is that for higher values of  $Z$  the electrons in the K shell would fail to satisfy certain fundamental conditions attaching to periodic motion, according to the particular unitary theory of gravitation and quantum phenomena favoured by the writer (Flint, 1932). Even though the truth of this contention were granted, it is difficult to see why such limitation should be decisive in respect of the types of nucleus or atom which may exist. At the worst it might be supposed to imply that the normal state of such hypothetical heavy atoms would be a doubly charged state, with the neutral atom having as transient an existence as, say, a doubly charged negative ion in a gas; or that its place in the periodic table would be two higher than would accord with its nuclear charge. But, truth to tell, I have little sympathy with these hypotheses in any form.

I return, then, to theories of nuclear structure. These have taken on their present aspect since the discovery of the neutron ( $A=1, Z=0$ ) by Chadwick in 1932 (Chadwick, 1932). As a result of that discovery it became possible for the first time to think of nuclei as constituted entirely of heavy particles, neutrons and protons ( $A=1, Z=1$ ), to the great simplification of the whole picture. By determining the "exact" masses of neutron and proton in the free state, and of the complex nuclei, with as high an accuracy as could be achieved, it became possible, too, to estimate, on the basis of this model, the mass defect ( $\Delta M$ ) or energy of binding ( $\Delta M \cdot c^2$ ) for each nucleus. Fig. 1 shows the results of these determinations. It is an important fact that the mass defect of the neutron-proton model increases roughly linearly with the mass number—which is also the total number of heavy particles in the nucleus. Taken together

with the facts that the numbers of nuclear protons and neutrons are not widely different for most stable nuclei (in all cases, except  ${}^3_2\text{He}$ , when there is an excess of one type of particle, it is neutrons which are present in the greater number), and that amongst the lightest nuclei it is the group  ${}^4_2\text{He}$ ,  ${}^{12}_6\text{C}$ ,  ${}^{16}_8\text{O}$  \* which comprises the most tightly bound systems, this result leads to the conclusion that intra-nuclear forces show saturation (as the chemists understand the term), with the  $\alpha$ -particle,  ${}^4_2\text{He}$ , as the quasi-saturated unit. With a structure of this type we should further expect that the volume occupied by the nuclear particles, in any case, would

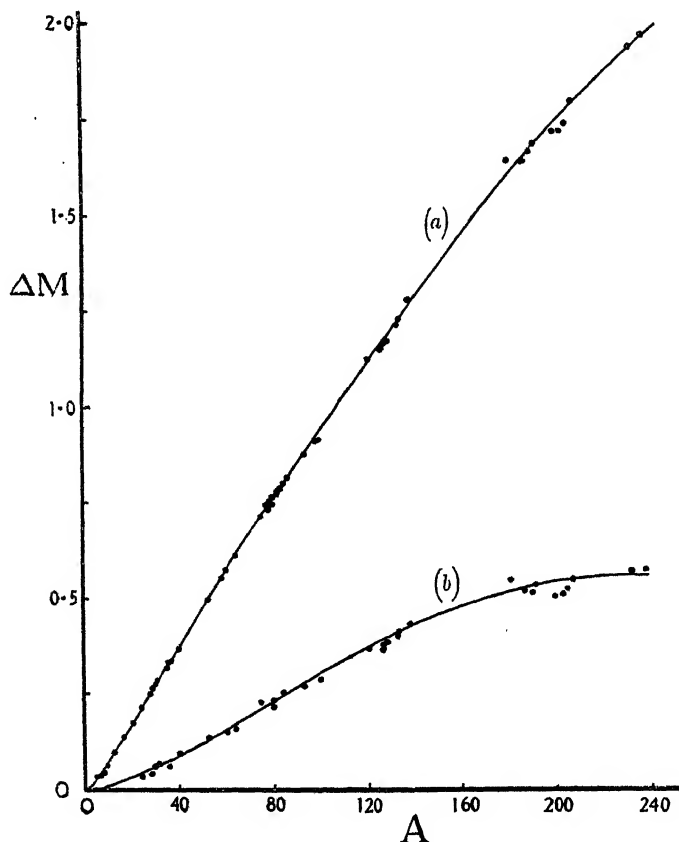


FIG. 1.—Curve (a), neutron-proton model; curve (b), maximum number of  $\alpha$ -sub-units assumed.

be fairly closely proportional to the number of particles involved, so that, if  $r$  is the effective radius,  $A/r^3$  should be approximately constant for all nuclei. There is independent evidence for the general truth of this result.

We have built up a picture, therefore, which has considerable resemblance to that of a minute drop of liquid, except that the number of constituent particles is, relatively, very small. If such a picture is valid, we should be able for many purposes to represent the intra-nuclear forces with sufficient accuracy in terms of surface tension and intrinsic pressure, and, when this is necessary, take count of the fact that the nucleus is a charged body by assuming a uniform surface charge on the drop. A liquid-drop model of the nucleus of this type has been

\*  ${}^8_4\text{Be}$  is not found in nature, but there is no serious problem raised by this fact. There is good reason to believe that in the ground state this nucleus would be tightly bound in relation to neutrons and protons, but just not stable in relation to division into two  $\alpha$ -particles.

much used since it was introduced by Gamow in 1929 (Gamow, 1929). We shall have recourse to its aid at a later stage in our discussion; meanwhile let us look further at the facts concerning those types of nucleus which are known to be stable.\*

As I have already said, stable nuclei are not known for which  $Z=43$  and  $Z=61$ ; that is, no stable nucleus is known containing 43 or 61 protons in its structure. Similarly, a survey of existing types shows that the following values of  $(A-Z)$  are not represented, viz.

$$19, 21, 35, 39, 45, 61, (65), (71), 89, 111, 115, 123 \dagger$$

or nuclei containing these numbers of neutrons are unknown amongst the existing elements (our survey covers the range  $Z < 84$ ,  $A < 210$ ). In the first place all these "missing" numbers are odd, which is in line with our conclusion regarding the tendency towards saturation of the forces (saturation being achieved with groups of two protons and two neutrons in every case), and in the second place "missing" (odd) numbers of neutrons are more in evidence than "missing" (odd) numbers of protons. This latter result appears to indicate a definite asymmetry, not connected with the purely electrostatic forces between the protons in the nucleus; in another aspect it is a reflection of the fact that stable nuclei (with  $Z > 7$ ) are unknown having  $Z$  odd and  $A$  even, and that, when  $Z$  is even, odd values of  $A$  are less likely than even values.

The survey which we have just made has served to establish certain empirical results which may be systematised and understood, after a fashion, but it does not answer the natural question, "What are the properties of the 'missing' nuclei, and in particular of those nuclei of which the non-occurrence upon the earth interrupts the sequence of isotopes of the different elements?" In the latter category are the "missing" even isotopes (*i.e.* even  $A$ ) intermediate in mass between the two odd isotopes which many of the elements of odd  $Z$  possess, and many odd isotopes belonging to elements of even  $Z$ . The answer to this question has, within the last ten years, been completely given by experiment. Previously, it was thought sufficient to say that since elements with  $Z > 83$  are all spontaneously radioactive, so with the lapse of time they must all disappear entirely from the earth; now we can add that we know, as the result of direct experiment, that the "missing" nuclei of smaller mass are, almost without exception, also radioactive, but with shorter periods than have uranium or thorium. Thus it is inevitable that these nuclei are "missing" on the earth to-day, whatever may have been the early history of our planet. Fig. 2 gives some idea of the growth—and present near completeness—of our knowledge in this respect, so far as concerns a small range of values of  $Z$ . It need only be added that information of like extent is available for all the elements; the discovery by Curie and Joliot in 1934 (Curie and Joliot, 1934) that an "artificially radioactive" phosphorus could be obtained by bombarding aluminium with  $\alpha$ -particles has rapidly been followed by the discovery and study of some hundreds of similar radioelements formed by irradiating ordinary materials with  $\alpha$ -particles, neutrons, protons, deuterons, electrons and high-energy X-radiation. Chemical and other tests have served to identify the new products as to  $A$  and  $Z$ , and a study of the energy of the radiations which they have been found to emit has given us an accurate index of the extent to which the previously "missing" nuclei are in fact unstable. Let us look at this last point in more detail, first remarking upon the nature of the disintegration processes involved.

With the "classical" radioelements, the descendants of uranium and thorium, two types of radioactive change were early identified: disintegration with the emission of  $\alpha$ -particles (helium nuclei) on the one hand, and the emission of  $\beta$ -particles (negative electrons) on the other. With the "artificially" produced radioelements  $\alpha$ -particle emission has not so far been discovered,† but  $\beta$ -emission of two types has been found. Broadly speaking, when species are produced in which the number of nuclear neutrons is in excess of what is normal for a stable nucleus, negative electrons are emitted; when protons are in relative excess positive electrons are given out (or the nucleus captures one of the extra-nuclear electrons, in a process

\* It is not possible to be completely certain of the absolute stability of the heavier nuclei; as an empirical rule we can regard a nucleus as stable if the half-value period for any spontaneous disintegration to which it may be liable is greater than  $10^{14}$  years.

† Brackets refer to doubtful cases, the doubt being whether both members of certain pairs of "neighbouring" isobars are in fact stable according to our definition.

‡ Except with element 85 (*v. infra*).

which is generally referred to as K-electron capture \*). In fig. 2 species of the former type are represented by open circles in general above and to the left, and those of the latter type by open circles generally below and to the right, of the straggled array of full circles which represent the stable nuclei. This statement is indefinite chiefly as concerns certain unstable species which are isobaric with two stable nuclei for which  $\Delta Z = 2$  (non-neighbouring stable isobars—*v. sup.*). When the nuclear charge of such an unstable nucleus is intermediate between the nuclear charges of the stable isobars it would appear that either positive or negative electron emission might occur. Sometimes such a radioelement is found to exhibit the phenomenon of radioactive branching (positron-negatron branching) as this conclusion would suggest, sometimes only one mode is evident—to the accuracy with which present investigations

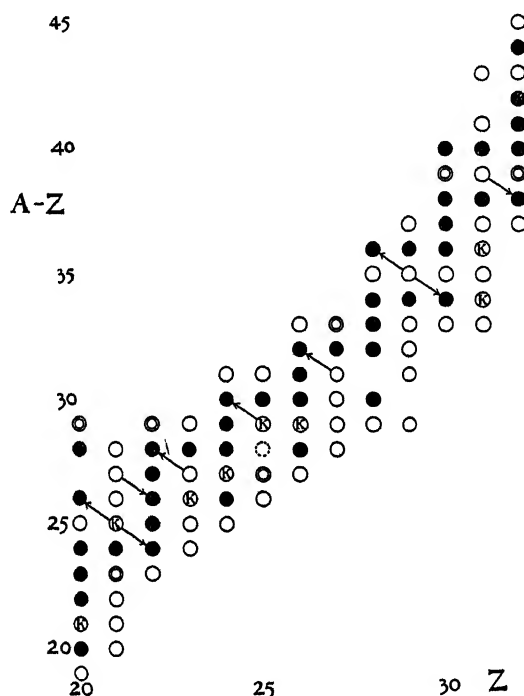


FIG. 2.

have gone. (Examples of the former type are indicated in fig. 2 by oblique arrows pointing in opposite directions, those of the latter by a single arrow, in one direction or the other.) Finally, a survey of "artificially" produced radioelements has greatly increased the number of known examples of nuclei having long-lived excited states (metastable states). A nucleus in such a state is radioactively quite distinct from the same nucleus in its lowest energy (ground) state. Before 1934 the only recognised instance of nuclear isomerism of this type was provided by the pair of bodies uranium  $X_2$  and uranium  $Z$  (for both of which  $A = 234$ ,  $Z = 92$ ); already, in the range of  $Z$  covered by fig. 2, for example, the seven species, represented by the double open circles in the diagram, have been found to show this effect.

Returning now to a survey of the amounts of energy liberated in the various disintegrations, we are in a position to correlate this information as to the degree of instability of unstable nuclei in general with what has already been said concerning the regularities exhibited by the  $A$  and  $Z$  values of the stable species. If we take the whole of the information available concerning negative-electron-active bodies (and parallel statements are almost certainly valid for the positron-active bodies, also, though the experimental information is at present less complete for them) we find that this clear regularity emerges. When a species having  $A$  and  $Z$  both

\* Energetically, K-capture is favoured, in relation to positron emission, by there being available about 1 MeV more energy for the former process than for the latter in all cases, but not in every case, by any means, is it the more probable mode of disintegration in actual fact.

even gives rise by  $\beta$ -disintegration to a daughter product which is itself  $\beta$ -active, then the disintegration energy of the first change ( $Z$  even  $\rightarrow$   $Z$  odd) is in general less than the disintegration energy of the second ( $Z$  odd  $\rightarrow$   $Z$  even), whereas, when there are two consecutive  $\beta$ -active bodies the former of which has  $A$  odd and  $Z$  even, then the reverse is the case and the disintegration energy is greater for the first body than for the second. We can obviously extend this empirical rule by including cases in which the disintegration energy is negative and so introduce stable species into our considerations. Then our rule requires that, if we have neighbouring isobars of even mass number, one stable and the other unstable (negative-electron-active), it is the species of even charge number which must be stable (unless we admit of the occurrence of neighbouring stable isobars for this value of  $A$ ). And, in the alternative case, the rule allows statements under two heads. First, if we have neighbouring isobars of odd mass number of which the species of smaller charge number is stable and that of greater charge number is negative-electron-active, then the charge number of the stable species is odd. Secondly, if for such neighbouring isobars the negative-electron-active species is the one of smaller  $Z$  and the stable one that of greater  $Z$ , then the stable species in question can either have  $Z$  even or odd. These three deductions from the extended rule obviously correspond exactly with our former statements that there are in general no stable species with  $Z$  odd and  $A$  even, whereas stable species with  $Z$  odd and  $A$  odd as also with  $Z$  even and  $A$  both even and odd may occur.

We have come full circle, therefore, and linked up our attempts to systematise for the stable and unstable species for which  $Z < 84$ ; and others, as for example Fuchs (Fuchs, 1939), have gone somewhat further, interpreting (or describing) the regularities to which I have drawn attention in more formal terms. I cannot burden you now with such detailed considerations; however, for  $Z < 84$ , it is clear, I think, that the problem of the number of the elements—or of the number of stable nuclear species—turns essentially on the question of stability towards  $\beta$ -disintegration.\* My last remarks, then, deal with the range  $Z > 83$ .

As I have already indicated, the customary assumption here is that there are no stable species of higher  $Z$  because of the onset of  $\alpha$ -instability. Disintegration with the emission of  $\alpha$ -particles decreases  $Z$  by two units; thus the problem of the stability of the heaviest elements has come to be regarded as primarily a question of the possibility or otherwise of  $\alpha$ -emission. But this "explanation" is not, and has never been, very satisfactory, chiefly because there is a very clear general tendency for the energy of  $\alpha$ -disintegration, that is for the degree of instability, to decrease as  $Z$  increases amongst the classical radioelements. In fact, if this were not so, radioactive series (of heavy elements) would not be found in the world. There would be no gap of 8 or 10 units in  $Z$  between the stable end-product of  $\alpha$ -transformation and the "nearest" active species to possess a lifetime comparable with the age of the solid earth. No, it is clear that we must search for another explanation. This was provided by the discovery of neutron-induced fission in 1939. In that year it was found by Meitner and Frisch (Meitner and Frisch, 1939; Frisch, 1939)—and the radiochemical work of Hahn and Strassmann supported these findings (Hahn and Strassmann, 1939)—that capture of a neutron by a uranium (or thorium) nucleus could lead to the division of the system into two parts of comparable mass and charge. These "fission fragments" were unstable nuclei of intermediate mass, which after a succession of  $\beta$ -transformations became stable nuclei belonging to the known isotopes of the mid-table elements. Evidence for a great many modes of division was found,  $\beta$ -active fission products being discovered, in one type of fission or another, with almost all values of  $Z$  from 35 (bromine) to 58 (cerium). Then, in the following year, Petrzhak and Flerov (Petrzhak and Flerov, 1940) showed that, in the natural state, uranium undergoes fission spontaneously: for every million atoms of ordinary uranium which disintegrate in the "normal" way, with the emission of  $\alpha$ -particles, roughly one divides into two fragments, according to one or other mode of spontaneous fission—and in that sense a relatively large number of additional radioactive series (series of successive  $\beta$ -active bodies) must be added to the three series of

\* One species, it is true, in this range of  $Z$  is  $\alpha$ -active and of long lifetime ( $^{148}_{62}\text{Sm}$ ), but no essential failure in our ideas is involved in this fact. Four other long-lived species ( $^{40}_{19}\text{K}$ ,  $^{87}_{37}\text{Rb}$ ,  $^{176}_{71}\text{Lu}$  and  $^{187}_{76}\text{Os}$ ) may be regarded as relics of "artificially active" bodies, of more usual type, left over from the primal mixture of elements of which the earth's substance was formed at an early epoch in cosmic evolution.

"classical" radioelements which are found on the earth. Now the point about spontaneous fission is just this: in relation to fission a uranium nucleus is already, energetically, in a very favourable state—with anything up to 170 MeV of energy available for the process. But the nucleus does not break up "instantaneously" into two charged parts just because energy is available: as in  $\alpha$ -disintegration it disintegrates according to the "laws of chance", with the probability of disintegration (reckoned per unit time) depending very markedly upon the difference between the available energy and an energy characteristic of the original nucleus and the fragments concerned. On a simple liquid-drop model (with constant density and "surface tension") this difference is likely to be a function of  $Z^2/A$ , making spontaneous fission increase extremely rapidly in probability as  $Z^2/A$  increases. No doubt the model is much too crude for

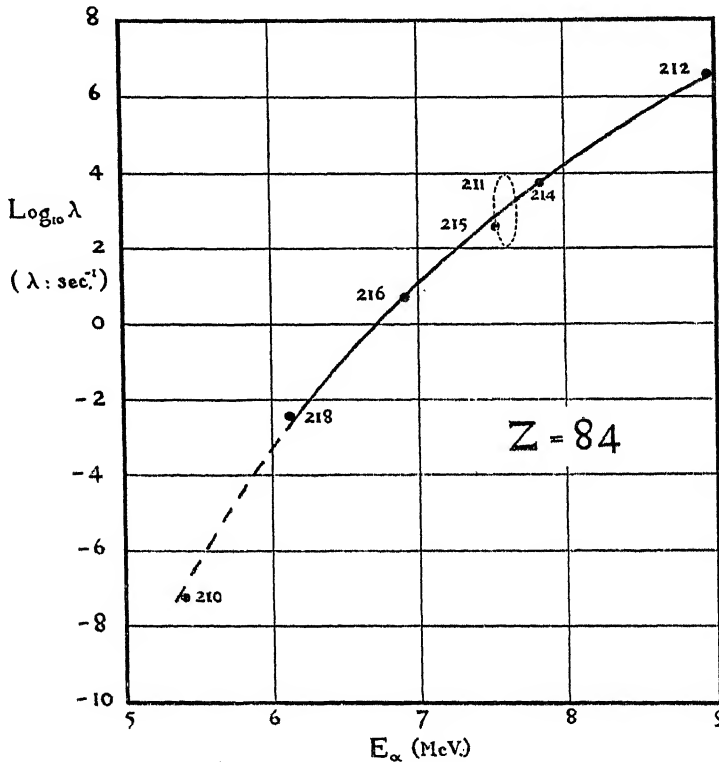


FIG. 3.

detailed calculation, but it is clear enough that for  $Z$  a few units greater than 92 spontaneous fission must be extremely rapid—and eventually "instantaneous", even for nuclei in the ground state.\* When that stage is reached it is unquestionably true to say that nuclei of higher charge cannot possibly exist. It is only necessary to suppose, therefore, that for the immediate range of  $Z$  just greater than 92 no long-lived  $\alpha$ -bodies occur, to understand why uranium is the last terrestrially known element of the table: beyond that range spontaneous fission will take care of the rest.

Having discussed the ultimate limitation to the number of the elements in the true sense (limitation as to  $Z$ ), it is of interest to inquire finally into the limitations in respect of  $A$  in the domain of the heavy radioelements. This can be done most simply by examining the way in which  $\alpha$ -disintegration energy  $E$  (or decay constant  $\lambda$ ) varies with mass number for a given  $Z$ . For this purpose the case of  $Z = 84$  provides the most clear-cut information. In fig. 3 the Geiger-Nuttall curve for the  $\alpha$ -active isotopes of this particular charge number is plotted in the form  $\log_{10} \lambda$  against  $E$ . Three radioelements of the radium series (RaA, RaC' and RaF)

\* The "instantaneous" neutron-induced fission of uranium is, of course, fission of a compound nucleus in a highly excited state.

and two each of the thorium and actinium series (ThA, ThC', and AcA and AcC') are represented, and consideration of the experimental data is simplified by the fact that so far as is known the  $\alpha$ -disintegration of each body takes place according to a single mode only (and change of nuclear spin in the process is, in each case, most probably zero). The points lie on a good smooth curve, but the important observation is that whereas, as  $A$  decreases from 218 to 212 (no information is available for  $A=217$  and  $A=213$ ) the disintegration energy (or  $\alpha$ -instability) progressively increases, the disintegration energy decreases again even more rapidly as  $A$  decreases further through 211 to 210 (polonium).<sup>\*</sup> The curve for  $Z=83$  shows similar features,  $E$  increases as  $A$  decreases from 214 (RaC) through 212 (ThC) to 211 (AcC), but 210 (RaE) is  $\beta$ -active and 209 (ordinary bismuth) is stable. The parallel would be even closer if it were found that radium E is  $\alpha$ -active in a rare mode (all the other  $\alpha$ -active species having this value of  $Z$  show  $\alpha/\beta$  branching), but it is striking enough as it is.

Until recently no radioelements with  $Z=85$  or  $Z=87$  were known, but within the last few years three  $\alpha$ -active isotopes of the former element and one  $\beta$ -active species with the latter charge number have been reported in the literature.<sup>†</sup> Information concerning the active isotopes with  $Z=85$  is sufficient once more to show the feature we have been discussing, in outline at least: the  $\alpha$ -disintegration energy for  $\left( \begin{smallmatrix} 211 \\ 85 \end{smallmatrix} \right)$  is less than, and that for  $\left( \begin{smallmatrix} 216 \\ 85 \end{smallmatrix} \right)$  greater than, the  $\alpha$ -disintegration energy for  $\left( \begin{smallmatrix} 218 \\ 85 \end{smallmatrix} \right)$ . For values of  $Z$  greater than 85, however, the  $\alpha$ -active species of which we have knowledge all belong to mass-sequences for which  $E$  increases as  $A$  decreases (for  $Z$  constant), but it would seem reasonable to suppose that, in this range of  $Z$  also, the class of  $\alpha$ -active species is limited, in the direction of  $A$  increasing, by the fact that negative-electron-emission becomes more probable than  $\alpha$ -emission—and, in the direction of  $A$  decreasing, by the rapid decrease of the probability of  $\alpha$ -emission in the limit in favour of K-electron capture or the emission of positrons. Outside certain limits of  $A$ , for a given  $Z$ , the energy available for  $\alpha$ -disintegration would thus appear to be too small for the process to be an important disintegration mode—and within these limits the disintegration energy would appear in each case to vary smoothly with  $A$ , but unsymmetrically, with  $E$  greatest for a value of  $A$  near the lower limit of the  $\alpha$ -range.

### SUMMARY

The development of the idea of the chemical element is traced from its early beginnings, and the importance for this development of the Newtonian concept of invariable mass is emphasised. The emergence of the nuclear atom model is outlined, and the discovery of the complex (isotopic) nature of the majority of known chemical elements is described. Nuclear charge ( $Z$ ) and mass ( $A$ ) numbers are defined. Previously recognised regularities concerning mass and charge numbers of existing stable species are shown to have exact counterparts in regularities relating to the degree of instability (as measured by the energy of disintegration) of  $\beta$ -active species ("naturally" and "artificially" radioactive species). Naturally occurring  $\alpha$ -active species are regarded as the analogues of the stable species for charge numbers greater than 83, and for charge numbers both greater and less than this value the limitation to the number of stable or quasi-stable isotopes of a given element (limitation of  $A$  values for a given  $Z$ ) is established as essentially a question of nuclear stability as against  $\beta$ -emission (positive and negative electron emission). Finally, reasons are given for supposing that the number of possible chemical elements is limited (limitation to  $Z$  in the direction of  $Z$  increasing) by the susceptibility to spontaneous nuclear fission of species of sufficiently high nuclear charge.

<sup>\*</sup> Since the lecture was delivered I have discovered that the device of plotting separate Geiger-Nuttall curves for each value of  $Z$  has been adopted by Berthelot (Berthelot, 1942), and that the "polonium anomaly" discussed above has been remarked on by him.

<sup>†</sup>  $\left( \begin{smallmatrix} 218 \\ 85 \end{smallmatrix} \right)$  and  $\left( \begin{smallmatrix} 216 \\ 85 \end{smallmatrix} \right)$  formed in rare modes of  $\beta$ -disintegration of RaA and ThA respectively,  $\left( \begin{smallmatrix} 211 \\ 85 \end{smallmatrix} \right)$  by an ( $\alpha, 2n$ ) reaction on  $^{209}_{83}\text{Bi}$ , and  $\left( \begin{smallmatrix} 223 \\ 87 \end{smallmatrix} \right)$  in a hitherto unnoticed  $\alpha$ -mode with Ac  $\left( \begin{smallmatrix} 227 \\ 89 \end{smallmatrix} \right)$ .



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XXIV.—Time-Scales in Relativity. By A. G. Walker, M.A., D.Sc., Department of Pure Mathematics, University of Liverpool. *Communicated by Sir EDMUND WHITTAKER, F.R.S.*

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1. *Introduction*.—One of the most important of Milne's discoveries is undoubtedly the significance of time-scale regradautions and, in particular, the relation between atomic  $t$ -time and gravitational  $\tau$ -time. Although in Milne's work  $t$ -time is more fundamental than  $\tau$ -time, this relationship is not inevitable, as was shown in an axiomatic development of cosmology given recently by the author.<sup>†</sup> There the  $\tau$ -scale was the more fundamental, and it was not found necessary to introduce Milne's  $t$ -scale. The object of the present paper is to discuss this primitive  $\tau$ -scale still further, and to show how the  $t$ -scale may be introduced by means of an axiom. The unpublished work mentioned above is not required for this purpose because the cosmological models considered here were described in earlier papers (Walker, 1937, 1940 *b*). We also examine the various constants, absolute and conventional, which are connected with the different scales of time and length, and with different models.

We refer to "axioms" rather than "hypotheses" in this work in order to stress the fact that we are not examining the external world but are constructing model universes. An hypothesis enters later when we assume that one of these models approximates to the external world. This leads us to consider certain astronomical observables (red-shift, distance, distribution, etc.) and the theoretical relations between them. It is hoped that these relations, when compared with actual observations, may eventually test our cosmological theory and determine those world-constants which are at present unknown.

2. *Absolute  $\tau^*$ -time*.—It has been established that clocks attached to fundamental particles can be so graduated that they are congruent in a well-defined sense and that the distance between any two particles is constant in time. This distance is by definition  $\frac{1}{2}c(\tau_2 - \tau_1)$ , where  $c$  is constant and  $\tau_1, \tau_2$  are instants of emission and reception at one particle of a light signal reflected at the other. These clocks are constructed from ideal experiments, *i.e.* experiments which are definable although not practicable, and are determinate except for an arbitrary world-wide change of zero and unit ( $\tau' = a\tau + b$ ). When the unit is changed ( $\tau' = a\tau$ ), all distances are affected proportionately.

We also know that, for given  $\tau$ -clocks and for a given value of  $c$ , each fundamental particle corresponds to a point of a 3-dimensional Riemannian space,  $\Sigma_3$ , in such a way that certain linear sets of particles correspond to geodesics, the distance between any two particles is given by the geodesic arc between the corresponding points, and angles are given correctly by the Riemannian formula.  $\Sigma_3$  is a space of constant curvature, say  $K_0$ , and convenient co-ordinates  $r, \theta, \phi$  can be defined so that any given particle O is at  $r=0$  and such that the metric of  $\Sigma_3$  is

$$d\epsilon^2 = \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}{(1 + \frac{1}{2} K_0 r^2)^2}. \quad (1)$$

All light paths are null geodesics in the 4-space with metric

$$d\sigma^2 = d\tau^2 - c^2 d\epsilon^2. \quad (2)$$

Suppose now that a clock as described above is given, that the value of  $c$  is given, and that  $K_0$  is the curvature of the appropriate  $\Sigma_3$ . Let particles O, P, Q be such that  $OP = OQ = l$  and angle  $POQ = \theta$  for given  $l$  and  $\theta$ , and define  $\lambda = \lim_{\theta \rightarrow 0} \left( \frac{PQ}{\theta} \right)$  for fixed  $l$ . Then  $\lambda$  as a function of  $l$ , say  $\lambda = f(l)$ , is determinate from ideal experiments, *i.e.* the form of  $f$  is observable.

<sup>†</sup> To appear shortly in *Proc. Roy. Soc. Edin.*

Calculating  $l$  and  $\lambda$  in terms of  $r$  we find from (1)

$$l = \int_0^r \frac{dr}{1 + \frac{1}{4}K_0 r^2}, \quad \lambda = \frac{r}{1 + \frac{1}{4}K_0 r^2}.$$

Hence  $f$  is calculated to be

$$\begin{aligned} K_0 > 0, & \quad f(l) = \sin(l\sqrt{K_0})/\sqrt{K_0}; \\ K_0 = 0, & \quad f(l) = l; \\ K_0 < 0, & \quad f(l) = \sinh(l\sqrt{-K_0})/\sqrt{-K_0}. \end{aligned} \quad (3)$$

Since  $f(l)$  is observable and is one of these alternatives, it follows at once that the value of  $K_0$  can be deduced. Hence:

*For a given  $\tau$ -clock and value of  $c$ ,  $K_0$  is determinate.*

The cases  $K_0 = 0$  and  $K_0 \neq 0$  must now be treated separately, and from now until § 7 we shall consider only  $K_0 \neq 0$ .

We shall now prove that when  $K_0 \neq 0$  there exists an absolute unit of  $\tau$ -time. First choose  $c = 1$ . Then  $K_0$  depends upon the unit of  $\tau$ -time, and with a change of unit given by  $\tau' = a\tau$ , all distances including  $l$  and  $\lambda$  are multiplied by  $a$ , so that from (3),  $K_0$  is replaced by  $K_0/a^2$ . Choosing  $a = |K_0|^{1/2}$ , then the new value of  $K_0$  is either 1 or -1, and the new unit of time which gives  $K_0$  this value is unique. Hence we have:

*When  $K_0 \neq 0$ , there is a scale of  $\tau$ -time which is unique except for arbitrary zero.*

*With this scale,  $K_0 = \pm 1$  when  $c = 1$ .*

This will be called the *absolute scale* and written  $\tau^*$ . The corresponding value of  $K_0$  when  $c = 1$  will be written  $k$ , so that  $k = 1$  or  $-1$ . The value  $k = 0$  will also be included later to cover the case  $K_0 = 0$ .

Since  $\tau^*$  is an absolute scale of time, it follows that an absolute measure of distance is given by the formula  $l^* = \frac{1}{2}(\tau_2^* - \tau_1^*)$ , and that the appropriate 3-space as described above has the metric (1) with  $K_0 = k$ . These absolute measures of time and distance intervals can now be used in other ideal experiments.

3. *Conventional Scales.*—A complete study of our models can now be carried out using the absolute scales of time and distance defined in the last section. When, however, we come to suppose that one of the models approximates to the external world, we must recognize that the units of these absolute scales are probably not the units of time and distance (seconds, centimetres, etc.) used in physics to describe this world. It is necessary, therefore, to introduce *conventional units*, which are supposed to agree with physical units at the present instant.

Conventional units can be determined from ideal experiments in which we are supposed to possess a clock reading absolute  $\tau^*$ -time and another reading the time of physics, *e.g.* seconds or years, and also a scale for measuring absolute lengths and another for measuring the lengths of physics, *e.g.* centimetres or miles. Suppose we find that a unit interval of absolute time is  $\tau_0$  units of conventional time, and that a unit of absolute length is  $c\tau_0$  units of conventional length. Then  $\tau_0$  and  $c$  are observables, and the absolute and conventional scales are connected by the transformations

$$\tau = \tau_0 \tau^*, \quad l = c\tau_0 l^*. \quad (4)$$

In terms of light-signals,  $l^*$  is defined by  $\frac{1}{2}(\tau_2^* - \tau_1^*)$ , so that from (4),  $l = \frac{1}{2}c(\tau_2 - \tau_1)$ . This gives the relation between conventional distances and time intervals.

The above comparisons for the determination of  $\tau_0$  and  $c$  are carried out at the present instant, but we can imagine similar experiments being carried out at any other instant. We cannot, however, say anything about their results unless we adopt some rules for the construction of conventional scales at all instants. Certain rules are physically obvious, such as those using a periodic atomic or gravitational phenomenon to define a unit of time, and a crystal lattice or the wave-length of a particular Fraunhofer line to define a unit of length. Some of these will be considered later in connection with atomic time-scales, but others are too complex to be considered in the present discussion. One rule of particular interest and simplicity gives what will be called the *conventional  $\tau$ -clock*. This clock reads

conventional time at the present instant and is constructed so that  $\tau_0$  is constant. Also, conventional lengths at all instants are defined so that  $c$  is constant, having the value determined at the present instant. It then follows that the relations (4) hold at all instants. From the form taken by dynamical equations when these scales are used, it appears that the above definition of conventional  $\tau$ -time is equivalent to that in which periodic gravitational phenomena are used to provide a clock.

A fact that must be constantly remembered is that all experiments, actual or ideal, are carried out by ourselves, *i.e.* at one particular particle, O. The above  $\tau$ -clock therefore belongs to O, and the conventional lengths are measured from O. We can now, however, define conventional  $\tau$ -clocks and lengths for any other particle by means of the absolute  $\tau^*$ -clock for that particle and the relations

$$\tau = \tau_0 \tau^*, \quad L = \frac{1}{2} c (\tau_2 - \tau_1),$$

where  $\tau_0$  and  $c$  have the values determined at O. From this it follows that all conventional measures can be represented, as described in § 2, in the 3-space with metric (1), where

$$K_0 = k/c^2 \tau_0^2. \quad (5)$$

Constants such as  $\tau_0$  and  $c$ , which are not essential to the model and appear only when the model is compared with the external world, will be called *conventional constants*.

4. *The Dimensional Axiom.*—In  $\tau$ -time, the corresponding distances between fundamental particles are constant, and each model has a static appearance. An axiom which is fruitful and gives sensible results is the *dimensional axiom*, which states that *there is no absolute zero of  $\tau^*$ -time*, and hence of  $\tau$ -time. This implies that a model is static, at least statistically, and that any non-static phenomenon is repeated and distributed in such a way that the state of the whole is the same at all instants.

This axiom will be applied to atomic clocks in §§ 5, 6. Another application will be made when we come to consider observations on distant extra-galactic nebulae, for some assumptions must be made about these nebulae over the long intervals of time which their light takes to reach us. Our assumption is that we observe what may be called a *mean nebula*, *i.e.* one which is in all ways the mean of all nebulae in its neighbourhood. From the Cosmological Principle, or its equivalent, it follows that one mean nebula is similar to another, and from the dimensional axiom we see that *the characteristics (size, intrinsic brightness, etc.) of a mean nebula are constant in  $\tau$ -time*.

5. *t-time.*—A  $t$ -scale of time is here defined as one in which the fundamental particles have uniform relative motion, distances being measured in the usual way in terms of light signals. The relation which must exist between a scale of this kind and a  $\tau^*$ -scale has been well established by Milne and Whitrow, and is of the form  $t = a e^{\tau^*/\kappa}$ , where  $a$  and  $\kappa$  are positive constants. Since there is no absolute zero of  $\tau^*$ -time, it follows that *there is no absolute unit of t-time*, and the value of  $a$  is not significant. The essence of the above transformation is therefore given by the relations

$$\frac{dt}{t} = \frac{d\tau^*}{\kappa}, \quad \begin{aligned} t &= 0 \text{ when } \tau^* = -\infty, \\ t &= \infty \text{ when } \tau^* = \infty. \end{aligned} \quad (6)$$

The zero of  $t$ -time is fixed and may be called the *instant of creation*; this will be written  $T_0$ .

For a given  $\kappa$ , a  $t$ -clock is determined except for changes of unit. It is accelerating in  $\tau^*$ -time and there is an instant, T, at which the  $\tau^*$ - and  $t$ -clocks are running at the same rate, *i.e.*  $dt = d\tau^*$ . From (6), T is given by  $t = \kappa$ , so that  $\kappa$  is the  $t$ -measure of the interval  $T_0 T$ . When  $\kappa$  remains fixed but the unit of  $t$ -time is changed, then T varies in such a way that  $T_0 T$  is still  $\kappa$  in the new unit. This demonstrates that  $\kappa$  is a pure number.

For each  $\kappa$  there is a unique *conventional t-clock*, this being by definition a  $t$ -clock which runs at the same rate as the conventional clock of physics at the present instant  $T_0$ . From (4) and (6) we see that the interval  $T_0 T_0$  as measured by this clock is

$$t_0 = \kappa T_0, \quad (7)$$

this, then, being the “present age of the universe” in the conventional  $t$ -time given by  $\kappa$ .

Since the zero of  $\tau$ -time is at our disposal, it is sometimes convenient to take it to be at the present instant; the relation between the two conventional time-scales is then

$$t = t_0 e^{\tau/k_0}. \quad (8)$$

Milne's transformation from  $t$  to  $\tau$  differs from this only in that he sets the  $\tau$ -clock to read  $t_0$  at the present instant.

When the value of  $\kappa$  is changed to  $\kappa'$ , then there is a new conventional scale, say  $t'$ , and from (7) and (8) it can be seen that the relation between  $t$  and  $t'$  is

$$t' = \gamma t^\beta, \quad \beta = \kappa/\kappa', \quad \gamma = \beta^{-1}(\kappa\tau_0)^{1-\beta}.$$

To find a convenient geometrical representation to associate with  $t$ -time, consider (1) and (2) with  $\tau^*$ -time and  $c=1$ , so that  $K_0=k$ . Transforming from (6) and writing  $r=\kappa\rho$ , we see that the 4-space with metric  $d\sigma^2$  is conformal to the space  $S_4$  with metric  $ds^2$ , where

$$ds^2 = dt^2 - t^2 d\epsilon^2, \quad d\epsilon^2 = \frac{d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2}{(1 + \frac{1}{4}K\rho^2)^2}, \quad (9)$$

and

$$K = k\kappa^2. \quad (10)$$

Since null geodesics correspond to null geodesics in conformal spaces, it follows that the light paths are null geodesics in  $S_4$ .

This space  $S_4$  and the space  $S_3$  with metric  $d\epsilon^2$ , have frequently been used by the writer as maps suitable to  $t$ -time, and their properties have been described elsewhere (see, for example, Walker, 1940 *b*). In previous work,  $K$  appeared as an arbitrary constant, distinguishing between different models, and this, we shall see, is still the case. We now find, however, that only the sign of  $K$  is significant in  $\tau$ -time, and that the value of  $|K|$  depends directly upon the choice of  $t$ -scale. The fact that  $\kappa$ , and hence  $K$ , is unique for a particular model depends upon the *atomic axiom*, by means of which we can construct an ideal experiment for the determination of  $\kappa$ .

6. *The Atomic Axiom.*—The interpretation of red-shift in light received from a distance is closely connected with the problem of time keeping, and the phenomenon is reproduced in Milne's world-model by way of a hypothesis concerning atomic time. This, in general terms, states that for an oscillatory phenomenon of an atomic character, the periods are constant in some  $t$ -time. For example, the frequency of a recognizable Fraunhofer line is supposed constant in some  $t$ -time. A phenomenon of this kind gives rise to what Milne calls an *atomic clock*.

For our purpose the corresponding axiom can be stated thus:

*Axiom.*—At each fundamental particle there occur periodic phenomena, which shall be described as "atomic", such that (a) the frequency of each is constant in some  $t$ -time, (b) all frequencies in some interval occur at the same (any) instant other than  $T_0$ , and (c) atomic clocks with the same frequency at any one instant keep the same  $t$ -time.

It follows from (a) that a particular value of  $\kappa$  is associated with each atomic clock. To determine this value for a particular atomic clock, set the  $\tau^*$ -clock to read zero at the present instant  $T_0$  and measure the frequency  $\nu^*$  of the given oscillation in  $\tau^*$ -time continuously for an interval of time. Then from (6) and (a) above, we get a functional relation of the form

$$\nu^* = \alpha e^{\tau^*/\kappa}, \quad (11)$$

and since this is observable, it follows that  $\kappa$  and  $\alpha$  are determinate. If the experiment is repeated at another instant, then the zero but not the unit of  $\tau^*$  alters and from (11) we see that  $\alpha$  would be affected but not  $\kappa$ . Hence,  $\kappa$  is a determinate pure constant for the given atomic clock, but  $\alpha$  depends upon the instant at which the experiment is performed.

The atomic axiom does not assert that all atomic clocks keep the same  $t$ -time, *i.e.* that they all give rise to the same value of  $\kappa$ . This important property does, however, follow from the dimensional axiom. To prove this, consider all atomic clocks at the present instant. Then from (11),  $\alpha = \nu^*$  for each, and from (b) of the axiom we have clocks giving all values of  $\alpha$  in some interval. Also, from (c), there is just one  $\kappa$  for each  $\nu^*$  and therefore each  $\alpha$ , and there exists, therefore, an observable relation  $\kappa = f(\alpha)$  at the present instant.

The experiments leading to this relation can be repeated at other instants, but by the dimensional axiom the form of the function so obtained must be independent of the instant at which the experiment is performed. To obtain the form of  $f$  after an interval  $\theta$  of  $\tau^*$ -time, we see from (11) that the new form of this equation gives  $a'$  in place of  $a$  where  $a' = ae^{\theta/\kappa}$ . Hence from  $\kappa = f(a)$  we deduce that the new functional relation is given implicitly by  $\kappa = f(ae^{-\theta/\kappa})$ , whence, from the dimensional axiom,

$$f(a) = f\{ae^{-\theta/f(a)}\}$$

for all  $\theta$  and all  $a$  in some interval.

For given  $a$  and any number  $x > 0$ , choose  $\theta = -f(a) \log(x/a)$ . Then we must have  $f(x) = f(a)$  for fixed  $a$  and variable  $x$ , and the function  $f$  is therefore a constant. Thus the same value of  $\kappa$  emerges from all the experiments, *i.e.* we have proved that:

*All atomic clocks keep the same  $t$ -time, and  $\kappa$  is a constant of the model.*

The latter part of this statement follows partly from what we have proved and partly from the Cosmological Principle or its equivalent. We now see from (10) that:

*The constant  $K$  is determinate for a particular model, and that models in which the respective values of  $K$  differ are essentially different.*

7.  $\kappa = 0$ .—This case differs from the others in that there is no absolute  $\tau^*$ -scale and no constant  $\tau_0$ . There is still conventional  $\tau$ -time and distance related by a conventional constant  $c$ , and the dimensional axiom still applies, denying the existence of an absolute zero of  $\tau$ -time. The appropriate 3-space for geometrical representation is given by (1) with  $K_0 = \kappa = 0$ .

While  $t$ -time is still related exponentially to  $\tau$ -time, there is not now a unique  $\kappa$  associated with a given  $t$ -clock. Conventional  $t$ -time exists as before and is related to conventional  $\tau$ -time by (8),  $t_0$  now being a constant having any positive value; a conventional  $t$ -scale exists for each  $t_0$ , which is the age of the universe in this scale. The argument which led to the constancy of  $\kappa$  in § 6 can be applied with conventional  $\tau$  in place of  $\tau^*$  and  $t_0$  in place of  $\kappa$ ; the dimensional axiom now leads to the fact that  $t_0$  has the same value for all atomic clocks which read conventional time at the present instant,  $t_0$  thus being a determinate conventional constant as before.

8. To sum up the results so far, we have proved the existence of an observable model constant  $K$ , which can take any value  $\dagger$  including zero, and which discriminates between different models. When  $K \neq 0$ , there is a  $\tau^*$ -clock which has a unique unit but an arbitrary zero, and a  $t$ -clock which has a unique zero but an arbitrary unit. The relation between these two clocks involves  $|K|$ , and the existence of the unique  $t$ -clock and the significance of  $|K|$  depend upon both the dimensional and the atomic axioms. When a model is compared with the external world, there arise conventional  $\tau$ - and  $t$ -clocks and conventional constants  $\tau_0$ ,  $c$ ,  $t_0$ . When  $K = 0$ , there are no absolute clocks, but there are conventional clocks as before and conventional constants  $c$ ,  $t_0$ .

Comparing results with those given by Milne, it appears that the world-model constructed by Milne is similar to our model in which  $K = -1$ , *i.e.*  $\kappa = -1$  and  $\kappa = 1$ . This model is interesting mathematically because the space  $S_4$  with metric (9) is flat when and only when  $K = -1$ , and co-ordinates can then be defined in relation to each particle such that the transformation from one particle to another is Lorentz. This property implies the uniqueness of the  $t$ -clock (except for unit), a fact which appears to form a major part of the arguments for a unique  $t$ -scale given by Milne and Whitrow. Such a derivation of the  $t$ -scale requires, however, the prior establishment or adoption of the Lorentz transformations (or of  $K = -1$  in our notation), and it is not clear, at least to the author, what physical argument or assumption produces these transformations. $\ddagger$

$\dagger$  The inclusion of the case  $K > 0$  (*i.e.*  $\kappa = 1$ ) depends upon the form of the early light-signal axioms. In this case, a light signal emitted at one particle is received infinitely often at a second particle, contrary to an axiom which is regarded as desirable by Milne and by the writer.

$\ddagger$  The existence of a unique  $t$ -scale, *i.e.* the exclusion of regradauations of the form  $t' = t^a$ ,  $a \neq 1$ , is fundamental in Milne's work, but it is difficult to trace its precise derivation. It is not given in Milne and Whitrow, 1938, where only the general  $\tau$ -scales, and hence  $t$ -scales, are derived, and although it emerges from the derivation of metric in Milne, 1940, there is, on p. 68, a supposition equivalent, in the author's opinion, to the assumption  $K = -1$ . In other arguments, the Lorentz transformations are assumed outright.

It is interesting to note that Milne regards  $t$ -time as more fundamental than  $\tau$ -time, and refers to  $\tau$ -time as "temporary . . . with its ephemeral constant  $t_0$ " (Milne, 1943, p. 17). This differs from the present view, for we have shown that  $\tau$ -time is definable *before*  $t$ -time, that its definition does *not* require  $t_0$ , and that the uniqueness of  $t$ -time requires axioms which are not involved in the definition of  $\tau$ -time. Also,  $t_0$  is seen to be on the same logical footing as  $c$ , both being observable conventional constants.

9. *Actual Observables*.—Assuming now that one of our models approximates to the external world, a problem of obvious importance is the determination of the constants  $h$ ,  $\kappa$ ,  $c$ ,  $\tau_0$ ,  $t_0$  which have arisen, *i.e.* of  $K$ ,  $c$ , and  $t_0$  since knowledge of these is sufficient. It is now necessary to construct actual instead of ideal experiments; one, for the determination of  $c$ , is already well known, so we may assume that  $c$  is known and that there remain  $K$  and  $t_0$  to be determined. For this purpose we can use such material as observations of spectral red-shift, apparent size, apparent brightness, distribution, and orientation of extra-galactic nebulae, and observation of sky brightness. There are also various determinations of the age of the universe, which therefore give  $t_0$  provided atomic phenomena are used; these will not be discussed here, but an account of this work has been given recently (Bok, 1946). With regard to observables connected with extra-galactic nebulae, we can calculate theoretical expressions for them and deduce correlations which can be compared with the correlations actually observed. The data necessary to give  $K$  and  $t_0$  are not yet available, but it can reasonably be expected that before long observations will not only give these constants but will also provide tests of the various relativistic theories of cosmology.

Calculations of this kind, particularly in relation to the Lemaitre models of general relativity, have frequently been studied by the author and by other writers (McCrea, 1935; Milne, 1935; Walker, 1934, 1940 *a*), and it will not, therefore, be necessary to give full details in the present paper. Our chief concern will be to find the degree of accuracy necessary to produce  $K$  and  $t_0$ , and to point out the agreement between calculations based on  $\tau$ -time and those based on  $t$ -time, an agreement which is clearly to be expected since the two systems of time-keeping are logically consistent. The use of a particular time-scale depends upon whether nebulae are regarded as being at relative rest or in relative uniform motion, the former being given by the  $\tau$ -scale and the latter by the  $t$ -scale. It is thus not sensible to say that one of these motions is the "true" motion, and this is confirmed by the agreement mentioned above. Other writers have not taken this view and have given theoretical correlations which appear to make a true motion determinate.† In our opinion this is due to an inadequate description of the supposed mean nebula; this point will be mentioned again in §§ 11, 12.

10. *Red-shift*.—For the purpose of correlating observables we shall express them in terms of the co-ordinate distance  $r$  from  $O$  in the space  $\Sigma_3$  associated with conventional  $\tau$ -time. From (5) and (7) the metric of  $\Sigma_3$  is (1) with  $K_0 = K/c^2 t_0^2$ , and this holds in all cases. The geodesic distance from  $O$  is

$$l = \int_0^r \frac{dr}{\sqrt{1 + K r^2 / 4 c^2 t_0^2}} = r - \frac{K r^3}{12 c^2 t_0^2} + O(r^5). \quad (12)$$

In conventional  $\tau$ -time with zero at the present instant  $T_0$ , light which reaches us at  $T_0$  left a nebula at distance  $l$  at an instant  $T$  given by  $\tau = -l/c$ . The frequency at  $T$  of a Fraunhofer line which has present frequency  $\nu_0$  was therefore  $\nu = \nu_0 e^{-l/ct_0}$ , and the Doppler ratio for this line is, from (12),

$$D = \nu_0/\nu = e^{l/ct_0} = 1 + \frac{r}{ct_0} + \frac{r^2}{2c^2 t_0^2} + O(r^3). \quad (13)$$

Thus  $D > 1$  and there is a red-shift, of amount depending upon  $l$ .

The proportionate shift at wave-length  $\lambda$  is  $\delta\lambda/\lambda = D - 1$ , and this, we see, is independent of  $\lambda$ . Observations agree with this independence and therefore with the constancy of  $\kappa$  for all frequencies, deduced by us from the dimensional axiom.

The calculation of  $D$  can also be carried out in  $t$ -time and gives the familiar result

$$D = t_0/t, \quad (13')$$

† See particularly Hubble and Tolman, 1935.

where  $t$  is the instant at which the light was emitted. This agrees with (13) because of (8) with  $\tau = -l/c$ .

11. *Distance from Apparent Size*.—This observable is

$$\Delta = (A/\Omega)^{1/2},$$

where  $\Omega$  is the solid angle subtended by the distant nebula at the observer, and  $A$  is the actual normal area of the nebula. In practice,  $A$  is estimated from near nebulae, a fact which must be taken into account in the theoretical calculation.

Previous calculations of  $\Delta$  in a Lemaitre model (Walker, 1934, § 7) can be applied to both  $\tau$ -time and  $t$ -time. They were carried out in space-time with metric (after transformation)  $R^2(d\tau^2 - c^{-2}d\epsilon^2)$ , where  $d\epsilon$  is given by (1) with  $K_0 = K/c^2 t_0^2$ , and  $R$  is a function of  $\tau$ ; we now have  $R \equiv 1$  for  $\tau$ -time and  $R \equiv t/t_0$  (and  $d\tau/t_0 = dt/t$ ) for  $t$ -time. In the Lemaitre model, distance from apparent size is  $R\tau(1 + Kr^2/4c^2 t_0^2)^{-1}$ , the area  $A$  being assumed measured at the distant nebula in the appropriate local scale of length. Applying this result to  $\tau$ -time,  $A$  for a mean nebula is constant by the dimensional axiom, and can therefore be measured at any instant. Putting  $R = 1$ , therefore, we get

$$\Delta = \frac{r}{1 + Kr^2/4c^2 t_0^2} = r - \frac{Kr^3}{4c^2 t_0^2} + O(r^5). \quad (14)$$

In  $t$ -time,  $R = t/t_0 = 1/D$ , and the Lemaitre distance is therefore the above  $\Delta$  multiplied by  $D^{-1}$ . Now, however, lengths which were constant in  $\tau$ -time are increasing with  $t$ , and  $A$  as measured at the nebula is proportional to  $t^2$ . The observer's estimate of  $A$  from nebulae in his vicinity is therefore  $(t_0/t)^2 = D^2$  times the area of the distant nebula, and since the  $A$  in  $\Delta = (A/\Omega)^{1/2}$  is the observer's estimate, we see that the Lemaitre distance must be multiplied by  $D$ . Thus  $\Delta$  is given by (14) as before.

12. *Distance from Apparent Brightness*.—This observable,  $\Delta'$ , is obtained by comparing the apparent with the absolute brightness of a nebula. The observed (apparent) brightness is corrected for what Hubble calls the "energy effects", due to the red-shift but independent of any particular theory as to its cause. The absolute brightness is constant in  $\tau$ -time by the dimensional axiom, and is in practice estimated from nebulae in the observer's vicinity. In  $\tau$ -time the calculation giving the theoretical formula for  $\Delta'$  is straightforward and similar to that for  $\Delta$ ; it is easily verified that

$$\Delta' = \Delta. \quad (15)$$

In  $t$ -time the absolute brightness, being a time-rate of emission of energy, decreases as  $t$  increases and is proportional to  $1/t$ . Local estimates must therefore be increased by the factor  $t_0/t = D$  to give the true brightness at the instant of emission. But the observed apparent brightness must also be increased by the factor  $D$  to allow for the "dimming factor", *i.e.* the diminution in the rate at which photons are received due to the recession of the source. These two corrections therefore cancel in the calculation of  $\Delta'$ , and we finally get the same formula as in  $\tau$ -time. A more detailed discussion of this problem has been given elsewhere (Walker, 1946).

13. *Number Counts*.—Mean nebulae are supposed distributed according to the Cosmological Principle, and the corresponding points in  $\Sigma_3$  are therefore distributed uniformly. If  $n$  is the number per unit proper volume, then  $n$  is constant in time by the dimensional axiom, and if  $N(r)$  is the number of nebulae within co-ordinate distance  $r$  of  $O$ , then from the metric of  $\Sigma_3$  we find

$$N(r) = 4\pi n \int_0^r \frac{r'^2 dr'}{(1 + Kr'^2/4c^2 t_0^2)^3} = \frac{4}{3}\pi n r^3 \left( 1 - \frac{9Kr^2}{20c^2 t_0^2} \right) + O(r^7). \quad (16)$$

14. *Sky Brightness*.—From the definitions of  $\Delta'$  and  $n$ , the rate at which energy is received on unit normal area from all nebulae within a small volume  $dV$  of  $\Sigma_3$  is  $D^{-1}\Delta'^{-2}In dV$ , where  $I$  and  $n$  are constants and  $\Delta'$  is the distance of  $dV$ . The factor  $D^{-1}$  is included to allow for the diminution of energy due to red-shift, and  $I$  is the rate at which a mean nebula radiates energy in unit solid angle. It follows from (15) and (14) that the energy received from all nebulae within co-ordinate distance  $r$  and small solid angle  $d\Omega$  is  $L(r)d\Omega$  where



$$L(r) = I_n \int_0^r \frac{D^{-1} \Delta'^{-2} r^2 dr}{(1 + Kr^2/4c^2 t_0^2)^3} = I_n \int_0^r \frac{D^{-1} dr}{1 + Kr^2/4c^2 t_0^2}.$$

From (12) and (13),  $(1 + Kr^2/4c^2 t_0^2)^{-1} = ct_0 D^{-1} dD/dr$ , and we find

$$L(r) = ct_0 I_n (1 - D^{-1}). \quad (17)$$

Since  $D > 1$  for all  $r$ , we see at once that the total sky brightness is finite when  $K \leq 0$ , and also when  $K > 0$  provided it is assumed in this case that each nebula is seen only once, the alternative being infinite sky brightness. When  $K \leq 0$ , the limiting value of  $L$  is  $ct_0 I_n$ .

15. *Orientation*.—It has been shown by the author (1940 a) that observations of nebular orientation may give useful information. Two observable angles  $\Theta$ ,  $\Phi$  and the galactic latitude  $\Lambda$  are involved, and the theoretical results are

$$\begin{aligned} K > 0, \quad \Theta &= \pm \frac{r\sqrt{K}}{ct_0} + O(r^2), \quad \Phi = 0; \\ K = 0, \quad \Theta &= 0, \quad \Phi = 0; \\ K < 0, \quad \Theta &= 0, \quad \Phi = \frac{r}{ct_0} \cos \Lambda \cdot \sqrt{-K} + O(r^2). \end{aligned} \quad (18)$$

16. *Correlations*.—Observables  $D$ ,  $\Delta$ ,  $\Delta'$ ,  $N$ ,  $\Theta$ ,  $\Phi$  have been expressed in terms of  $r$ , and several correlations are obtained by the elimination of  $r$ . These are observable relations, which can provide tests for our theory and enable us to determine  $K$  and  $t_0$ . That these constants are determinate is corroborated by the fact that they would emerge from a complete knowledge of, for example, the  $D$ - $\Delta$  correlation.

The correlations will not be considered in detail here, but some conclusions can be drawn. We see from (13), (14), (15), and (16) that first and second approximations to the relations between  $D$ ,  $\Delta$ ,  $\Delta'$ , and  $N$  give only  $t_0$ . Third order approximations are necessary before these relations will yield  $K$ , even in sign, and it will be a long time before observations are sufficiently accurate for this purpose. It has been suggested (Hubble and Tolman, 1935) that the  $D$ - $N$  correlation might be a way of getting  $K$ , at least in sign, and this appears at first sight to be confirmed by the form of (16). We find, however, from (12), (13); and (16), writing  $\delta = D - 1$ , that

$$N = \text{const} \left( \delta^3 - \frac{3}{2} \delta^4 + \left( \frac{7}{4} - \frac{K}{5} \right) \delta^5 + \dots \right).$$

This indicates how good the observations of  $\delta$  and  $N$  would have to be to give information about  $K$ , and supports my conclusion.

The position is improved when observables  $\Theta$  and  $\Phi$  are admitted, for if the orientation theory leading to (16) is confirmed, then correlations between  $\Theta$ ,  $\Phi$ ,  $D$  and  $\Delta$  (or  $\Delta'$ ) to the first order give both  $t_0$  and  $K$ . This appears to be the most promising method of obtaining the desired information.

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XXV.—Some Continuant Determinants arising in Physics and Chemistry. By D. E. Rutherford, M.A., B.Sc.. D.Math., United College, University of St Andrews. (With Four Text-figures.)

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## I. INTRODUCTION

In many problems in physics and chemistry certain determinants of large order and of a special type require to be evaluated. The determinants in question usually arise from some kind of secular equation, and in most cases they owe their origin to a system of particles of one kind or another which, in their equilibrium positions, form a regular lattice, each particle being acted upon by its nearest neighbours and perhaps by a fixed boundary. These forces may be assumed to be elastic in character, to the degree of approximation required. For simplicity we shall refer to these particles as atoms although they may be electrons, molecules or even material particles in the Newtonian sense, according to the nature of the problem under investigation. In particular we may mention the occurrence of these determinants in problems involving the solution of Schrödinger's Wave Equation for permitted energy levels and in determining the distribution of electric charge in crystals, metals and large molecules. They have also been used by Born in his investigations of crystal structure by means of X-rays. It is hoped therefore that the results achieved in this paper will be of interest to the physicist and chemist as well as to the mathematician.

In § 2 some types arising from one-dimensional problems are considered, while with the aid of the mathematical analysis of § 3 certain two-dimensional types are dealt with in § 4. The hexagonal chains of common occurrence in organic chemistry give rise to determinants which are special cases of those evaluated in § 4. Certain three-dimensional types may be treated in the same way as the corresponding two-dimensional types, but others have so far proved intractable.

## 2. ONE-DIMENSIONAL TYPES

Of the determinants which we are about to consider we shall distinguish between what we may call the *fixed boundary* type and the *free boundary* type. These types of determinants may arise in different ways, but we give them these names because they apply appropriately to simple mechanical models of lattices. These models consist of  $n$  equal material particles which are constrained to move in a straight line and which are linked together by equal elastic springs as shown in the following diagrams:—

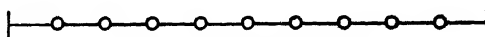


FIG. 1.—Fixed boundary.

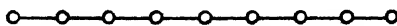


FIG. 2.—Free boundary.

The secular determinants which give the normal periods of vibration for the two cases illustrated above take the forms:

$$\left| \begin{array}{cccc} x & 1 & & \\ 1 & x & 1 & \\ & 1 & x & 1 \\ \dots & \dots & \dots & \dots \\ & & 1 & x & 1 \\ & & & 1 & x \end{array} \right|, \quad \left| \begin{array}{cccc} (x+1) & 1 & & \\ & 1 & x & 1 \\ & & 1 & x & 1 \\ \dots & \dots & \dots & \dots \\ & & & 1 & x & 1 \\ & & & & 1 & (x+1) \end{array} \right| \quad (2.1)$$

It will be observed that the free boundary type is distinguished from the fixed boundary type by the presence of additional elements in the top-left and bottom-right corners. In the simple models described above, the  $n$  atoms are supposed to be identical, but we shall also consider the case where the atoms are of two kinds arranged alternately in the model. All the cases so far mentioned give rise to matrices of the form  $S_m(u, v, a, b)$ , where we define

$$S_{2n}(u, v, a, b) \equiv \begin{bmatrix} (u+b) & 1 & & & & \\ & 1 & v & 1 & & \\ & & 1 & u & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & v & 1 \\ & & & & 1 & u \\ & & & & & 1 \\ & & & & & & 1 & (v+a) \end{bmatrix}_{(2n)}, \quad S_{2n+1}(u, v, a, b) \equiv \begin{bmatrix} (u+b) & 1 & & & & \\ & 1 & v & 1 & & \\ & & 1 & u & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & u & 1 \\ & & & & 1 & v \\ & & & & & 1 \\ & & & & & & 1 & (u+a) \end{bmatrix}_{(2n+1)}$$

We shall also use the further abbreviations

$$R_m(x, a, b) \equiv S_m(x, x, a, b), \quad P_m(x) \equiv R_m(x, 0, 0).$$

With this notation it is clear that the determinants (2.1) take the forms

$$|P_m(x)|, \quad |R_m(x, 1, 1)|$$

respectively.

It was shown by Wolstenholme (Muir, p. 401) that if we write

$$x = 2 \cos \theta,$$

then

$$|P_m(x)| = \frac{\sin(m+1)\theta}{\sin \theta}. \quad (2.2)$$

Using the addition theorem for determinants we deduce that

$$\begin{aligned} |R_m(x, a, b)| &= |P_m(x)| + (a+b) |P_{m-1}(x)| + ab |P_{m-2}(x)| \\ &= \frac{\sin(m+1)\theta + (a+b) \sin m\theta + ab \sin(m-1)\theta}{\sin \theta}. \end{aligned} \quad (2.3)$$

With the aid of elementary trigonometry we can derive the following particular cases of the last formula:—

$$\begin{aligned} |R_m(x, 1, 1)| &= \frac{2 \sin m\theta(1 + \cos \theta)}{\sin \theta}, \\ |R_m(x, 1, 0)| &= \frac{\sin(2m+1)\theta/2}{\sin \theta/2}. \end{aligned}$$

It is clear from (2.2) that  $|P_m(x)|$  vanishes when and only when

$$\theta = \frac{k\pi}{m+1}, \quad [k=1, \dots, m],$$

that is, when

$$x = 2 \cos \frac{k\pi}{m+1}, \quad [k=1, \dots, m].$$

These are in fact the  $m$  distinct roots of the equation  $|P_m(x)| = 0$ . Since  $|P_m(x)|$  is a polynomial in  $x$  of degree  $m$  and since the coefficient of  $x^m$  in this polynomial is  $+1$ , we conclude that

$$|P_m(x)| = \prod_{k=1}^m \left( x - 2 \cos \frac{k\pi}{m+1} \right). \quad (2.4)$$

Similarly, it may be shown that

$$|R_m(x, 1, 1)| = \prod_{k=1}^m \left( x - 2 \cos \frac{k\pi}{m} \right), \quad (2.5)$$

and that

$$|R_m(x, 1, 0)| = \prod_{k=1}^m \left( x - 2 \cos \frac{2k\pi}{2m+1} \right).$$

The determinant of the matrix  $S_m(u, v, 0, 0)$  is easily evaluated by multiplying each row

in which  $u$  occurs by  $\sqrt{v}$ , each row in which  $v$  occurs by  $\sqrt{u}$ , and dividing each column containing  $u$  by  $\sqrt{u}$ , and each column containing  $v$  by  $\sqrt{v}$ . This done, it is patent that

$$|S_{2n}(u, v, o, o)| = |P_{2n}(\sqrt{uv})| = \frac{\sin(2n+1)\theta}{\sin \theta},$$

$$|S_{2n+1}(u, v, o, o)| = \sqrt{\frac{u}{v}} |P_{2n+1}(\sqrt{uv})| = \sqrt{\frac{u}{v}} \cdot \frac{\sin(2n+2)\theta}{\sin \theta} = \frac{u \sin(2n+2)\theta}{\sin 2\theta},$$

where we now write  $\sqrt{(uv)} = 2 \cos \theta$ . Using the addition theorem once more, we can obtain formulæ for  $|S_m(u, v, a, b)|$ . In particular

$$\begin{aligned} |S_{2n}(u, v, 1, 1)| &= |S_{2n}(u, v, o, o)| + |S_{2n-1}(v, u, o, o)| + |S_{2n-1}(u, v, o, o)| + |S_{2n-2}(v, u, o, o)| \\ &= \frac{\sin(2n+1)\theta}{\sin \theta} + \frac{(u+v) \sin 2n\theta}{\sin 2\theta} + \frac{\sin(2n-1)\theta}{\sin \theta} \\ &= \frac{(uv+u+v) \sin 2n\theta}{\sin 2\theta}, \\ |S_{2n+1}(u, v, 1, 1)| &= \frac{u \sin(2n+2)\theta + 2\sqrt{(uv)} \sin(2n+1)\theta + v \sin 2n\theta}{\sin 2\theta} \\ &= \frac{v \left\{ \frac{u}{v} \sin(2n+2)\theta + 2\sqrt{\frac{u}{v}} \cdot \sin(2n+1)\theta + \sin 2n\theta \right\}}{\sin 2\theta} \\ &= \frac{2\{\sqrt{u} \sin(n+1)\theta + \sqrt{v} \sin n\theta\} \{\sqrt{u} \cos(n+1)\theta + \sqrt{v} \cos n\theta\}}{\sin 2\theta}. \end{aligned}$$

### 3. THE LATENT ROOTS OF CERTAIN MATRICES

Let  $L(x)$  be a square matrix of order  $\lambda$  such that

$$L(x) = L(o) + xI,$$

where  $I$  denotes the unit matrix. Now  $L(o)$  can be reduced to its classical canonical form  $\Lambda(o)$  by a non-singular matrix  $H$ , such that

$$HL(o)H^{-1} = \Lambda(o),$$

from which it follows that

$$HL(x)H^{-1} = \Lambda(o) + xI.$$

Thus  $H$ , which does not involve  $x$ , also reduces  $L(x)$  to its canonical form

$$\Lambda(x) \equiv \Lambda(o) + xI.$$

Now, if the equation

$$|L(x)| = |L(o) + xI| = 0$$

has roots  $l_1, \dots, l_\lambda$ , then the latent roots of  $L(x)$ , and therefore of  $\Lambda(x)$ , are

$$\kappa_i(x) \equiv x - l_i, \quad [i = 1, \dots, \lambda],$$

for the characteristic equation of  $L(x)$  is

$$|L(x) - \kappa I| = |L(o) + (x - \kappa)I| = |L(x - \kappa)| = 0.$$

Let  $M(x)$  be another square matrix of order  $\mu$  such that

$$M(x) = M(o) + xI,$$

and let  $L(M(x))$  be the partitioned matrix

$$L(M(x)) = L(o) \langle I \rangle + I \langle M(x) \rangle.$$

In the last formula we use the notation  $A \langle B \rangle$  to denote the direct product matrix which is more frequently denoted by  $A \times B$ . Thus, if

$$A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B \equiv \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then

$$A \langle B \rangle = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}.$$

$L(M(x))$  is in fact the matrix of order  $\lambda\mu$  which is obtained from  $L(x)$  by replacing every element  $x$  by the submatrix  $M(x)$  and every element  $i$  by the unit matrix  $I$ . Using a theorem on the product of direct product matrices (MacDuffee, p. 82), we find that

$$\begin{aligned} [H \langle I \rangle] [L(M(x))] [H^{-1} \langle I \rangle] &= [H \langle I \rangle] [L(o \langle I \rangle + I \langle M(x) \rangle)] [H^{-1} \langle I \rangle] \\ &= [HL(o)H^{-1}] \langle I \rangle + [HIH^{-1}] \langle M(x) \rangle \\ &= \Lambda(o) \langle I \rangle + I \langle M(x) \rangle \\ &= \Lambda(M(x)). \end{aligned}$$

It follows that

$$\begin{aligned} |L(M(x))| &= |\Lambda(M(x))| \\ &= \prod_i |\kappa_i(M(x))| \\ &= \prod_i |M(x) - I_i I| \\ &= \prod_i |M(x - I_i)| \\ &= \prod_{i,j} (x - I_i - m_j), \end{aligned}$$

where  $m_1, \dots, m_\mu$  are the roots of the equation  $|M(x)| = 0$ . Since the matrix  $L(M(x))$  is itself the sum of a matrix independent of  $x$  and a matrix which is  $x$  times the unit matrix of order  $\lambda\mu$ , we have the following result:—

**THEOREM:** If the matrix  $L(x)$  is of the form  $L(o) + xI$  and the matrix  $M(x)$  is of the form  $M(o) + xI$ , then the latent roots of the matrix  $L(M(o))$  are  $I_i + m_j$ ;  $i = 1, \dots, \lambda$ ;  $j = 1, \dots, \mu$ , where  $I_1, \dots, I_\lambda$  are the latent roots of  $L(o)$  and  $m_1, \dots, m_\mu$  are the latent roots of  $M(o)$ .

#### 4. TWO-DIMENSIONAL TYPES

In the light of the results of § 3 it is now possible to deal with certain two-dimensional problems associated with both square and hexagonal lattices. Figs. 3 and 4 represent these cases when a fixed boundary is present and when there are two kinds of atoms arranged alternately. The bonds, that is to say the elastic forces, connecting the atoms to each other and to the boundary are represented by lines in the diagrams.

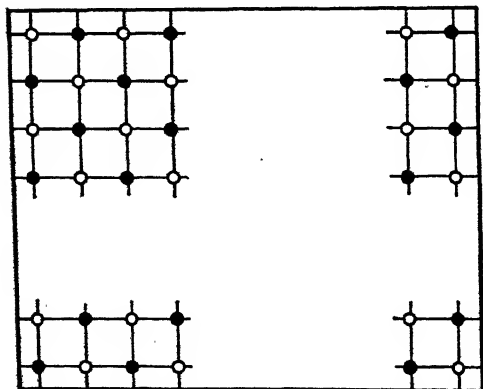


FIG. 3.—Square lattice of  $2m \times 2n$  atoms of two kinds, fixed boundary.

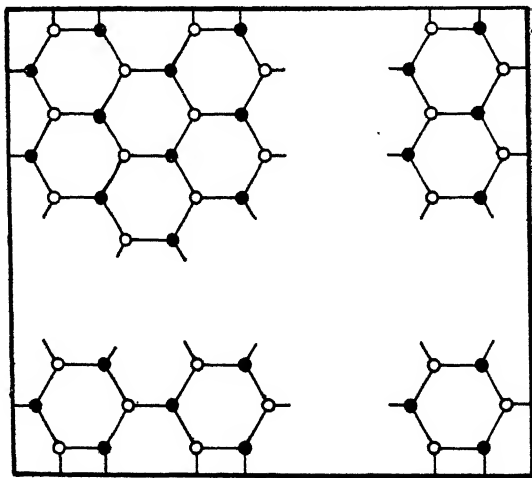


FIG. 4.—Hexagonal lattice of  $(2n+1) \times 2m$  atoms of two kinds, fixed boundary.

The corresponding models of these lattices with free boundaries are obtained from the above by removing the fixed boundaries and the bonds connecting them to the atoms.

*Square Lattice.*—Since the matrices  $P_m(x)$  and  $R_m(x, 1, 1)$  are both of the type discussed in § 3, we conclude from (2.4) and (3.1) that

$$|P_m(P_n(x))| = \prod_{k=1}^m \prod_{j=1}^n \left( x - 2 \cos \frac{k\pi}{m+1} - 2 \cos \frac{j\pi}{n+1} \right), \quad (4.1)$$

and from (2.5) and (3.1) that

$$|R_m(R_n(x, 1, 1), 1, 1)| = \prod_{k=1}^m \prod_{j=1}^n \left( x - 2 \cos \frac{k\pi}{m} - 2 \cos \frac{j\pi}{n} \right). \quad (4.2)$$

The formulæ (4.1) and (4.2) apply to square lattices with fixed and free boundaries respectively in which all the atoms are of the same kind. These results may be extended to the three-dimensional case without difficulty.

If the boundary is fixed and there are two kinds of atoms present, the appropriate determinant is

$$|S_m(S_n(u, v, o, o), S_n(v, u, o, o), o, o)|.$$

On multiplying and dividing the rows and columns of this determinant by  $\sqrt{u}$  and  $\sqrt{v}$  in the same way as those of  $|S_n(u, v, o, o)|$ , we may show that the value of the former determinant is

$$|P_m(P_n(\sqrt{uv}))| = \prod_{k=1}^m \prod_{j=1}^n \left( \sqrt{uv} - 2 \cos \frac{k\pi}{m+1} - 2 \cos \frac{j\pi}{n+1} \right)$$

or

$$\sqrt{(u/v)} |P_m(P_n(\sqrt{uv}))| = \sqrt{(u/v)} \prod_{k=1}^m \prod_{j=1}^n \left( \sqrt{uv} - 2 \cos \frac{k\pi}{m+1} - 2 \cos \frac{j\pi}{n+1} \right)$$

according as the number of atoms is even or odd.

*Hexagonal Lattice.*—Let us write

$$D_{2n}(\phi, q, r, s) = \begin{bmatrix} \phi I_n & qI_n + rJ' \\ qI_n + rJ & sI_n \end{bmatrix}, \quad D_{2n+1}(\phi, q, r, s) = \begin{bmatrix} \phi I_{n+1} & qH' + rK' \\ qH + rK & sI_n \end{bmatrix},$$

where

$$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(n)}, \quad J = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}_{(n)}$$

$$H = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(n \times (n+1))}, \quad K = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}_{(n \times (n+1))}$$

where the dash (') denotes a transposed matrix and where  $\phi, q, r, s$  are non-zero scalars. Taking determinants of both sides of the following matrix equation

$$\begin{bmatrix} I & O \\ -(qI + rJ) & \phi I \end{bmatrix} \begin{bmatrix} \phi I & qI + rJ' \\ qI + rJ & sI \end{bmatrix} = \begin{bmatrix} \phi I & qI + rJ' \\ O & (\phi s - q^2)I - qr(J + J') - r^2JJ' \end{bmatrix},$$

we find that

$$|\phi I| |D_{2n}(\phi, q, r, s)| = |\phi I| |(\phi s - q^2)I - qr(J + J') - r^2JJ'|.$$

If, after cancelling out the factor  $|\phi I|$  on each side, we multiply each odd row and each odd column of the remaining determinant on the right-hand side by  $-1$ , this determinant becomes

$$|(\phi s - q^2)I + qr(J + J') - r^2JJ'|.$$

Further, the matrix  $JJ'$  is the same as the unit matrix  $I$  except that the bottom-right element is zero. It follows that

$$|D_{2n}(\phi, q, r, s)| = (qr)^n \left| R_n \left( \frac{\phi s - q^2 - r^2}{qr}, \frac{r}{q}, 0 \right) \right|.$$

A similar argument shows that

$$|D_{2n+1}(\phi, q, r, s)| = \phi(qr)^n \left| P_n \left( \frac{\phi s - q^2 - r^2}{qr} \right) \right|.$$

Thus, the determinant  $|D_m|$  can be evaluated with the help of (2.2) and (2.3).

The matrix associated with the hexagonal lattice illustrated in fig. 4 takes the form  $S_{2n+1}(U, V, O, O)$ , where we write

$$U = \begin{bmatrix} uI & & & & \\ & Iv & & & \\ & & uI & & \\ & & & Iv & \\ & & & & \ddots \\ & & & & & uI \\ & & & & & & Iv \\ & & & & & & & uI \\ & & & & & & & & Iv \end{bmatrix}_{(2m)}, \quad V = \begin{bmatrix} v & & & & \\ & uI & & & \\ & & Iv & & \\ & & & \ddots & \\ & & & & uI \\ & & & & & Iv \\ & & & & & & uI \\ & & & & & & & Iv \end{bmatrix}_{(2m)}.$$

By rearranging rows and columns it may be shown that

$$\begin{aligned} |S_{2n+1}(U, V, O, O)| &= \begin{vmatrix} I_{n+1}\langle U \rangle & (H' + K')\langle I_{2m} \rangle \\ (H + K)\langle I_{2m} \rangle & I_n\langle V \rangle \end{vmatrix} \\ &= |D_{2n+1}(U, I, I, V)|. \end{aligned}$$

This determinant may be treated similarly to  $|D_{2n+1}(\phi, q, r, s)|$ . Taking determinants of both sides of the matrix equation,

$$\begin{aligned} \begin{bmatrix} I_{n+1}\langle I_{2m} \rangle & 0 \\ -(H + K)\langle I_{2m} \rangle & I_n\langle U \rangle \end{bmatrix} \begin{bmatrix} I_{n+1}\langle U \rangle & (H' + K')\langle I_{2m} \rangle \\ (H + K)\langle I_{2m} \rangle & I_n\langle V \rangle \end{bmatrix} \\ = \begin{bmatrix} I_{n+1}\langle U \rangle & (H' + K')\langle I_{2m} \rangle \\ 0 & I_n\langle UV \rangle - \{(H + K)(H' + K')\}\langle I \rangle \end{bmatrix}, \end{aligned}$$

we find that

$$|U|^n |D_{2n+1}(U, I, I, V)| = |U|^{n+1} |I_n\langle UV \rangle - \{(H + K)(H' + K')\}\langle I \rangle|.$$

Now it is readily verified that  $HH' = KK' = I$  and that  $KH' = J$ ,  $HK' = J'$ . Thus

$$\begin{aligned} |S_{2n+1}(U, V, O, O)| &= |U| |P_n(UV - zI)| \\ &= |U| \left| \prod_{k=1}^n UV - zI - 2 \cos \frac{k\pi}{n+1} I \right| \\ &= |U| \left| \prod_{k=1}^n UV - 4 \cos^2 \frac{k\pi}{(2n+1)} I \right|. \end{aligned}$$

Again, an argument similar to the preceding one shows that

$$|D_2(U, zI, O, V)| = |UV - z^2 I|.$$

On the other hand, if we rearrange the rows and columns of  $D_2(U, zI, O, V)$  in the order  $2m+1, 2, 2m+3, 4, \dots, 2m, 1, 2m+2, 3, 2m+4, \dots, 4m$ , it will be seen that

$$\begin{aligned} |D_2(U, zI, O, V)| &= \begin{vmatrix} vI & zI + J' \\ zI + J & uI \end{vmatrix} \\ &= |D_{4m}(v, z, I, u)| \\ &= z^{2m} \left| R_{2m} \left( \frac{uv - z^2 - 1}{z}, \frac{1}{z}, 0 \right) \right|. \end{aligned}$$

Hence, if we write

$$z_k = 2 \cos \frac{k\pi}{2(n+1)}, \quad \frac{uv - z_k^2 - 1}{z_k} = 2 \cos \phi_k,$$

then

$$\begin{aligned} |S_{2n+1}(U, V, O, O)| &= |U| \prod_{k=1}^n |D_2(U, z_k I, O, V)| \\ &= |U| \prod_{k=1}^n z_k^{2m} |R_{2m}(2 \cos \phi_k, z_k^{-1}, 0)| \\ &= (uv - 1)^m \prod_{k=1}^n \frac{z_k^{2m+1} \sin(2m+1)\phi_k + z_k^{2m} \sin 2m\phi_k}{z_k \sin \phi_k}. \end{aligned}$$

Two particular cases are of special interest. If  $n=1$ , the determinant is

$$|S_3(U, V, O, O)| = 2^m (uv - 1)^m \left\{ \frac{\sin(2m+1)\phi + 2^{-\frac{1}{2}} \sin 2m\phi}{\sin \phi} \right\},$$

where  $2 \cos \phi = (uv - 3)/\sqrt{2}$ . This determinant is the appropriate one for a hexagonal chain of  $6m$  atoms.

On the other hand, if we take  $m=1$ , we have the determinant

$$\begin{aligned} |S_{2n+1}(U, V, O, O)| &= (uv - 1) \prod_{k=1}^n \frac{z_k^3 \sin 3\phi_k + z_k^2 \sin 2\phi_k}{z_k \sin \phi_k} \\ &= (uv - 1) \prod_{k=1}^n (4z_k^2 \cos^2 \phi_k + 2z_k \cos \phi_k - z_k^2) \\ &= (uv - 1) \prod_{k=1}^n \{(uv - z_k^2)^2 - uv\}. \end{aligned}$$

Hence, if we write

$$t_k = \sqrt{(1 + 4z_k^2)} = \sqrt{\left(9 + 8 \cos \frac{k\pi}{(n+1)}\right)},$$

we obtain in the case  $m=1$

$$\begin{aligned} |S_{2n+1}(U, V, O, O)| \\ = (uv - 1) \prod_{k=1}^n \left\{ \sqrt{uv} + \frac{1}{2}(1 + t_k) \right\} \left\{ \sqrt{uv} - \frac{1}{2}(1 + t_k) \right\} \left\{ \sqrt{uv} + \frac{1}{2}(1 - t_k) \right\} \left\{ \sqrt{uv} - \frac{1}{2}(1 - t_k) \right\}. \end{aligned}$$

This determinant arises from a hexagonal chain of  $4n+2$  atoms.

## 5. CONCLUSION

In this paper we have evaluated a number of determinants of a type which are likely to occur in theoretical physics. Some of the simpler cases have in fact been already discussed by other writers. Thus Goodwin (1939) deals with an equation of the type  $|R_m(x, a, a)| = 0$  and discusses its roots at some length. Lennard-Jones (1937) has evaluated a determinant which, on rearranging the rows and columns, becomes  $|D_{2\nu}(\epsilon, \beta_a, \beta_s, \epsilon)|$ . Lennard-Jones and Turkevich (1937) considered a determinant which reduces to

$$|D_{2\nu}(\epsilon, \beta_s, \beta_a, \epsilon)| = -\beta_a^2 |D_{2\nu-2}(\epsilon, \beta_a, \beta_s, \epsilon)| - 2(\beta_a \beta_s)^2.$$

These and other cases have also been treated by Coulson (1938) while Born (1942) dealt with  $|S_m(u, v, 1, 1)|$ .

No doubt the methods described here can be extended to cover other cases but in some cases, such as that of the hexagonal lattice with a free boundary, there appears at present to be an insuperable difficulty. Nevertheless, as Ledermann (1944) has shown, the conditions at the boundary can be modified without effecting an essential alteration in the density of the roots of the secular equation. Thus, if the density of the roots rather than their actual value is required, the hexagonal lattice with a free boundary can be replaced by one



with a fixed boundary. Again, although the boundary of the hexagonal lattice in fig. 4 is a rectangular one, we may imagine two opposite edges joined together to form a sort of cylindrical network. This cylinder can then be cut along a different line and opened out again to form a hexagonal lattice with a boundary in the shape of a parallelogram. In this case also there will be no essential modification of the density of the roots of the secular equation, provided the total number of atoms in the lattice is large compared with the number of atoms whose immediate neighbourhood has been modified.

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XXVI.—The Universal Integral Invariants of Hamiltonian Systems and Application to the Theory of Canonical Transformations. By Hwa-Chung Lee, Ph.D. (Edin.), Wuhan University, China. *Communicated by* Sir EDMUND WHITTAKER, F.R.S.

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1. *Introduction.*—Consider a Hamiltonian system of differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad (i = 1, \dots, n), \quad (\text{I.I})$$

where  $H$  is a function of the  $2n$  variables  $q_i$  and  $p_i$  involving in general also the time  $t$ . For each given Hamiltonian function  $H$  the system (1.1) possesses infinitely many absolute and relative integral invariants of every order  $r=1, \dots, 2n$ , which can all be written out when (1.1) is integrated.\* Our interest now is not in these integral invariants, which are possessed by *one* Hamiltonian system, but in those which are possessed by *all* Hamiltonian systems. Such an integral invariant, which is independent of the Hamiltonian  $H$ , is said to be *universal*. A universal relative integral invariant of order  $r$  is  $\oint \omega$ , where  $\omega$  is the *Pfaffian form* †.

$$\omega = \sum_{i=1}^n p_i \delta q_i, \quad (1.2)$$

and consequently (by Stokes's theorem) a universal absolute integral invariant of order 2 is  $\int \omega'$ , where  $\omega'$  is the *exterior derivative* of  $\omega$ , namely

$$\omega' = \sum_i \delta p_i \delta q_i, \quad (1.3)$$

and then the integrals  $\int \omega'^2, \int \omega'^3, \dots, \int \omega'^n$  are universal absolute integral invariants of orders  $4, 6, \dots, 2n$  respectively,<sup>†</sup> where

$$\left. \begin{aligned} \omega'^2 &= \sum_{i,j} \delta p_i \delta q_i \delta p_j \delta q_j = 2! \sum_{i<j} \delta p_i \delta q_i \delta p_j \delta q_j, \\ \omega'^3 &= \sum_{i,j,k} \delta p_i \delta q_i \delta p_j \delta q_j \delta p_k \delta q_k = 3! \sum_{i<j<k} \delta p_i \delta q_i \delta p_j \delta q_j \delta p_k \delta q_k, \\ . &. . . . . \\ \omega'^n &= \sum_{i_1, \dots, i_n} \delta p_{i_1} \delta q_{i_1} \dots \delta p_{i_n} \delta q_{i_n} = n! \delta p_{i_1} \delta q_{i_1} \dots \delta p_{i_n} \delta q_{i_n}, \end{aligned} \right\} \quad (\text{I.4})$$

and the integrals  $\oint \omega^1, \oint \omega^2, \dots, \oint \omega^{n-1}$  are universal relative integral invariants of orders 3, 5,  $\dots, 2n-1$  respectively, where

[illegible]

\* For  $r=1$ , see E. T. Whittaker, *Analytical Dynamics*, 2nd ed., § 117. For any  $r$ , see E. Goursat, *Leçons sur le problème de Pfaff*, p. 214.

† Whittaker, *l.c.*, p. 272.

‡ Goursat, *l.c.*, p. 231.

§ This follows from a known theorem (Goursat, *l.c.*, p. 212) by noting that  $(\omega\omega')' = \omega'^2$ ,  $(\omega\omega'^2)' = \omega'^3$ , . . . ,  $(\omega\omega'^{n-1})' = \omega'^n$ .

We shall show that these Poincaré's integrals are essentially the only universal integral invariants,\* *i.e.* there is no universal absolute integral invariant of odd order, and the only universal absolute integral invariants of even orders  $2, 4, 6, \dots, 2n$  are, to within constant multiples, respectively the following:—

$$\int \omega', \quad \int \omega'^2, \quad \int \omega'^3, \quad \dots, \quad \int \omega'^n, \quad (1.6)$$

and there is no universal relative integral invariant of even order, and the only universal relative integral invariants of odd orders  $1, 3, 5, \dots, 2n-1$  are respectively constant multiples of

$$\oint \omega, \quad \oint \omega \omega', \quad \oint \omega \omega'^2, \quad \dots, \quad \oint \omega \omega'^{n-1}. \quad (1.7)$$

Our method is tensorial, as this is much more powerful than the method generally used in questions connected with Hamilton's canonical equations.

2. *Methodical Preparations.*—For unification of notation the  $2n$  variables  $q_1, \dots, q_n, p_1, \dots, p_n$  will be written as  $x^1, \dots, x^n, x^{n+1}, \dots, x^{2n}$  respectively, thus

$$x^i = q_i, \quad x^{n+i} = p_i, \quad (i = 1, \dots, n). \quad (2.1)$$

Reserving the Latin indices  $i, j, \dots$  for the range  $1, \dots, n$ , we shall use Greek indices for the double range  $1, \dots, 2n$ . The summation convention for Greek indices will be understood. Thus (1.1) may be condensed into

$$\frac{dx^a}{dt} = \epsilon^{ab} H_b \quad \left( H_b = \frac{\partial H}{\partial x^b} \right), \quad (2.2)$$

where

$$\epsilon^{i,j} = 0, \quad \epsilon^{i,n+j} = \delta_j^i, \quad \epsilon^{n+j,i} = -\delta_j^i, \quad \epsilon^{n+i,n+j} = 0, \quad (2.3)$$

a delta (with two indices) having value 1 or 0 according as the two indices are equal or unequal.

In view of (2.3) we may represent  $\epsilon^{ab}$  by the skew-symmetrical matrix

$$\epsilon^{ab} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (2.4)$$

where  $O$  and  $I$  denote the zero and unit matrices of order  $n$  respectively. This skew-symmetrical matrix, of order  $2n$ , is non-singular, and so has an inverse which is again skew-symmetrical and which we write

$$\epsilon_{ab} = \begin{pmatrix} O & -I \\ I & 0 \end{pmatrix}, \quad (2.5)$$

whence

$$\epsilon_{i,j} = 0, \quad \epsilon_{i,n+j} = -\delta_j^i, \quad \epsilon_{n+j,i} = \delta_j^i, \quad \epsilon_{n+i,n+j} = 0. \quad (2.6)$$

As  $\epsilon^{ab}$  and  $\epsilon_{ab}$  are inverse to each other, we have the relations

$$\epsilon^{a\sigma} \epsilon_{\sigma\beta} = \delta_\beta^a, \quad \epsilon_{a\sigma} \epsilon^{\sigma\beta} = \delta_a^\beta. \quad (2.7)$$

The skew-symmetry of the two epsilons is expressed by the equations

$$\epsilon^{ab} = -\epsilon^{ba}, \quad \epsilon_{ab} = -\epsilon_{ba}. \quad (2.8)$$

We shall have to make use of the *variational equations*† of the Hamiltonian system (2.2), which are obtained by operating on (2.2) by the differential symbol  $\delta$ . The two differential symbols  $d$  and  $\delta$  being commutative, the variational system of (2.2) is then

$$\frac{d(\delta x^a)}{dt} = \epsilon^{a\beta} H_{\beta\gamma} \delta x^\gamma \quad \left( H_{\beta\gamma} = \frac{\partial H_\beta}{\partial x^\gamma} = \frac{\partial^2 H}{\partial x^\beta \partial x^\gamma} \right). \quad (2.9)$$

3. *Absolute Integral Invariants.*—Let

$$\Omega_r = A_{a_1 \dots a_r} \delta x^{a_1} \dots \delta x^{a_r} \quad (3.1)$$

be an exterior differential form of degree  $r$  ( $1 \leq r \leq 2n$ ), where  $A_{a_1 \dots a_r}$  is skew-symmetrical in the indices  $a_1, \dots, a_r$ , and is a collection of functions of the variables  $x$  and of the time  $t$ .

\* That  $\oint \omega$  is the only relative integral invariant of the first order for every Hamiltonian system was believed to be true and posed to me by Professor J. S. Wang, to whom I am indebted.

† Whittaker, *L.C.*, p. 269.

By definition, the  $r$ -ple integral  $\int \Omega_r$  is an *absolute integral invariant* (of order  $r$ ) of the Hamiltonian system (2.2) if, in consequence of this system (2.2) and of its variational system (2.9), we have  $\frac{d}{dt} \int \Omega_r = 0$  for an arbitrary ( $r$ -dimensional) domain of integration, whence

$$\frac{d}{dt} \Omega_r = 0$$

or

$$\frac{d}{dt} (A_{a_1 \dots a_r} \delta x^{a_1} \dots \delta x^{a_r}) = 0.$$

In consequence of (2.9), this may be written:

$$\left[ \frac{dA_{a_1 \dots a_r}}{dt} + \epsilon^{\lambda \mu} (A_{\lambda a_2 \dots a_r} H_{\mu a_1} + \dots + A_{a_1 \dots a_{r-1} \lambda} H_{\mu a_r}) \right] \delta x^{a_1} \dots \delta x^{a_r} = 0.$$

Now it is easily seen that the expression inside the brackets ( ) is skew-symmetrical in the indices  $a_1, \dots, a_r$ , so that the whole expression inside the brackets [ ] is skew-symmetrical in these indices. Since the ( $r$ -dimensional) volume-element  $\delta x^{a_1} \dots \delta x^{a_r}$  is an arbitrary skew-symmetrical quantity, its coefficient in the above identity must vanish:

$$\frac{dA_{a_1 \dots a_r}}{dt} + \epsilon^{\lambda \mu} (A_{\lambda a_2 \dots a_r} \delta_{a_1}^\nu + \dots + A_{a_1 \dots a_{r-1} \lambda} \delta_{a_r}^\nu) H_{\mu \nu} = 0,$$

which, when the first term on the left is differentiated in consequence of (2.2), becomes

$$\frac{\partial A_{a_1 \dots a_r}}{\partial t} + \frac{\partial A_{a_1 \dots a_r}}{\partial x^\lambda} \epsilon^{\lambda \mu} H_\mu + \epsilon^{\lambda \mu} (\delta_{a_1}^\nu A_{\lambda a_2 \dots a_r} + \dots + \delta_{a_r}^\nu A_{a_1 \dots a_{r-1} \lambda}) H_{\mu \nu} = 0. \quad (3.2)$$

This is the condition that  $\int \Omega_r$  be an absolute integral invariant of the Hamiltonian system (2.2).

In order that this integral invariant  $\int \Omega_r$  be universal, (3.2) must hold for an arbitrary function  $H$ , whence

$$\frac{\partial A_{a_1 \dots a_r}}{\partial t} = 0, \quad \frac{\partial A_{a_1 \dots a_r}}{\partial x^\lambda} \epsilon^{\lambda \mu} = 0, \quad (3.3)$$

$$\epsilon^{\lambda \mu} (\delta_{a_1}^\nu A_{\lambda a_2 \dots a_r} + \dots + \delta_{a_r}^\nu A_{a_1 \dots a_{r-1} \lambda}) + \epsilon^{\lambda \nu} (\delta_{a_1}^\mu A_{\lambda a_2 \dots a_r} + \dots + \delta_{a_r}^\mu A_{a_1 \dots a_{r-1} \lambda}) = 0. \quad (3.4)$$

Of the two conditions in (3.3), the first implies that  $A_{a_1 \dots a_r}$  is independent of  $t$ , and the second (where  $\mu$  is a free index) is equivalent to  $\frac{\partial A_{a_1 \dots a_r}}{\partial x^\lambda} = 0$ , so that  $A_{a_1 \dots a_r}$  is also independent of the  $x$ 's. Thus the  $A$ 's are all constants.

For the solution of (3.4) we consider first a few simple cases.

4. The Case  $r=1$ .—In this case, (3.1) is a Pfaffian form  $\Omega_1 = A_a \delta x^a$ , and (3.4) becomes

$$\epsilon^{\lambda \mu} \delta_a^\nu A_\lambda + \epsilon^{\lambda \nu} \delta_a^\mu A_\lambda = 0.$$

Contracting this for  $\nu$  and  $a$  we find  $(2n+1)\epsilon^{\lambda \mu} A_\lambda = 0$ , whence (transvecting by  $\epsilon_{\mu \alpha}$ )

$$(2n+1)A_a = 0,$$

i.e.  $A_a = 0$ , which is the only solution of (3.4) for  $r=1$ . We have then

THEOREM I. *There is no universal absolute integral invariant of the first order.*

5. The Case  $r=2$ .—In this case, (3.1) is of the form  $\Omega_2 = A_{a\beta} \delta x^a \delta x^\beta$ , and (3.4) becomes

$$\epsilon^{\lambda \mu} (\delta_a^\nu A_{\lambda \beta} + \delta_\beta^\nu A_{a \lambda}) + \epsilon^{\lambda \nu} (\delta_a^\mu A_{\lambda \beta} + \delta_\beta^\mu A_{a \lambda}) = 0. \quad (5.1)$$

Contracting this for  $\nu$  and  $\beta$  we find

$$2n\epsilon^{\lambda \mu} A_{a \lambda} + \delta_a^\mu A = 0 \quad (A = A_{a\beta} \epsilon^{a\beta}),$$

whence (transvecting by  $\epsilon_{\mu\beta}$ )

$$2nA_{\alpha\beta} + \epsilon_{\alpha\beta}A = 0,$$

or

$$A_{\alpha\beta} = -\frac{A}{2n}\epsilon_{\alpha\beta}, \quad (5.2)$$

which is the most general solution of (5.1), *i.e.* of (3.4) for  $r=2$ , where  $A$  is an arbitrary constant. Now by (2.6) and (2.1) we find

$$\epsilon_{\alpha\beta}\delta x^\alpha\delta x^\beta = \Sigma(\delta p_i\delta q_i - \delta q_i\delta p_i) = 2\Sigma\delta p_i\delta q_i, \quad (5.3)$$

so that (5.2) implies

$$\Omega_2 = A_{\alpha\beta}\delta x^\alpha\delta x^\beta = -\frac{A}{2n}\epsilon_{\alpha\beta}\delta x^\alpha\delta x^\beta = -\frac{A}{n}\Sigma\delta p_i\delta q_i,$$

and we have

**THEOREM 2.** *Apart from an arbitrary constant factor, there is only one universal absolute integral invariant of the second order, namely  $\int \Sigma \delta p_i \delta q_i$ .*

6. *The Case  $r=3$ .*—In this case, (3.1) has the form  $\Omega_3 = A_{\alpha\beta\gamma}\delta x^\alpha\delta x^\beta\delta x^\gamma$ , and (3.4) becomes

$$\epsilon^{\lambda\mu}(\delta_\alpha^\nu A_{\lambda\beta\gamma} + \delta_\beta^\nu A_{\alpha\lambda\gamma} + \delta_\gamma^\nu A_{\alpha\beta\lambda}) + \epsilon^{\lambda\nu}(\delta_\alpha^\mu A_{\lambda\beta\gamma} + \delta_\beta^\mu A_{\alpha\lambda\gamma} + \delta_\gamma^\mu A_{\alpha\beta\lambda}) = 0.$$

Contracting this for  $\nu$  and  $\gamma$  we find

$$(2n-1)\epsilon^{\lambda\mu}A_{\alpha\beta\lambda} - \delta_\alpha^\mu A_\beta + \delta_\beta^\mu A_\alpha = 0 \quad (A_\alpha = A_{\alpha\beta\gamma}\epsilon^{\beta\gamma}),$$

whence (transvecting by  $\epsilon_{\mu\gamma}$ )

$$(2n-1)A_{\alpha\beta\gamma} - \epsilon_{\alpha\gamma}A_\beta + \epsilon_{\beta\gamma}A_\alpha = 0.$$

Transvecting this by  $\epsilon^{\alpha\beta}$  we find

$$(2n+1)A_\gamma = 0,$$

*i.e.*  $A_\gamma = 0$ , and therefore the above equation implies  $A_{\alpha\beta\gamma} = 0$ , which is the only solution of (3.4) for  $r=3$ . Hence

**THEOREM 3.** *There is no universal absolute integral invariant of the third order.*

7. *The Case  $r=4$ .*—In this case, (3.1) has the form  $\Omega_4 = A_{\alpha\beta\gamma\sigma}\delta x^\alpha\delta x^\beta\delta x^\gamma\delta x^\sigma$ , and (3.4) becomes

$$\begin{aligned} & \epsilon^{\lambda\mu}(\delta_\alpha^\nu A_{\lambda\beta\gamma\sigma} + \delta_\beta^\nu A_{\alpha\lambda\gamma\sigma} + \delta_\gamma^\nu A_{\alpha\beta\lambda\sigma} + \delta_\sigma^\nu A_{\alpha\beta\gamma\lambda}) \\ & + \epsilon^{\lambda\nu}(\delta_\alpha^\mu A_{\lambda\beta\gamma\sigma} + \delta_\beta^\mu A_{\alpha\lambda\gamma\sigma} + \delta_\gamma^\mu A_{\alpha\beta\lambda\sigma} + \delta_\sigma^\mu A_{\alpha\beta\gamma\lambda}) = 0. \end{aligned} \quad (7.1)$$

Contracting this for  $\nu$  and  $\sigma$  we find

$$(2n-2)\epsilon^{\lambda\mu}A_{\alpha\beta\gamma\lambda} + \delta_\alpha^\mu A_{\beta\gamma} - \delta_\beta^\mu A_{\alpha\gamma} + \delta_\gamma^\mu A_{\alpha\beta} = 0 \quad (A_{\alpha\beta} = A_{\alpha\beta\gamma\sigma}\epsilon^{\gamma\sigma}),$$

whence (transvecting by  $\epsilon_{\mu\sigma}$ )

$$(2n-2)A_{\alpha\beta\gamma\sigma} + \epsilon_{\alpha\sigma}A_{\beta\gamma} + \epsilon_{\beta\sigma}A_{\gamma\alpha} + \epsilon_{\gamma\sigma}A_{\alpha\beta} = 0.$$

Transvecting this by  $\epsilon^{\alpha\beta}$  we find

$$2nA_{\gamma\sigma} + \epsilon_{\gamma\sigma}A = 0 \quad (A = A_{\alpha\beta}\epsilon^{\alpha\beta} = A_{\alpha\beta\gamma\sigma}\epsilon^{\alpha\beta}\epsilon^{\gamma\sigma}),$$

whence

$$A_{\alpha\beta} = -\frac{A}{2n}\epsilon_{\alpha\beta}$$

and therefore

$$A_{\alpha\beta\gamma\sigma} = \frac{A}{2n(2n-2)}(\epsilon_{\alpha\sigma}\epsilon_{\beta\gamma} + \epsilon_{\beta\sigma}\epsilon_{\gamma\alpha} + \epsilon_{\gamma\sigma}\epsilon_{\alpha\beta}), \quad (7.2)$$

which is the most general solution of (7.1), *i.e.* of (3.4) for  $r=4$ ,  $A$  being an arbitrary constant.

Now, by (7.2),

$$\Omega_4 = A_{\alpha\beta\gamma\sigma} \delta x^\alpha \delta x^\beta \delta x^\gamma \delta x^\sigma = \frac{3A}{2n(2n-2)} (\epsilon_{\alpha\beta} \delta x^\alpha \delta x^\beta) (\epsilon_{\gamma\sigma} \delta x^\gamma \delta x^\sigma) \\ = \frac{3A}{n(n-1)} (\sum \delta p_i \delta q_i) (\sum \delta p_i \delta q_i) \quad \text{by (5.3),}$$

and we have

THEOREM 4. *Apart from an arbitrary constant factor, there is only one universal absolute integral invariant of the fourth order, namely  $\int \sum \delta p_i \delta q_i \delta p_i \delta q_i$ .*

8. *The General Case.*—Continuing the solution of (3.4) for any  $r$  in the same way it is clear that we have

THEOREM 5. *There is no universal absolute integral invariant of any odd order, and apart from an arbitrary constant factor there is only one universal absolute integral invariant of every even order  $2s$ , namely  $\int \sum \delta p_{i_1} \delta q_{i_1} \dots \delta p_{i_s} \delta q_{i_s}$ .*

In fact, contracting (3.4) for  $v$  and  $a_r$  we find

$$(2n-r+2)\epsilon^{\lambda\mu} A_{a_1 \dots a_{r-1} \lambda} \\ + (-1)^{r-2} \delta_{a_1}^\mu A_{a_2 a_3 \dots a_{r-1}} + (-1)^{r-3} \delta_{a_2}^\mu A_{a_1 a_3 \dots a_{r-1}} + \dots + \delta_{a_{r-1}}^\mu A_{a_1 a_2 \dots a_{r-2}} = 0 \\ \text{(where } A_{a_1 \dots a_{r-2}} = A_{a_1 \dots a_r} \epsilon^{a_{r-1} a_r}),$$

whence (transvecting by  $\epsilon_{\mu a_r}$ )

$$(2n-r+2)A_{a_1 \dots a_r} \\ + (-1)^{r-2} \epsilon_{a_1 a_r} A_{a_2 a_3 \dots a_{r-1}} + (-1)^{r-3} \epsilon_{a_2 a_r} A_{a_1 a_3 \dots a_{r-1}} + \dots + \epsilon_{a_{r-1} a_r} A_{a_1 a_2 \dots a_{r-2}} = 0.$$

Transvecting this by  $\epsilon^{a_1 a_2}$  we have

$$(2n-r+4)A_{a_3 \dots a_r} \\ + (-1)^{r-4} \epsilon_{a_3 a_r} A_{a_1 a_2 \dots a_{r-1}} + (-1)^{r-5} \epsilon_{a_4 a_r} A_{a_1 a_3 \dots a_{r-1}} + \dots + \epsilon_{a_{r-1} a_r} A_{a_1 a_2 \dots a_{r-2}} = 0 \\ \text{(where } A_{a_1 \dots a_{r-4}} = A_{a_1 \dots a_{r-2}} \epsilon^{a_{r-3} a_{r-2}} = A_{a_1 \dots a_r} \epsilon^{a_{r-3} a_{r-2}} \epsilon^{a_{r-1} a_r}).$$

Continue to transvect this by  $\epsilon^{a_3 a_4}$ , then the result again by  $\epsilon^{a_5 a_6}$ , and so on, we see that if  $r$  is odd, the result after transvection by  $\epsilon^{a_{r-2} a_{r-1}}$  is

$$(2n+1)A_{a_r} = 0,$$

whence (by tracing the equations back successively)

$$A_{a_r} = 0, \quad A_{a_{r-2} a_{r-1} a_r} = 0, \quad A_{a_{r-4} \dots a_r} = 0, \quad \dots$$

and finally  $A_{a_1 \dots a_r} = 0$  which is the only solution of (3.4) in this odd case. The first half of Theorem 5 is proved.

On the other hand, if  $r$  is even, the result after transvection by  $\epsilon^{a_{r-3} a_{r-2}}$  is

$$2n A_{a_{r-1} a_r} + \epsilon_{a_{r-1} a_r} A = 0 \quad (A = A_{a_1 \dots a_r} \epsilon^{a_1 a_2} \dots \epsilon^{a_{r-1} a_r})$$

or

$$A_{a_{r-1} a_r} = -\frac{A}{2n} \epsilon_{a_{r-1} a_r} \quad \text{i.e.} \quad A_{a\beta} = -\frac{A}{2n} \epsilon_{a\beta},$$

whence (tracing back successively) we obtain the solution of (3.4) in this even case, which involves the arbitrary constant  $A$  and is of the form

$$A_{a_1 \dots a_r} = \frac{(-1)^{\frac{r}{2}} A}{2n(2n-2) \dots (2n-r+2)} (\epsilon_{a_1 a_2} \dots \epsilon_{a_{r-1} a_r} + \dots)$$

where the unwritten terms in the brackets  $()$  are of the same type as the first term and are such

that the entire expression inside the brackets is skew-symmetrical in the indices  $\alpha_1, \dots, \alpha_r$ ; thus there are  $(r-1)(r-3)\dots 3\cdot 1$  terms in all. Hence

$$\begin{aligned}\Omega_r &= A_{\alpha_1 \dots \alpha_r} \delta x^{\alpha_1} \dots \delta x^{\alpha_r} = \frac{(-1)^{\frac{r}{2}} 1 \cdot 3 \dots (r-1) A}{2n(2n-2) \dots (2n-r+2)} (\epsilon_{\alpha_1 \alpha_2} \delta x^{\alpha_1} \delta x^{\alpha_2}) \dots (\epsilon_{\alpha_{r-1} \alpha_r} \delta x^{\alpha_{r-1}} \delta x^{\alpha_r}) \\ &= (-1)^s \frac{1 \cdot 3 \dots (2s-1) A}{n(n-1) \dots (n-s+1)} (\sum \delta p_{i_1} \delta q_{i_1}) \dots (\sum \delta p_{i_s} \delta q_{i_s}),\end{aligned}$$

where we set  $r=2s$ . The second half of Theorem 5 is also proved.

9. *Relative Integral Invariants*.—If  $\oint \Omega_r$  is a relative integral invariant of order  $r$ , then  $\int \Omega'_r$  is an absolute integral invariant of order  $r+1$ , and conversely,\* where  $\Omega'_r$  is the exterior derivative of  $\Omega_r$ . Hence, when universality is understood, we have by § 8

$$\Omega'_r = 0 \quad \text{for even } r, \quad (9.1)$$

$$\Omega'_r = C \sum_{i_1, \dots, i_s} \delta p_{i_1} \delta q_{i_1} \delta q_{i_2} \delta q_{i_2} \dots \delta p_{i_s} \delta q_{i_s} \quad \text{for odd } r=2s-1, \quad (9.2)$$

where  $C$  is a constant.

If (9.1) is the case, then  $\dagger \Omega_r$  itself must be the exterior derivative of another exterior form  $\Lambda$  (of degree  $r-1$ ):  $\Omega_r = \Lambda'$ , so that

$$\begin{aligned}\oint \Omega_r &= \int \Omega'_r && \text{(by Stokes's theorem)} \\ &= \int \Lambda'' = 0 && \text{(since } \Lambda'' = 0 \text{ by Poincaré's theorem),}\end{aligned}$$

and we have

THEOREM 6. *There is no universal relative integral invariant of even order.*

If (9.2) is the case, we may write it in the form

$$(\Omega_r - C \sum_{i_1, \dots, i_s} p_{i_1} \delta q_{i_1} \delta p_{i_2} \delta q_{i_2} \dots \delta p_{i_s} \delta q_{i_s})' = 0,$$

whence, by the same reason, the expression inside brackets must be of the form  $\Lambda'$ , so that

$$\Omega_r = C \sum p_{i_1} \delta q_{i_1} \delta q_{i_2} \delta q_{i_2} \dots \delta p_{i_s} \delta q_{i_s} + \Lambda',$$

therefore

$$\oint \Omega_r = C \oint \sum p_{i_1} \delta q_{i_1} \delta p_{i_2} \delta q_{i_2} \dots \delta p_{i_s} \delta q_{i_s} \quad (\text{since } \oint \Lambda' = \int \Lambda'' = 0),$$

and we have

THEOREM 7. *Apart from an arbitrary constant factor, there is only one universal relative integral invariant of odd order  $2s-1$ , namely*

$$\oint \sum p_{i_1} \delta q_{i_1} \delta p_{i_2} \delta q_{i_2} \dots \delta p_{i_s} \delta q_{i_s} \quad (s=1, \dots, n).$$

10. *Systems which possess a given Integral Invariant.*† We now study the converse problem suggested by the preceding results, namely that of determining all the systems of differential equations which possess one of the integral invariants obtained above. Let

$$\frac{dq_i}{dt} = Q_i, \quad \frac{dp_i}{dt} = P_i$$

be a system of  $2n$  differential equations where the  $Q$ 's and  $P$ 's are unknown functions of the  $q$ 's,  $p$ 's, and  $t$ . We may always write such a system in the form §

$$\frac{dx^\alpha}{dt} = \epsilon^{\alpha\beta} X_\beta, \quad (10.1)$$

\* Goursat, *l.c.*, p. 212.

† E. Cartan, *Leçons sur les invariants intégraux*, p. 73. Also Goursat, *l.c.*, p. 105.

‡ Compare Whittaker, *l.c.*, § 116.

§ Where  $X_i = -P_i$ ,  $X_{n+i} = Q_i$ .

the  $X$ 's being unknown functions of the  $x$ 's and  $t$ . The variational system of (10.1) is

$$\frac{d(\delta x^\alpha)}{dt} = \epsilon^{\alpha\beta} \frac{\partial X_\beta}{\partial x^\gamma} \delta x^\gamma. \quad (10.2)$$

We find in consequence of (10.2)

$$\frac{d}{dt}(\epsilon_{\alpha\beta} \delta x^\alpha \delta x^\beta) = Y_{\alpha\beta} \delta x^\alpha \delta x^\beta, \quad (10.3)$$

where

$$Y_{\alpha\beta} = \frac{\partial X_\alpha}{\partial x^\beta} - \frac{\partial X_\beta}{\partial x^\alpha},$$

which is skew-symmetrical in  $\alpha$  and  $\beta$ . If then  $\int \epsilon_{\alpha\beta} \delta x^\alpha \delta x^\beta$  is an absolute integral invariant of the system (10.1), the left-hand side of (10.3) vanishes, so that  $Y_{\alpha\beta} = 0$  or

$$\frac{\partial X_\alpha}{\partial x^\beta} - \frac{\partial X_\beta}{\partial x^\alpha} = 0, \quad (10.4)$$

which implies the existence of a function  $H$  of the  $x$ 's and  $t$  such that

$$X_\beta = \frac{\partial H}{\partial x^\beta}, \quad (10.5)$$

and therefore (10.1) is a Hamiltonian system. We have thus given a short proof of the known theorem that a system possessing the absolute integral invariant  $\int \Sigma \delta p_i \delta q_i (= \frac{1}{2} \int \epsilon_{\alpha\beta} \delta x^\alpha \delta x^\beta)$  is necessarily a Hamiltonian system. We wish to prove more, namely

**THEOREM 8.** *Every system which possesses one of the integrals*

$$\int \Sigma \delta p_{i_1} \delta q_{i_1} \dots \delta p_{i_s} \delta q_{i_s} \quad (s = 1, \dots, n)$$

*as absolute integral invariant is a Hamiltonian system.*

Since by (5.3)

$$2^s \Sigma \delta p_{i_1} \delta q_{i_1} \dots \delta p_{i_s} \delta q_{i_s} = (\epsilon_{\alpha_1 \beta_1} \delta x^{\alpha_1} \delta x^{\beta_1}) \dots (\epsilon_{\alpha_s \beta_s} \delta x^{\alpha_s} \delta x^{\beta_s}),$$

we have

$$\frac{d}{dt} \{ (\epsilon_{\alpha_1 \beta_1} \delta x^{\alpha_1} \delta x^{\beta_1}) \dots (\epsilon_{\alpha_s \beta_s} \delta x^{\alpha_s} \delta x^{\beta_s}) \} = 0,$$

or, by (10.3),

$$(Y_{\alpha_1 \beta_1} \delta x^{\alpha_1} \delta x^{\beta_1}) \dots (\epsilon_{\alpha_s \beta_s} \delta x^{\alpha_s} \delta x^{\beta_s}) + \dots + (\epsilon_{\alpha_1 \beta_1} \delta x^{\alpha_1} \delta x^{\beta_1}) \dots (Y_{\alpha_s \beta_s} \delta x^{\alpha_s} \delta x^{\beta_s}) = 0.$$

The  $s$  terms on the left of this equation are equal to one another, so that we have simply

$$(Y_{\alpha_1 \beta_1} \delta x^{\alpha_1} \delta x^{\beta_1}) \dots (\epsilon_{\alpha_s \beta_s} \delta x^{\alpha_s} \delta x^{\beta_s}) = 0,$$

or, removing the arbitrary skew-symmetrical quantity  $\delta x^{\alpha_1} \delta x^{\beta_1} \dots \delta x^{\alpha_s} \delta x^{\beta_s}$ ,

$$Y_{\alpha_1 \beta_1} \epsilon_{\alpha_2 \beta_2} \dots \epsilon_{\alpha_s \beta_s} + \dots = 0,$$

where the unwritten terms of the sum on the left are of the same type as the first term and are such that the sum as a whole is a skew-symmetrical quantity. Transvecting the above equation by  $\epsilon^{\alpha_1 \beta_1} \dots \epsilon^{\alpha_s \beta_s}$ , the result may be written  $Y_{\alpha_1 \beta_1} = 0$ , whence we have again (10.4) and therefore (10.5). Theorem 8 is proved.

**THEOREM 9.** *Every system which possesses one of the integrals*

$$\oint \Sigma p_{i_1} \delta q_{i_1} \delta p_{i_2} \delta q_{i_2} \dots \delta p_{i_s} \delta q_{i_s} \quad (s = 1, \dots, n)$$

*as relative integral invariant is a Hamiltonian system.*

For, the  $s$ th integral in Theorem 9 equals the  $s$ th integral in Theorem 8 by Stokes's theorem, so that if a system possesses the  $s$ th integral in Theorem 9 as relative integral invariant it also possesses the  $s$ th integral in Theorem 8 as absolute integral invariant, whence the system is Hamiltonian by Theorem 8. Theorem 9 is proved.



11. *Application to the Theory of Canonical Transformations.*—Making use of the above results, we can determine the canonical transformations by a very simple argument. By definition a transformation from the variables  $q_1, \dots, q_n, p_1, \dots, p_n$  to the new variables  $\bar{q}_1, \dots, \bar{q}_n, \bar{p}_1, \dots, \bar{p}_n$  is called *canonical* if it changes every Hamiltonian system (1.1) again into a Hamiltonian system

$$\frac{d\bar{q}_i}{dt} = \frac{\partial \bar{H}}{\partial \bar{p}_i}, \quad \frac{d\bar{p}_i}{dt} = -\frac{\partial \bar{H}}{\partial \bar{q}_i}, \quad (i=1, \dots, n). \quad (11.1)$$

Then, since  $\int \Sigma \delta p_i \delta q_i$  and  $\int \Sigma \delta \bar{p}_i \delta \bar{q}_i$  are both universal absolute integral invariants of the second order, they must coincide except for a constant factor (Theorem 2). Hence a canonical transformation must satisfy the condition

$$\Sigma \delta \bar{p}_i \delta \bar{q}_i = c \Sigma \delta p_i \delta q_i, \quad (11.2)$$

where  $c$  is a constant.\*

This condition, being necessary, is also sufficient. To prove this we first write (11.2) in the form

$$\epsilon_{\rho\sigma} \delta \bar{x}^\rho \delta \bar{x}^\sigma = c \epsilon_{\alpha\beta} \delta x^\alpha \delta x^\beta \quad [\text{see (5.3)}],$$

whence

$$\epsilon_{\rho\sigma} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \frac{\partial \bar{x}^\sigma}{\partial x^\beta} = c \epsilon_{\alpha\beta}. \quad (11.3)$$

An equivalent form of this is

$$\epsilon^{\alpha\beta} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \frac{\partial \bar{x}^\sigma}{\partial x^\beta} = c \epsilon^{\rho\sigma}. \quad (11.4)$$

The sufficiency proof is to show that if (11.2) is satisfied, (1.1) is transformed into (11.1). We establish this by deducing (11.1) from (1.1) making use of (11.2) as follows:—

$$\begin{aligned} \frac{d\bar{x}^\rho}{dt} &= \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \frac{dx^\alpha}{dt} + \frac{\partial \bar{x}^\rho}{\partial t} \\ &= \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \epsilon^{\alpha\beta} \frac{\partial H}{\partial x^\beta} + \frac{\partial \bar{x}^\rho}{\partial t} \quad [\text{by (1.1) condensed in the form (2.2)}] \\ &= c \epsilon^{\rho\sigma} \frac{\partial x^\beta}{\partial \bar{x}^\sigma} \frac{\partial H}{\partial x^\beta} + \frac{\partial \bar{x}^\rho}{\partial t} \quad [\text{by (11.2) in the equivalent form (11.4)}] \\ &= c \epsilon^{\rho\sigma} \frac{\partial (cH)}{\partial \bar{x}^\sigma} + \frac{\partial \bar{x}^\rho}{\partial t} = \epsilon^{\rho\sigma} \frac{\partial \bar{H}}{\partial \bar{x}^\sigma}, \end{aligned}$$

which is a condensed form of (11.1), where

$$\bar{H} = cH + \phi, \quad (11.5)$$

and  $\phi$  is defined by

$$\epsilon^{\rho\sigma} \frac{\partial \phi}{\partial \bar{x}^\sigma} = \frac{\partial \bar{x}^\rho}{\partial t}$$

or

$$\frac{\partial \phi}{\partial x^\alpha} = \epsilon^{\rho\sigma} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \frac{\partial \bar{x}^\sigma}{\partial t}, \quad (11.6)$$

which is *completely integrable* since its condition of integrability is satisfied because of (11.3):

$$\frac{\partial}{\partial x^\beta} \left( \frac{\partial \phi}{\partial x^\alpha} \right) - \frac{\partial}{\partial x^\alpha} \left( \frac{\partial \phi}{\partial x^\beta} \right) = \frac{\partial}{\partial t} \left( \epsilon^{\rho\sigma} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \frac{\partial \bar{x}^\sigma}{\partial x^\beta} \right) = \frac{\partial}{\partial t} (c \epsilon_{\alpha\beta}) = 0.$$

The sufficiency proof is complete.†

\* A different method (without using integral invariants) of obtaining this condition, in the equivalent form (11.3), and of deducing the equations (11.5) and (11.6), is indicated in H. C. Lee, "Sur les transformations des congruences hamiltoniennes," *Comptes rendus* (Paris), CCVI (1938), p. 1431. A generalisation of the method is given in H. C. Lee, "On even-dimensional skew-metric spaces and their groups of transformations," *Amer. Journ. of Math.*, LXVII (1945), p. 327.

† We have tacitly supposed for generality that the canonical transformation involves the time  $t$ . If in particular the new variables  $\bar{x}$  as functions of the old  $x$ 's are independent of  $t$ , (11.6) reduces to  $\frac{\partial \phi}{\partial x^\alpha} = 0$ , whence  $\phi$  is a constant (or a function of  $t$ ). Since to a Hamiltonian function can be added an arbitrary function of  $t$  without affecting the corresponding Hamiltonian system, the transformation law (11.5) in this case may simply be taken in the form  $\bar{H} = cH$ .

Now, (11.2) may be written  $(\sum \bar{p}_i \delta \bar{q}_i - c \sum \bar{p}_i \delta q_i)' = 0$ , whence  $\sum \bar{p}_i \delta \bar{q}_i - c \sum \bar{p}_i \delta q_i$  is an exact differential  $\delta w$  of a function  $w(q_1, \dots, q_n, p_1, \dots, p_n, t)$ ,  $t$  being unvaried under the symbol  $\delta$ . Hence the condition (11.2) for canonical transformation is equivalent to

$$\sum \bar{p}_i \delta \bar{q}_i = c \sum \bar{p}_i \delta q_i + \delta w. \quad (11.7)$$

With the aid of  $w$ , we can actually integrate the system (11.6). For this purpose we first write the condition (11.7) in the form

$$\bar{g}_\rho \delta \bar{x}^\rho - c g_\alpha \delta x^\alpha = \delta w,$$

where  $g_i = p_i$ ,  $g_{n+i} = 0$ , and similarly  $\bar{g}_i = \bar{p}_i$ ,  $\bar{g}_{n+i} = 0$  ( $i = 1, \dots, n$ ). We find

$$\frac{\partial g_\alpha}{\partial x^\beta} - \frac{\partial g_\beta}{\partial x^\alpha} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -\epsilon_{\alpha\beta},$$

and similarly

$$\frac{\partial \bar{g}_\rho}{\partial \bar{x}^\sigma} - \frac{\partial \bar{g}_\sigma}{\partial \bar{x}^\rho} = -\epsilon_{\rho\sigma}.$$

Then, the above condition gives

$$\frac{\partial w}{\partial x^\alpha} = \bar{g}_\rho \frac{\partial \bar{x}^\rho}{\partial x^\alpha} - c g_\alpha,$$

whence

$$\begin{aligned} \frac{\partial^2 w}{\partial t \partial x^\alpha} &= \frac{\partial}{\partial t} \left( \bar{g}_\rho \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \right) - 0 = \bar{g}_\rho \frac{\partial^2 \bar{x}^\rho}{\partial t \partial x^\alpha} + \frac{\partial \bar{g}_\rho}{\partial t} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \\ &= \bar{g}_\rho \frac{\partial^2 \bar{x}^\rho}{\partial x^\alpha \partial t} + \left( \frac{\partial \bar{g}_\rho}{\partial \bar{x}^\sigma} - \frac{\partial \bar{g}_\sigma}{\partial \bar{x}^\rho} \right) \frac{\partial \bar{x}^\sigma}{\partial t} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} + \frac{\partial \bar{g}_\sigma}{\partial \bar{x}^\rho} \frac{\partial \bar{x}^\sigma}{\partial t} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \\ &= \bar{g}_\rho \frac{\partial^2 \bar{x}^\rho}{\partial x^\alpha \partial t} - \epsilon_{\rho\sigma} \frac{\partial \bar{x}^\sigma}{\partial x^\alpha} \frac{\partial \bar{x}^\rho}{\partial t} + \frac{\partial \bar{g}_\sigma}{\partial \bar{x}^\rho} \frac{\partial \bar{x}^\sigma}{\partial t} \frac{\partial \bar{x}^\rho}{\partial x^\alpha}, \end{aligned}$$

so that

$$\begin{aligned} \epsilon_{\mu\sigma} \frac{\partial \bar{x}^\rho}{\partial x^\alpha} \frac{\partial \bar{x}^\sigma}{\partial t} &= \bar{g}_\rho \frac{\partial^2 \bar{x}^\rho}{\partial x^\alpha \partial t} + \frac{\partial \bar{g}_\rho}{\partial \bar{x}^\sigma} \frac{\partial \bar{x}^\sigma}{\partial x^\alpha} \frac{\partial \bar{x}^\rho}{\partial t} - \frac{\partial^2 w}{\partial t \partial x^\alpha} \\ &= \frac{\partial}{\partial x^\alpha} \left( \bar{g}_\rho \frac{\partial \bar{x}^\rho}{\partial t} \right) - \frac{\partial^2 w}{\partial x^\alpha \partial t} = \frac{\partial}{\partial x^\alpha} \left( \bar{g}_\rho \frac{\partial \bar{x}^\rho}{\partial t} - \frac{\partial w}{\partial t} \right), \end{aligned}$$

and therefore (11.6) is integrated into

$$\phi = \bar{g}_\rho \frac{\partial \bar{x}^\rho}{\partial t} - \frac{\partial w}{\partial t} + \text{arbitrary function of } t.$$

Hence (11.5) may be written in the form

$$\bar{H} = cH - \frac{\partial w}{\partial t} + \sum_{i=1}^n \bar{p}_i \frac{\partial \bar{q}_i}{\partial t}, \quad (11.8)$$

dropping an additive function of  $t$ .

The above results may be stated in

**THEOREM 10.** *A necessary and sufficient condition for canonical transformation is (11.2), where  $c$  is a constant, which condition is equivalent to (11.7), where  $w$  is a function of  $(q_1, \dots, q_n, p_1, \dots, p_n, t)$ . The corresponding change of Hamiltonian function is given by (11.8).*

Note that by (2.3) the left-hand side of (11.4) may be written

$$\sum_{i=1}^n \left\{ \frac{\partial \bar{x}^\rho}{\partial q_i} \frac{\partial \bar{x}^\sigma}{\partial p_i} - \frac{\partial \bar{x}^\sigma}{\partial p_i} \frac{\partial \bar{x}^\rho}{\partial q_i} \right\},$$

which is the well-known *Poisson's bracket-expression*  $(\bar{x}^\rho, \bar{x}^\sigma)$ . Hence the condition (11.4) for canonical transformation decomposes into

$$(\bar{q}_i, \bar{q}_j) = 0, \quad (\bar{p}_i, \bar{p}_j) = 0, \quad (\bar{q}_i, \bar{p}_j) = \begin{cases} 0 & \text{when } i \neq j, \\ c & \text{when } i = j. \end{cases} \quad (11.9)$$

\* Many authors, using other methods, come to this conclusion with  $c=1$  only. See notably *Encykl. der Math. Wiss.*, III, 3, p. 454; *Handbuch der Physik*, Band V, p. 99; Forsyth, *Theory of Differential Equations*, part IV, vol. V, p. 399. Although the distinction between  $c=1$  and  $c \neq 1$  is slight and is not very important in application, it is a difference in theory.

Note also that another equivalent form of (11.3) is

$$\epsilon_{\rho\sigma} = \epsilon \epsilon_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\rho} \frac{\partial x^\beta}{\partial \bar{x}^\sigma} \quad (11.10)$$

of which the right-hand side may be written in the form  $-c[\bar{x}^\rho, \bar{x}^\sigma]$ , where  $[u, v]$  denotes the well-known *Lagrange's bracket-expression*

$$\sum_{i=1}^n \left\{ \frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right\}.$$

Hence the condition (11.10) for canonical transformation decomposes into

$$[\bar{q}_i, \bar{q}_j] = 0, \quad [\bar{p}_i, \bar{p}_j] = 0, \quad [q_i, \bar{p}_j] = \begin{cases} 0 & \text{when } i \neq j, \\ c^{-1} & \text{when } i = j. \end{cases} \quad (11.11)$$

It is seen that if the constant  $c$  is unity, a canonical transformation is a *contact transformation*.\* We emphasise this fact by saying that *the group of canonical transformations*† *contains the group of contact transformations as a proper subgroup*.

We can define a canonical transformation in a specific manner‡ by the equations

$$\left. \begin{aligned} M_s(q_1, \dots, q_n, \bar{q}_1, \dots, \bar{q}_n, t) &= 0 \quad (s=1, \dots, m; \ 0 \leq m \leq n), \\ \bar{p}_i &= \frac{\partial W}{\partial \bar{q}_i} + \sum_{s=1}^m \lambda_s \frac{\partial M_s}{\partial \bar{q}_i}, \\ -c\bar{p}_i &= \frac{\partial W}{\partial q_i} + \sum_{s=1}^m \lambda_s \frac{\partial M_s}{\partial q_i}, \end{aligned} \right\} (i=1, \dots, n), \quad (11.12)$$

where  $W$  is a function of  $(q_1, \dots, q_n, \bar{q}_1, \dots, \bar{q}_n, t)$  such that

$$W(q_1, \dots, q_n, \bar{q}_1, \dots, \bar{q}_n, t) = w(q_1, \dots, q_n, p_1, \dots, p_n, t).$$

We find

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial W}{\partial t} + \sum_{i=1}^n \frac{\partial W}{\partial \bar{q}_i} \frac{\partial \bar{q}_i}{\partial t}, \\ \sum_{i=1}^n \bar{p}_i \frac{\partial \bar{q}_i}{\partial t} &= \sum_{i=1}^n \left( \frac{\partial W}{\partial \bar{q}_i} + \sum_{s=1}^m \lambda_s \frac{\partial M_s}{\partial \bar{q}_i} \right) \frac{\partial \bar{q}_i}{\partial t} \\ &= \sum_{i=1}^n \frac{\partial W}{\partial \bar{q}_i} \frac{\partial \bar{q}_i}{\partial t} + \sum_{s=1}^m \lambda_s \left( \sum_{i=1}^n \frac{\partial M_s}{\partial \bar{q}_i} \frac{\partial \bar{q}_i}{\partial t} \right) \\ &= \sum_{i=1}^n \frac{\partial W}{\partial \bar{q}_i} \frac{\partial \bar{q}_i}{\partial t} - \sum_{s=1}^m \lambda_s \frac{\partial M_s}{\partial t}, \end{aligned}$$

so that (11.8) becomes in this case

$$\bar{H} = cH - \frac{\partial W}{\partial t} - \sum_{s=1}^m \lambda_s \frac{\partial M_s}{\partial t}. \quad (11.13)$$

In particular, if  $c=1$ , (11.13) is known as the law of change of Hamiltonian function § corresponding to the contact transformation (11.12). However, the form (11.13) of the law is less significant than the general form (11.8), because the multipliers  $\lambda_1, \dots, \lambda_m$  are quantities irrelevant to the canonical transformation, being merely parasites of the process of analysis.

\* For the conditions of a contact transformation in terms of Lagrange's and Poisson's bracket-expressions, see, e.g., Whittaker, *l.c.*, pp. 298–300.

† That the totality of canonical transformations forms a group can be seen from any one of the conditions (11.2), (11.3), (11.4), (11.7), and (11.10).

‡ Compare, for the special case of a contact transformation, Whittaker, *l.c.*, p. 295.

§ Whittaker, *l.c.*, p. 309.

## XXVII.—Expansions of Lamé Functions into Series of Legendre Functions.\*

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## INTRODUCTION

1. THE object of this paper is to investigate expansions of Lamé functions into series of associated Legendre functions. Representations of Lamé *polynomials* by (terminating) series of associated Legendre polynomials are not entirely new. In fact they are almost as old as the theory of Lamé polynomials itself. In a letter to Borchardt, published in Crelle's *Journal*, Heine (1859 *a*) points out that the characteristic feature of his own approach (as compared to Lamé's original treatment) is the representation of Lamé's polynomials by series of Legendre functions (Heine, 1859 *b*) instead of series of powers (as in Lamé's investigations). In Heine's investigations the series of Legendre functions appear quite naturally from some considerations on ellipsoidal surface harmonics and their connection with spherical surface harmonics. The main theoretical interest, in Heine's opinion, of his series comes from their coefficients being deducible from the coefficients of certain orthogonal substitutions.

It seems that Heine's developments, like so many other valuable parts of his *Handbuch der Kugelfunktionen*, have been overlooked by subsequent workers in the field. Darwin (1901) rediscovered, apparently independently from Heine, and used also numerically the expansions of Lamé polynomials into series of associated Legendre polynomials, but his work too was to share the fate of Heine's. In the modern presentations of the theory of Lamé polynomials known to me, one finds only the power-series. Humbert (1926) only mentions Darwin's work, which he thinks to be too involved, and Strutt (1932) remarks that Darwin gives a different representation of Lamé polynomials, without going into further details.

When I started on the investigations contained in this paper I was not aware of Heine's and Darwin's developments. This proved to be a fortunate circumstance, for so I chose quite a different approach providing a wider basis from which one could get not only the series dealt with by Heine and Darwin, but actually all the different types of series of associated Legendre functions representing (polynomial or transcendental) Lamé functions. My starting-point was the integral equation satisfied by Lamé functions, in connection with Ince's developments of Lamé functions into what he termed Fourier-Jacobi expansions (Ince, 1940 *b*).

Fourier-Jacobi expansions are not quite new in the theory of Lamé polynomials either, though I did not know this at that time. Heine (1878, pp. 372 and 377) has two different types of them, one (*cf.* also Heine, 1859 *b*, p. 95) closely related to his expansions into Legendre functions. Darwin (1901) too used these expansions. Hobson (1892) even discussed such expansions in connection with certain *transcendental* Lamé functions. However, all these beginnings have been more or less neglected by more recent authors. It is, in my opinion, an important step that Ince rediscovered these series and, going beyond his predecessors, realised the fundamental importance of the Fourier-Jacobi series in the theory of Lamé functions, in particular of transcendental Lamé functions, which he was the first to discuss as of equal standing with Lamé polynomials. It is sufficient to recall the analogous developments in the case of Mathieu's equation in order to see that the importance of this new type of expansion can hardly be overestimated.

Mathieu's equation, which may be written in the form

$$\frac{d^2 y}{du^2} + \{h - a \sin^2 u\} y = 0, \quad (1.1)$$

\* This paper was written in August 1940 and is published here without any alteration. Since then I have investigated similar expansions of the general solution of Lamé's equation, and also corresponding expansions of solutions of the Heun equation.

has been studied by earlier writers (Lindemann, 1883; Schubert, 1886) by expanding the solutions into series of powers of  $\sin u$  or  $\cos u$ . Though these researches gave valuable results on the solution of (1.1) with arbitrary values of  $h$ , they failed to give sufficient information about the periodic solutions. It is not too much to say that the theory of the *periodic solutions* of (1.1) is entirely based on the representation by Fourier series of these solutions, and hence it is only natural to expect that the corresponding Fourier-Jacobi series will prove equally successful in connection with Lamé's equation.

Also, the introduction of the Fourier-Jacobi series to the theory of Lamé functions has the advantage that it exhibits the close relation between the theory of Lamé functions on one hand and the theory of Mathieu functions on the other hand. It is hoped that it will be possible to develop a theory of Lamé functions analogous to the well-known theory of Mathieu functions. As a first example, Dr Ince himself studied relations between Lamé functions (Ince, 1940 *b*, § 6) which are analogous to, and generalisations of, certain relations between Mathieu functions. It seems well worth trying to develop the theory of Lamé functions so far that all important theorems on Mathieu functions shall appear as limiting cases of corresponding theorems on Lamé functions.

2. It is well known (Heine, 1878, § 106) that, beside the Fourier expansion of Mathieu functions, there are expansions in terms of Bessel functions—Neumann series—which are of great importance when dealing with functions of imaginary variable or with Mathieu functions of the second kind. These expansions have essentially the same coefficients as the Fourier expansions of the functions concerned, thus showing that the two types of expansions are intimately connected. This connection may be exhibited by the integral equations satisfied by Mathieu functions which convert the Fourier series into the corresponding Neumann series.

In a discussion following a lecture on Lamé functions by the late Dr Ince to the Edinburgh Mathematical Society, I pointed out that a similar procedure should be possible with Lamé functions, and is likely to give new series convergent in domains where the power-series or the Fourier-Jacobi series are not convergent. Dr Ince added that it should be expected that the new series would be in terms of Legendre functions. From what follows, the correctness of both conjectures will be seen.

The procedure I was thinking of at that time is as follows. Let us take Lamé's equation in the form

$$L[y] \equiv \frac{d^2 y}{du^2} + \{h - n(n+1)k^2 \operatorname{sn}^2(u, k)\}y = 0, \quad (2.1)$$

where  $k^2$  is the modulus of the elliptic functions;  $4K$  and  $2iK'$  are the primitive periods of  $\operatorname{sn}(u, k) = \operatorname{sn} u$ . For certain values of  $h$ , (2.1) has solutions which are periodic mod.  $2K$  or  $4K$ , and which will be called Lamé functions of first kind or Lamé functions simply. An even Lamé function of period  $2K$ , say,

$$y = \operatorname{dn} u \sum_{r=0}^{\infty} C_{2r} \cos(2r \operatorname{am} u) \quad (2.2)$$

[Ince, 1940 *b* (3.3)] satisfies the integral equation (Whittaker and Watson, 1927, § 23.6; Ince, 1940 *a*, § 10)

$$y(u) = \lambda \int_{-2K}^{2K} P_n(k \operatorname{sn} u \operatorname{sn} v) y(v) dv, \quad (2.3)$$

where  $P_n$  denotes the Legendre function of degree  $n$ . Substituting (2.2) on the right-hand side of (2.3), we have formally

$$y(u) = \lambda \sum_{r=0}^{\infty} C_{2r} \int_{-2K}^{2K} P_n(k \operatorname{sn} u \operatorname{sn} v) \cos(2r \operatorname{am} v) \operatorname{dn} v dv. \quad (2.4)$$

The integral appearing here can easily be shown to be a constant multiple of  $P_n^{2r}(\operatorname{dn} u)$ . Thus expansions of Lamé functions into series of associated Legendre functions have been obtained. In the main part of the paper, however, another approach will be preferred.

It will be seen later that any of the six quantities  $\operatorname{sn} u$ ,  $\operatorname{cn} u$ ,  $\operatorname{dn} u$ ,  $k \operatorname{sn} u$ ,  $ik \operatorname{cn} u/k'$ ,  $\operatorname{dn} u/k'$  may be taken as the variable of the Legendre functions and consequently there are six different types of expansions. Also it will be seen that these series converge for values of  $u$  for which the power-series and Fourier-Jacobi series are divergent, and hence the new series are a welcome means to investigate the analytic continuation of periodic solutions of Lamé's equation.

These series have another feature which renders them useful even in connection with Lamé *polynomials* where the question of convergence and analytic continuation does not arise at all. It will be seen that the series in associated Legendre functions remain still solutions of Lamé's equation when  $P_n^m$  is replaced by  $Q_n^m$  or by any linear combination  $c_1 P_n^m + c_2 Q_n^m$ . Hence we have a second solution of Lamé's equation the asymptotic behaviour of which for large values of the variable is readily obtained.

In the following pages only the general theory of the solutions is given, and no attempt has been made to utilise them for numerical work. Also I restricted myself for the sake of brevity to solutions of real periods  $2K$  and  $4K$ , though solutions of other real periods are likely to be accessible to the same method.

There are similar developments in connection with the Fuchsian equation of second order with four singularities—the so-called Heun equation.

## PRELIMINARIES

3. For the sake of brevity we shall write

$$s = \operatorname{sn} u, \quad c = \operatorname{cn} u, \quad d = \operatorname{dn} u. \quad (3.1)$$

All elliptic functions, unless the contrary is stated explicitly, have the modulus  $k^2$ . When Lamé functions appear in connection with potential problems (Whittaker and Watson, 1927, § 23.5),  $k$  is real and between 0 and 1. In this case we may put  $k = \sin \theta$ , where the acute angle  $\theta$  is the modular angle. The complementary modulus,  $k' = \sqrt{1 - k^2}$ , is equal to  $\cos \theta$  in this case.

There are the well-known formulæ (Whittaker and Watson, 1927, §§ 22.11 and 22.12)

$$s' = cd, \quad c' = -sd, \quad d' = -k^2 sc, \quad (3.2)$$

where, for instance,  $s' = \frac{ds}{du}$ . There is also the relation (Whittaker and Watson, 1927, § 22.11)

$$d^2 = 1 - k^2 s^2 = k'^2 + k^2 c^2 = c^2 + k'^2 s^2. \quad (3.3)$$

In the following sections  $z$  will denote any of the six quantities  $s$ ,  $c$ ,  $d$ ,  $ks$ ,  $ikc/k'$ ,  $d/k'$ . In these cases  $(1 - z^2)^{\frac{1}{2}}$  shall be determined uniquely by the relations

$$\begin{aligned} (1 - s^2)^{\frac{1}{2}} &= c, & (1 - c^2)^{\frac{1}{2}} &= s, & (1 - d^2)^{\frac{1}{2}} &= ks, & (1 - k^2 s^2)^{\frac{1}{2}} &= d, \\ (1 + k^2 c^2/k'^2)^{\frac{1}{2}} &= d/k', & (1 - d^2/k'^2)^{\frac{1}{2}} &= ikc/k', \end{aligned} \quad (3.4)$$

which are in agreement with (3.3).

4. Ferrer's definition of associated Legendre functions,

$$P_n^m(z) = (1 - z^2)^{\frac{1}{2}m} \frac{d^m P_n(z)}{dz^m}, \quad Q_n^m(z) = (1 - z^2)^{\frac{1}{2}m} \frac{d^m Q_n(z)}{dz^m}, \quad (4.1)$$

will be adopted.  $m$  shall be zero or a positive integer, while  $n$  is unrestricted. This is the definition adopted by Whittaker and Watson (1927) for real values of  $z$  between  $-1$  and  $+1$ . Having defined  $(1 - z^2)^{\frac{1}{2}}$  uniquely by (3.4), and  $m$  being an integer, we may use this definition of the associated Legendre functions without any ambiguity for all values of  $z$ . Hobson (1931) in his standard work uses a different definition. For  $-1 < z < 1$  and integer  $m$  his definition differs from (4.1) in a factor  $(-)^m$ .

We have (Whittaker and Watson, 1927, § 15.51)

$$P_n^{m'} = \frac{dP_n^m}{dz} = (1-z^2)^{-\frac{1}{2}} P_n^{m+1} - \frac{mz}{1-z^2} P_n^m, \quad (4.2)$$

denoting by dashes differentiation with respect to  $z$  and omitting the variable from the symbol of  $P_n^m$ . Also we have (Hobson, 1931, p. 291; the different sign is a consequence of the different definition of  $P_n^m$ )

$$2mz(1-z^2)^{-\frac{1}{2}} P_n^m = P_n^{m+1} + (n-m+1)(n+m)P_n^{m-1}. \quad (4.3)$$

From (4.2) and (4.3) we obtain

$$\begin{aligned} zP_n^{m'} &= z(1-z^2)^{-\frac{1}{2}} \{P_n^{m+1} + m(m-1)z(1-z^2)^{-\frac{1}{2}} P_n^m\} - \frac{m^2 z^2}{1-z^2} P_n^m \\ &= z(1-z^2)^{-\frac{1}{2}} \left\{ \frac{1}{2}(m+1)P_n^{m+1} + \frac{1}{2}(n-m+1)(n+m)(m-1)P_n^{m-1} \right\} - \frac{m^2 z^2}{1-z^2} P_n^m, \end{aligned}$$

and applying once more (4.3) to  $(m+1)P_n^{m+1}$  and  $(m-1)P_n^{m-1}$  respectively,

$$\begin{aligned} zP_n^{m'} &= \frac{1}{2}P_n^{m+2} + \frac{1}{2} \left\{ n(n+1) - \frac{1+z^2}{1-z^2} m^2 \right\} P_n^m \\ &\quad + \frac{1}{2}(n-m+1)(n-m+2)(n+m-1)(n+m)P_n^{m-2}. \end{aligned} \quad (4.4)$$

Similarly

$$\begin{aligned} 3mzP_n^{m'} &= z(1-z^2)^{-\frac{1}{2}} \{3mP_n^{m+1} + (m-2)(m-1)mz(1-z^2)^{-\frac{1}{2}} P_n^m\} - \frac{z^2}{1-z^2} (m^2+2)mP_n^m \\ &= z(1-z^2)^{-\frac{1}{2}} \left\{ \frac{1}{2}(m+1)(m+2)P_n^{m+1} \right. \\ &\quad \left. + \frac{1}{2}(m-2)(m-1)(n-m+1)(n+m)P_n^{m-1} \right\} - \frac{z^2}{1-z^2} (m^2+2)mP_n^m, \end{aligned}$$

and hence

$$\begin{aligned} 3mzP_n^{m'} &= \frac{1}{2}(m+2)P_n^{m+2} + \frac{1}{2} \left\{ n(n+1) - \frac{1+z^2}{1-z^2} (m^2+2) \right\} mP_n^m \\ &\quad + \frac{1}{2}(n-m+1)(n-m+2)(n+m-1)(n+m)(m-2)P_n^{m-2}. \end{aligned} \quad (4.5)$$

Also we have the differential equation

$$P_n^{m''} = \frac{2z}{1-z^2} P_n^{m'} - \left\{ \frac{n(n+1)}{1-z^2} - \frac{m^2}{(1-z^2)^2} \right\} P_n^m. \quad (4.6)$$

(4.4), (4.5) and (4.6) are still valid for functions of negative order defined by

$$P_n^{-m} = (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m. \quad (4.7)$$

All properties of  $P_n^m$  mentioned hitherto are shared by the Legendre functions of second kind,  $Q_n^m$ .

Further, we shall employ the addition theorem for Legendre functions (Whittaker and Watson, 1927, § 15.71)

$$P_n\{zt + (1-z^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}} \cos \omega\} = \sum_{m=-\infty}^{\infty} (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(z) P_n^m(t) \cos m\omega, \quad (4.8)$$

and the formula obtained by differentiating (4.8) with respect to  $\omega$ ,  
 $(1-z^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}} \sin \omega P_n'\{zt + (1-z^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}} \cos \omega\}$

$$= \sum_{m=-\infty}^{\infty} (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(z) P_n^m(t) m \sin m\omega. \quad (4.9)$$

In order to discuss the convergence of our series we need the asymptotic representation of associated Legendre functions for large (positive integer) values of  $m$ . We have [Hobson, 1931, p. 188 (11)]

$$\begin{aligned} P_n^m(z) &= (-)^m \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} P_n^{-m}(z) \\ &= (-)^m \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)\Gamma(m+1)} \left( \frac{1-z}{1+z} \right)^{\frac{1}{2}m} F\left(-n, n+1; m+1; \frac{1-z}{2}\right), \end{aligned}$$

and hence for large positive integer  $m$ ,

$$P_n^m(z) \sim \frac{(-)^m \Gamma(n+m+1)}{\Gamma(n-m+1)\Gamma(m+1)} \left( \frac{1-z}{1+z} \right)^{\frac{1}{2}m}. \quad (4.10)$$

Also [Hobson, 1931, p. 201 (26)],

$$\begin{aligned} Q_n^m(z) &= (-)^m \frac{\Gamma(n+m+1)\Gamma(m-n)}{\Gamma(m+1)} \left\{ \left( \frac{1+z}{1-z} \right)^{\frac{1}{2}m} F\left(-n, n+1; m+1; \frac{1+z}{2}\right) \right. \\ &\quad \left. - e^{\mp n\pi i} \left( \frac{1-z}{1+z} \right)^{\frac{1}{2}m} F\left(-n, n+1; m+1; \frac{1-z}{2}\right) \right\}, \end{aligned}$$

where the upper or lower sign has to be taken according as the imaginary part of  $z$  is positive or negative. Hence

$$Q_n^m(z) \sim (-)^m \frac{\Gamma(n+m+1)\Gamma(m-n)}{\Gamma(m+1)} \left\{ \left( \frac{1+z}{1-z} \right)^{\frac{1}{2}m} - e^{\mp n\pi i} \left( \frac{1-z}{1+z} \right)^{\frac{1}{2}m} \right\}. \quad (4.11)$$

It is worth while remarking that all the recurrence formulæ used in this section are satisfied by  $P_n^m(-z)$  as well as by  $P_n^m(z)$ , and if  $n$  is not an integer, these two solutions are different from each other.  $Q_n^m(z)$  is a linear combination of the two, namely that linear combination which behaves for large values of  $|z|$  like  $\text{const. } z^{-n-1}$ .

#### RECURRENCE RELATIONS. CONVERGENCE OF THE SERIES

5. It will be seen that Lamé's differential equation has solutions of the form

$$y = \sum_{m=0}^{\infty} X_m \Gamma(n-m+1) P_n^m(z) \quad (5.1)$$

and

$$y = \left( \frac{1-\kappa z^2}{1-z^2} \right)^{\frac{1}{2}} \sum_{m=1}^{\infty} X'_m \Gamma(n-m+1) m P_n^m(z), \quad (5.2)$$

where  $\kappa$  is some constant depending on  $k$  alone and the coefficients  $X_m$  and  $X'_m$  depend on  $k$ ,  $k$  and  $n$ .

Substituting (5.1) in (2.1), we shall arrive at the relation

$$\begin{aligned} \sum_{m=0}^{\infty} X_m \Gamma(n-m+1) \{ P_n^{m+2}(z) + (a-bm^2) P_n^m(z) \\ + (n-m+1)(n-m+2)(n+m-1)(n+m) P_n^{m-2}(z) \} = 0, \end{aligned} \quad (5.3)$$

which has to be satisfied identically in  $z$ . Here  $a$  and  $b$  are constants depending on  $k$ ,  $k$  and  $n$ , but independent of  $m$  and  $z$ . (5.3) may be written

$$\begin{aligned} \sum_{m=2}^{\infty} X_{m-2} \Gamma(n-m+3) P_n^m + \sum_{m=0}^{\infty} X_m \Gamma(n-m+1) (a-bm^2) P_n^m \\ + \sum_{m=-2}^{\infty} X_{m+2} \Gamma(n-m+1) (n+m+1)(n+m+2) P_n^m = 0. \end{aligned}$$

In the first two terms ( $m=-2$  and  $-1$ ) of the last sum we use (4.7) and then write the last equation in the form

$$\sum_{m=0}^{\infty} \Gamma(n-m+3) \{ \epsilon_{m-2} X_{m-2} - \beta_{m-2} X_m - \alpha_m X_{m+2} \} P_n^m(z) = 0. \quad (5.4)$$



Here

$$\epsilon_{-2} = \epsilon_{-1} = 0, \quad \epsilon_0 = 2 \quad \text{and} \quad \epsilon_m = 1 \quad \text{when } m \geq 1;$$

$$\alpha_m = -\frac{(n+m+1)(n+m+2)}{(n-m+1)(n-m+2)} \quad (5.5)$$

and

$$\beta_{m-2} = \frac{bm^2 - a}{(n-m+1)(n-m+2)} \quad \text{when } m \neq 1 \quad \text{while} \quad \beta_{-1} = \frac{b-a}{n(n+1)} + 1.$$

In order that (5.4) shall hold identically in  $z$ ,  $\{X_m\}$  must satisfy the system of recurrence relations

$$\epsilon_{m-2}X_{m-2} = \beta_{m-2}X_m + \alpha_m X_{m+2} \quad (m=0, 1, 2, \dots). \quad (5.6)$$

Obviously (5.6) consists of two independent systems of recurrence relations, namely

$$\begin{aligned} 0 &= \beta_{-2}X_0 + \alpha_0 X_2, & 2X_0 &= \beta_0 X_2 + \alpha_2 X_4, \\ X_{2m-2} &= \beta_{2m-2}X_{2m} + \alpha_{2m} X_{2m+2} & (m=2, 3, 4, \dots) \end{aligned} \quad (5.7)$$

for the coefficients with even subscript, and

$$\begin{aligned} 0 &= \beta_{-1}X_1 + \alpha_1 X_3, \\ X_{2m-1} &= \beta_{2m-1}X_{2m+1} + \alpha_{2m+1} X_{2m+3} & (m=1, 2, 3, \dots) \end{aligned} \quad (5.8)$$

for the coefficients with odd subscript.

Similarly, substituting (5.2) into (2.1), we shall arrive at a relation of the form

$$\begin{aligned} \sum_{m=1}^{\infty} X'_m \Gamma(n-m+1) \{ (m+2)P_n^{m+2}(z) + (a' - b'm^2)mP_n^m(z) \\ + (n-m+1)(n-m+2)(n+m-1)(n+m)(m-2)P_n^{m-2}(z) \} = 0, \end{aligned} \quad (5.9)$$

and this leads to the recurrence relations

$$\begin{aligned} 0 &= \beta'_{-1}X'_1 + \alpha_1 X'_3, & 0 &= \beta'_0 X'_2 + \alpha_2 X'_4, \\ X'_{2m-2} &= \beta'_{2m-2}X'_{2m} + \alpha_{2m} X'_{2m+2} & (m=3, 4, 5, \dots), \end{aligned} \quad (5.10)$$

where

$$\beta'_{-1} = \frac{b' - a'}{n(n+1)} - 1 \quad \text{and} \quad \beta'_{m-2} = \frac{b'm^2 - a'}{(n-m+1)(n-m+2)} \quad (m=2, 3, 4, \dots). \quad (5.11)$$

(5.10) consists of two independent systems of recurrence relations, namely

$$\begin{aligned} 0 &= \beta'_{-1}X'_1 + \alpha_1 X'_3, \\ X'_{2m-1} &= \beta'_{2m-1}X'_{2m+1} + \alpha_{2m+1} X'_{2m+3} & (m=1, 2, 3, \dots), \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} 0 &= \beta'_0 X'_2 + \alpha_2 X'_4, \\ X'_{2m-2} &= \beta'_{2m-2}X'_{2m} + \alpha_{2m} X'_{2m+2} & (m=2, 3, 4, \dots), \end{aligned} \quad (5.13)$$

for coefficients with even and odd subscripts respectively.

6. From (5.6) the infinite continued fraction (Perron, 1913, § 57)

$$\frac{X_m}{X_{m-2}} = \frac{1}{\beta_{m-2} + \frac{\alpha_m}{\beta_m + \frac{\alpha_{m+2}}{\beta_{m+2} + \dots}}} \quad (6.1)$$

follows. Since  $\lim_{m \rightarrow \infty} \alpha_m = -1$  and  $\lim_{m \rightarrow \infty} \beta_m = b$  ( $m \rightarrow \infty$ ), this continued fraction is convergent, at least for sufficiently large values of  $m$  (Perron, 1913, § 56, Satz 41), whenever the quadratic equation  $\rho^2 - b\rho + 1 = 0$  has two roots of different moduli;  $\lim_{m \rightarrow \infty} X_m/X_{m-2}$  ( $m \rightarrow \infty$ ) is equal to that root of  $\rho^2 - b\rho + 1 = 0$ , the modulus of which is smaller. Hence the infinite continued fraction (6.1) is convergent unless  $b$  is real and  $-2 \leq b \leq 2$ . Also

$$\lim_{m \rightarrow \infty} \frac{X_m}{X_{m-2}} = \frac{1}{2} \{ b \pm \sqrt{b^2 - 4} \}, \quad (6.2)$$

whichever has the smaller modulus. In particular, if  $b$  is real (and  $b^2 > 4$ ), then in (6.2) the sign opposite to that of  $b$  must be taken.

The system (5.7) is consistent and  $\lim X_m = 0$  ( $m \rightarrow \infty$ ) only if the expression derived from (6.1) for  $X_2/X_0$  is equal to the value of this quotient as given by the first equation (5.7). Hence the condition of consistency

$$\beta_{-2} + \frac{\alpha_0}{\beta_0} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_4}{\beta_4} + \dots = 0, \quad (6.3)$$

which is a transcendental equation for  $h$ .

Similarly from (5.8) the equation for  $h$ ,

$$\beta_{-1} + \frac{\alpha_1}{\beta_1} + \frac{\alpha_3}{\beta_3} + \frac{\alpha_5}{\beta_5} + \dots = 0, \quad (6.4)$$

is readily derived.

Similar considerations in connection with expansions of the type (5.2) show that the infinite continued fractions arising from (5.10) are convergent whenever the two roots of the quadratic equation  $\rho'^2 - \delta'\rho' + 1 = 0$  have different moduli, *i.e.* whenever  $\delta'$  is either complex, or real with  $\delta'^2 > 4$ . Also

$$\lim_{m \rightarrow \infty} \frac{X'_m}{X'_{m-2}} = \frac{1}{2} \{ \delta' \pm \sqrt{(\delta'^2 - 4)} \}, \quad (6.5)$$

whichever has the smaller modulus.

The system of recurrence formulæ (5.12) is consistent, and  $\lim X'_m = 0$  ( $m \rightarrow \infty$ ) if  $h$  is a root of the transcendental equation

$$\beta'_{-1} + \frac{\alpha_1}{\beta'_1} + \frac{\alpha_3}{\beta'_3} + \frac{\alpha_5}{\beta'_5} + \dots = 0, \quad (6.6)$$

and the system of recurrence relations (5.13) has solutions different from zero if  $h$  satisfies the transcendental equation

$$\beta'_0 + \frac{\alpha_2}{\beta'_2} + \frac{\alpha_4}{\beta'_4} + \frac{\alpha_6}{\beta'_6} + \dots = 0. \quad (6.7)$$

All these conclusions hold when  $n$  is not an integer. When  $n$  is an integer, it may be taken to be a positive integer. In this case there are polynomial solutions in which  $m$  ranges from zero to  $n$ , and transcendental Lamé functions [Ince, 1940 *a*, § 2 (ii)] with which  $m$  ranges from  $n+1$  to infinity. Though  $P_n^m$  vanishes when  $m$  and  $n$  are both not negative integers and  $m > n$ , the product  $\Gamma(n-m+1)P_n^m$  tends to a finite limit when  $n$  tends to a positive integer  $< m$ . For these transcendental Lamé functions (6.2) and (6.5) still hold for integer  $n$ , though the equations of consistency are modified (see Ince, *l.c.*). Ince also discussed the question of common roots of two equations of consistency (1940 *b*, § 5).

In the rest of the paper it will sometimes be assumed that  $n$  is not an integer. The results for integer values of  $n$  then easily follow by a limiting process.

7. Turning to the question of the convergence of the series (5.1), it is seen from (4.10) that for large  $m$ ,

$$X_m \Gamma(n-m+1) P_n^m(z) \sim (-)^m X_m \frac{\Gamma(n+m+1)}{\Gamma(m+1)} \left( \frac{1-z}{1+z} \right)^{\frac{1}{2}m},$$

and hence with  $m$  tending to infinity, the ratio of the moduli of the  $m$ th and  $(m-2)$ th term of (5.1) tends to

$$\left| \frac{1-z}{1+z} \right| \lim_{m \rightarrow \infty} \left| \frac{X_m}{X_{m-2}} \right| = \frac{1}{2} \left| \delta \pm \sqrt{(\delta^2 - 4)} \right| \left| \frac{1-z}{1+z} \right|. \quad (7.1)$$

Hence the series is convergent when

$$\left| \frac{1-z}{1+z} \right| < \frac{2}{\left| \delta \pm \sqrt{(\delta^2 - 4)} \right|} = \frac{1}{2} \left| \delta \mp \sqrt{(\delta^2 - 4)} \right|. \quad (7.2)$$

In the last expression that sign must be chosen which makes  $\frac{1}{2} |\delta \mp \sqrt{(\delta^2 - 4)}|$  larger. This larger value always exceeding unity, the domain of convergence in the  $z$ -plane will be the finite part of the plane outside the circle  $C_1$  with centre

$$z_1 = \frac{2 + |\delta \mp \sqrt{(\delta^2 - 4)}|^2}{2 - |\delta \mp \sqrt{(\delta^2 - 4)}|^2} \quad (7.3)$$

and radius

$$r_1 = \frac{4 |\delta \mp \sqrt{(\delta^2 - 4)}|}{|\delta \mp \sqrt{(\delta^2 - 4)}|^2 - 2}. \quad (7.4)$$

Together with (5.1),

$$y = \sum_{m=0}^{\infty} X_m \Gamma(n-m+1) P_n^m(-z) \quad (7.5)$$

is a solution of Lamé's equation, distinct from (5.1) if  $n$  is not an integer. This solution is convergent outside the circle  $C_2$  of the  $z$ -plane with centre  $z_2 = -z_1$  and radius  $r_2 = r_1$ .

Another significant solution is

$$y = \sum_{m=0}^{\infty} X_m \Gamma(n-m+1) Q_n^m(z), \quad (7.6)$$

convergent in that part of the  $z$ -plane which is outside both  $C_1$  and  $C_2$ .

The same results hold for (5.2) and the associated series,  $\delta$  being replaced throughout by  $\delta'$ .

It is worth while noting that all series are convergent on the entire imaginary axis of the  $z$ -plane.

#### SOLUTIONS IN SERIES OF $P_n^m(d)$ AND $P_n^m(d/k')$

8. Besides (2.3) there is also the integral equation

$$y(u) = \lambda \int_{-2K}^{+2K} P_n\left(\frac{ik}{k'} \operatorname{cn} u \operatorname{cn} v\right) y(v) dv \quad (8.1)$$

(Whittaker and Watson, 1927, § 23.6), which is satisfied by certain Lamé functions. Now, putting in (4.8)

$$t=0, z=a, \quad (1-z^2)^{\frac{1}{2}} = ks, \quad \omega = \frac{\pi}{2} - am v, \quad (8.2)$$

we have the expansion of the nucleus of (2.3)

$$P_n(ks \sin am v) = \sum_{m=-\infty}^{\infty} (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(0) P_n^m(d) \cos m(\tfrac{1}{2}\pi - am v). \quad (8.3)$$

Similarly,

$$t=0, z=d/k', \quad (1-z^2)^{\frac{1}{2}} = ikc/k', \quad \omega = am v$$

yields for the nucleus of (8.1) the expansion

$$P_n(ikc \cos am v/k') = \sum_{m=-\infty}^{+\infty} (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(0) P_n^m(d/k') \cos(m am v). \quad (8.4)$$

Inserting (8.3) and (8.4) into (2.3) and (8.1) respectively, we obtain formally the expansions

$$y(u) = \lambda \sum (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(0) P_n^m(d) \int_{-2K}^{+2K} y(v) \cos m(\tfrac{1}{2}\pi - am v) dv \quad (8.5)$$

and

$$y(u) = \lambda \sum (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(0) P_n^m\left(\frac{d}{k'}\right) \int_{-2K}^{+2K} y(v) \cos(m am v) dv, \quad (8.6)$$

representing certain Lamé functions.

Again, there is a slightly different type of integral equation (Lambe and Ward, 1934, equation (4.16); Ince, 1940 *a*, § 10, III),

$$y(u) = \lambda \int_{-2K}^{2K} \operatorname{cn} u \operatorname{cn} v P'_n(k \operatorname{sn} u \operatorname{sn} v) y(v) dv. \quad (8.7)$$

In order to obtain the expansion of the nucleus of this integral equation, let us put in (4.9) the values (8.2), thus obtaining

$$ks \operatorname{cn} v P'_n(ks \operatorname{sn} v) = \sum_{m=-\infty}^{+\infty} (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(o) P_n^m(d) m \sin m(\frac{1}{2}\pi - am v). \quad (8.8)$$

Introducing this into (8.7), the expansion

$$y(u) = \lambda \frac{c}{ks} \sum (-)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(o) P_n^m(d) \int_{-2K}^{2K} y(v) \sin m(\frac{1}{2}\pi - am v) dv \quad (8.9)$$

is easily obtained.

It is not necessary, at this stage of the work, to enter into questions of convergence or to discuss which type of Lamé functions are represented by the above series.

9. The expansion (8.5) suggests trying a solution of Lamé's equation in form of the series

$$y(u) = \sum_{m=0}^{\infty} A_m \Gamma(n-m+1) P_n^m(d). \quad (9.1)$$

In order to introduce this series into Lamé's equation, let us first compute

$$\frac{dy}{du} = -k^2 s c \sum_{m=0}^{\infty} A_m \Gamma(n-m+1) P_n^{m'}(d)$$

and

$$\begin{aligned} \frac{d^2 y}{du^2} &= \sum_{m=0}^{\infty} A_m \Gamma(n-m+1) \{ k^4 s^2 c^2 P_n^{m''}(d) - k^2 d (c^2 - s^2) P_n^{m'}(d) \} \\ &= \sum_{m=0}^{\infty} A_m \Gamma(n-m+1) \left\{ k^2 d P_n^{m'}(d) - \left( n(n+1) k^2 c^2 - m^2 \frac{c^2}{s^2} \right) P_n^m(d) \right\}, \end{aligned}$$

using (4.6). Hence

$$L[y] = \sum_{m=0}^{\infty} A_m \Gamma(n-m+1) \{ k^2 d P_n^{m'}(d) + (h - n(n+1) k^2 + m^2 c^2 / s^2) P_n^m(d) \}$$

and, employing (4.4),

$$\begin{aligned} L[y] &= \sum_{m=0}^{\infty} A_m \Gamma(n-m+1) \{ \frac{1}{4} k^2 P_n^{m+2}(d) + \frac{1}{2} [2h - n(n+1) k^2 - (2 - k^2) m^2] P_n^m(d) \\ &\quad + \frac{1}{4} k^2 (n-m+1)(n-m+2)(n+m-1)(n+m) P_n^{m-2}(d) \} = 0. \end{aligned} \quad (9.2)$$

This is a relation of the form (5.3), and hence for the coefficients we obtain the recurrence relations (5.6) with

$$X_m = A_m, \quad a = 4h/k^2 - 2n(n+1), \quad b = 2(2/k^2 - 1). \quad (9.3)$$

From the results of § 5 it follows that a solution of the form

$$\sum A_{2r} \Gamma(n-2r+1) P_n^{2r}(d) \quad \text{or} \quad \sum A_{2r+1} \Gamma(n-2r) P_n^{2r+1}(d)$$

exists if  $h$  satisfies the transcendental equation (6.3) or (6.4) respectively. The coefficients of these expansions are given by the recurrence formulæ (5.7) and (5.8) respectively, always with above values of  $a$  and  $b$ .

The continued fractions connected with these equations are convergent unless  $b$  is real and  $-2 < b < 2$ , i.e. unless  $k^2$  is real and  $> 1$ . Hence our solution is valid in the whole plane of the complex parameter  $n$ ; in the plane of the complex parameter  $k^2$  we have a branch-cut

from 1 along the positive real axis to infinity. According to (7.2), the series is convergent in that region of the complex variable  $u$  in which

$$\left| \frac{1-d}{1+d} \right| < \left| \frac{2}{k^2} - 1 \pm \frac{2k'}{k^2} \right| = \left| \frac{1 \pm k'}{k} \right|^2, \quad (9.4)$$

whichever sign gives the larger value.

In the applications the most important case is that in which  $k = \sin \theta$ ,  $k' = \cos \theta$ , and the modular angle is (real and) acute. In this case the domain of convergence is

$$\left| \frac{1-d}{1+d} \right| < \cot^2 \frac{1}{2} \theta. \quad (9.5)$$

The solution suggested by (8.6) will be taken in the form

$$\gamma(u) = \sum_{m=0}^{\infty} A_m^* \Gamma(n-m+1) P_n^m(d/k'). \quad (9.6)$$

In this case we obtain

$$\begin{aligned} L[\gamma] = & - \sum_{m=0}^{\infty} A_m^* \Gamma(n-m+1) \left\{ \frac{1}{4} k^2 P_n^{m+2}(d/k') - \frac{1}{2} [2h - n(n+1)k^2 - (2-k^2)m^2] P_n^m(d/k') \right. \\ & \left. + \frac{1}{4} k^2 (n-m+1)(n-m+2)(n+m-1)(n+m) P_n^{m-2}(d/k') \right\}. \end{aligned} \quad (9.7)$$

Hence the coefficients satisfy the recurrence formulæ (5.6) with

$$X_m = A_m^*, \quad a = 2n(n+1) - 4h/k^2, \quad b = 2(1 - 2/k^2). \quad (9.8)$$

The continued fractions are again convergent in the cut  $k^2$ -plane. The domain of convergence in the  $u$ -plane is different, being determined by

$$\left| \frac{k'-d}{k'+d} \right| < \left| \frac{1 \pm k'}{k} \right|^2 \quad (9.9)$$

or

$$\left| \frac{k'-d}{k'+d} \right| < \cot^2 \frac{1}{2} \theta \quad (9.10)$$

respectively, according to whether  $k$  is complex or real (and a proper fraction).

10. The solution of (2.1),

$$\gamma = \frac{c}{ks} \sum_{m=1}^{\infty} A'_m \Gamma(n-m+1) m P_n^m(d) \quad (10.1)$$

is obviously of the form (5.2) with  $z=d$ ,  $\kappa=k'^{-2}$ , and is suggested by (8.9). We have

$$\frac{d\gamma}{du} = \sum_{m=1}^{\infty} m A'_m \Gamma(n-m+1) \left\{ -k c^2 P_n^{m'}(d) - \frac{d}{k s^2} P_n^m(d) \right\}$$

and

$$\begin{aligned} \frac{d^2 \gamma}{du^2} = & \sum_{m=1}^{\infty} m A'_m \Gamma(n-m+1) \left\{ k^3 s c^3 P_n^{m''}(d) + \frac{kcd}{s} (1+2s^2) P_n^{m'}(d) + \frac{c}{ks} \frac{k^2 s^2 + 2d^2}{s^2} P_n^m(d) \right\} \\ = & \frac{c}{ks} \sum_{m=1}^{\infty} m A'_m \Gamma(n-m+1) \left\{ 3k^2 d P_n^{m'}(d) - \left[ n(n+1)k^2 c^2 - k^2 - \frac{2d^2}{s^2} - m^2 \frac{c^2}{s^2} \right] P_n^m(d) \right\}, \end{aligned}$$

using (4.6). Hence

$$L[\gamma] = \frac{c}{ks} \sum_{m=1}^{\infty} m A'_m \Gamma(n-m+1) \left\{ 3k^2 d P_n^{m'}(d) + \left[ h + k^2 - n(n+1)k^2 + \frac{2d^2}{s^2} + m^2 \frac{c^2}{s^2} \right] P_n^m(d) \right\},$$

or, employing (4.5),

$$\begin{aligned} L[\gamma] = & \frac{c}{ks} \sum_{m=1}^{\infty} A'_m \Gamma(n-m+1) \left\{ \frac{1}{4} k^2 (m+2) P_n^{m+2}(d) + [h - \frac{1}{2} n(n+1)k^2 - \frac{1}{2} (2-k^2)m^2] m P_n^m(d) \right. \\ & \left. + \frac{1}{4} k^2 (n-m+1)(n-m+2)(n+m-1)(n+m)(m-2) P_n^{m-2}(d) \right\}. \end{aligned} \quad (10.2)$$

This is of the form (5.9) with

$$X'_m = A'_m, \quad a' = 4h/k^2 - 2n(n+1), \quad b' = 2(2/k^2 - 1), \quad (10.3)$$

and hence we obtain recurrence formulæ (5.10) for these values.  $b'$  having the value of  $b$  of (9.3), the series (10.1) and the continued fractions related to it are convergent in the same domains of  $n, k, u$  as the series (9.1) and the corresponding continued fractions.

Again there is another solution,

$$y = \frac{s}{i\tilde{k}c} \sum_{m=1}^{\infty} A_m^{*'} T(n-m+1) m P_n^m\left(\frac{d}{k'}\right) \quad (10.4)$$

with

$$X'_m = A_m^{*'}, \quad a' = 2n(n+1) - 4h/k^2, \quad b' = 2(1 - 2/k^2). \quad (10.5)$$

The convergence properties of this series are identical with those of (9.6).

#### SERIES OF $P_n^m(s)$ , $P_n^m(k's)$ , $P_n^m(c)$ AND $P_n^m(i\tilde{k}c/k')$

11. Besides the integral equations used hitherto there are other integral equations satisfied by Lamé polynomials (Whittaker and Watson, 1927, § 23.6). They suggest trying instead of (9.1) and (9.6) the solutions

$$y = \sum_{m=0}^{\infty} B_m \Gamma(n-m+1) P_n^m(s) \quad (11.1)$$

and

$$y = \sum_{m=0}^{\infty} B_m^{*'} \Gamma(n-m+1) P_n^m(k's). \quad (11.2)$$

Proceeding in the same way as in § 9, we obtain the recurrence formulæ (5.6) with

$$X_m = B_m, \quad a = 2n(n+1) - 4[n(n+1) - k]/k'^2, \quad b = 2(1 - 2/k'^2) \quad (11.3)$$

or

$$X_m = B_m^{*'}, \quad a = 4[n(n+1) - k]/k'^2 - 2n(n+1), \quad b = 2(2/k'^2 - 1) \quad (11.4)$$

respectively.

The continued fractions originating from these recurrence formulæ are convergent unless  $k'^2$  is real and  $> 1$ . Hence in this case the branch-cut in the  $k^2$ -plane goes from 0 along the negative real axis to  $-\infty$ . Hence this type of expansion will be still valid for  $k=1$  where the expansions of §§ 8-10 cease to have a meaning.

The expansion (11.1) is convergent in that domain of the complex plane  $u$  in which

$$\left| \frac{1-s}{1+s} \right| < \left| \frac{1 \pm k}{k'} \right|^2 \quad \text{or} \quad \left| \frac{1-s}{1+s} \right| < \cot^2 \left( \frac{1}{4}\pi - \frac{1}{2}\theta \right), \quad (11.5)$$

the former formula being valid for complex  $k$ , with that sign which yields the larger modulus of  $1 \pm k$ , the latter for real and positive acute modular angles. The expansion (11.2) is convergent in the domain

$$\left| \frac{1-k's}{1+k's} \right| < \left| \frac{1 \pm k}{k'} \right|^2 \quad \text{or} \quad \left| \frac{1-k's}{1+k's} \right| < \cot^2 \left( \frac{1}{4}\pi - \frac{1}{2}\theta \right) \quad (11.6)$$

of the complex  $u$ -plane.

Similarly for the coefficients of the solutions

$$y = \frac{d}{c} \sum_{m=1}^{\infty} B'_m \Gamma(n-m+1) m P_n^m(s) \quad (11.7)$$

and

$$y = \frac{c}{d} \sum_{m=1}^{\infty} B_m^{*'} \Gamma(n-m+1) m P_n^m(k's), \quad (11.8)$$

the recurrence relations (5.10) hold with

$$X'_m = B'_m, \quad a' = a, \quad b' = b \quad (11.9)$$

from (11.3) and

$$X'_m = B_m^{**}, \quad a' = a, \quad b' = b \quad (11.10)$$

from (11.4) respectively. The convergence properties of these series and of the corresponding continued fractions are identical with the convergence properties of the series (11.1) and (11.2) and of the corresponding infinite continued fractions respectively.

Plainly the recurrence formulæ for the coefficients of series into  $P_n^m(kz)$  originate from the corresponding recurrence relations for the coefficients of series into  $P_n^m(d)$  by changing  $k$  and  $h$  into  $k'$  and  $n(n+1) - h$  respectively. The same change leads from the coefficients of series into  $P_n^m(d/k')$  to those of series into  $P_n^m(s)$ .

12. Finally we have to deal with the last quadruple of solutions which again may be obtained by a transformation of Lamé's equation. Let us assume the solutions

$$y = \sum_{m=0}^{\infty} C_m \Gamma(n-m+1) P_n^m(c), \quad (12.1)$$

$$y = \sum_{m=0}^{\infty} C_m^* \Gamma(n-m+1) P_n^m(ikc/k'), \quad (12.2)$$

$$y = \frac{d}{s} \sum_{m=1}^{\infty} C'_m \Gamma(n-m+1) m P_n^m(c), \quad (12.3)$$

and

$$y = \frac{s}{d} \sum_{m=1}^{\infty} C_m^{*'} \Gamma(n-m+1) m P_n^m\left(\frac{ikc}{k'}\right). \quad (12.4)$$

Inserting these series into Lamé's differential equation, an analysis similar to that of §§ 9 and 10 yields for the coefficients  $C_m$ ,  $C_m^*$  and  $C'_m$ ,  $C_m^{*'}$  the relations (5.6) and (5.10) with

$$X_m = C_m, \quad X'_m = C'_m, \quad a = a' = 4h - 2n(n+1), \quad b = b' = 4k^2 - 2 \quad (12.5)$$

and

$$X_m = C_m^*, \quad X'_m = C_m^{*'}, \quad a = a' = 2n(n+1) - 4h, \quad b = b' = 2 - 4k^2 \quad (12.6)$$

respectively.

The infinite continued fractions connected with the recurrence formulæ for the  $C_m$  and  $C_m^*$  are convergent *unless  $k^2$  is real and a proper fraction*. Hence just in the case which is most important in the applications, the series of this section present new difficulties. Thus they are not likely to be of much use except in connection with the polynomial solutions of Lamé's equation where the question of convergence does not arise.

For complex  $k$  (or real  $k > 1$  or  $< -1$ ), however, the continued fractions are convergent for any value of  $n$ . The series (12.1) and (12.3) are convergent in the domain

$$\left| \frac{1-c}{1+c} \right| < |k \pm ik'|^2, \quad (12.7)$$

whereas the series (12.2) and (12.4) are convergent in the domain

$$\left| \frac{k' - ikc}{k' + ikc} \right| < |k \pm ik'|^2 \quad (12.8)$$

of the complex  $u$ -plane. In both cases that sign has to be taken which assigns to  $|k \pm ik'|$  the larger value.

#### LAMÉ POLYNOMIALS

13. Each of the twelve types of series gives rise to two distinct types of solutions, one with even, the other with odd subscripts  $m$  of the coefficients. So, for instance, from (9.1) we have the two solutions

$$y = \sum_{r=0}^{\infty} A_{2r} \Gamma(n-2r+1) P_n^{2r}(d) \quad \text{and} \quad y = \sum_{r=0}^{\infty} A_{2r+1} \Gamma(n-2r) P_n^{2r+1}(d),$$

which will be distinguished as (9.1)*e* and (9.1)*o*.

First we shall deal with the simplest type of Lamé functions, which are known to be polynomials of degree  $n$  in  $s, c, d$ . These Lamé functions, generally called *Lamé polynomials*, have some features of their own.  $\Pi_n$  shall denote any polynomial of degree  $\frac{1}{2}n$  in  $s^2$  or  $c^2$  or  $d^2$ , and  $n$  shall be a positive integer in this and the following section.

In the case of *Lamé polynomials*, the coefficients of our series vanish whenever  $m > n$ . Hence the series terminate as well as the continued fractions associated with them and the question of convergence does not arise at all. No restrictions on  $k$  or  $u$  are necessary, and any of the twelve types of solutions may be used.

Corresponding to a given integer value of  $n \geq 0$ , there are in general exactly  $2n + 1$  Lamé polynomials corresponding to as many different values of  $h$ . There are eight different types of Lamé polynomials (Whittaker and Watson, 1927, § 23.5). For even  $n$  we have Lamé polynomials of the first species, of form  $\Pi_n$ , and of the third species, of forms  $cd\Pi_{n-2}$ ,  $ds\Pi_{n-2}$ , or  $sc\Pi_{n-2}$ . For odd values of  $n$  we have Lamé polynomials of second species, of forms  $s\Pi_{n-1}$ ,  $c\Pi_{n-1}$ , or  $d\Pi_{n-1}$ , and of fourth species,  $scd\Pi_{n-3}$ .

It will be seen that each type of Lamé polynomials may be represented in six different ways by a terminating series of associated Legendre polynomials.

Take for instance (9.1)*e*. It is of form  $\Pi_n$  if  $n$  is even, while it is of form  $d\Pi_{n-1}$  if  $n$  is odd. Correspondingly (9.1)*o* is of form  $sd\Pi_{n-2}$  if  $n$  is even, and of form  $s\Pi_{n-1}$  if  $n$  is odd. Dealing similarly with all twelve types, we obtain the following table of representations of Lamé polynomials:—

$n$	Form	Represented by					
Even	$\Pi_n$	(9.1) <i>e</i>	(9.6) <i>e</i>	(11.1) <i>e</i>	(11.2) <i>e</i>	(12.1) <i>e</i>	(12.2) <i>e</i>
Odd	$s\Pi_{n-1}$	(9.1) <i>o</i>	(10.4) <i>o</i>	(11.1) <i>e</i>	(11.2) <i>e</i>	(12.1) <i>o</i>	(12.4) <i>o</i>
	$c\Pi_{n-1}$	(9.6) <i>o</i>	(10.1) <i>o</i>	(11.1) <i>o</i>	(11.8) <i>o</i>	(12.1) <i>e</i>	(12.2) <i>e</i>
	$d\Pi_{n-1}$	(9.1) <i>e</i>	(9.6) <i>e</i>	(11.2) <i>o</i>	(11.7) <i>o</i>	(12.2) <i>o</i>	(12.3) <i>o</i>
Even	$cd\Pi_{n-2}$	(9.6) <i>o</i>	(10.1) <i>o</i>	(11.7) <i>e</i>	(11.8) <i>e</i>	(12.2) <i>o</i>	(12.3) <i>o</i>
	$ds\Pi_{n-2}$	(9.1) <i>o</i>	(10.4) <i>o</i>	(11.2) <i>o</i>	(11.7) <i>o</i>	(12.3) <i>e</i>	(12.4) <i>e</i>
	$sc\Pi_{n-2}$	(10.4) <i>e</i>	(10.1) <i>e</i>	(11.1) <i>o</i>	(11.8) <i>o</i>	(12.1) <i>o</i>	(12.4) <i>o</i>
Odd	$scd\Pi_{n-3}$	(10.1) <i>e</i>	(10.4) <i>e</i>	(11.7) <i>e</i>	(11.8) <i>e</i>	(12.3) <i>e</i>	(12.4) <i>e</i>

14. Each of the double periodic Lamé functions can be represented by six terminating series of associated Legendre polynomials  $P_n^m(z)$ ,  $z$  being  $s, c, d, ks, ikc/k', d/k'$  respectively in the six cases. There are certain interesting relations between two different series representing the same function.

Take for instance (9.1)*e* and (9.6)*e*, *i.e.*

$$\sum_{r=0}^{[n/2]} A_{2r} \Gamma(n-2r+1) P_n^{2r}(d) \quad \text{and} \quad \sum_{r=0}^{[n]} A_{2r}^* \Gamma(n-2r+1) P_n^{2r}(d/k'),$$

both representing solutions of the form  $\Pi_n$  if  $n$  is even, and of the form  $d\Pi_{n-1}$  if  $n$  is odd. The corresponding values of  $a$  and  $b$  differ only in sign, and hence the  $a_m$  are the same in both cases while the  $\beta_{m-2}$  differ only in sign if  $m \neq 1$  ( $\beta_{-1}$  is essentially different in both cases). Hence we obtain the same (terminating) continued fraction (6.3) in both cases and consequently the same values  $h$ .

Taking the same value of  $h$  in (9.1)*e* and (9.6)*e*, we obtain except for a constant factor the same periodic solution, for to each value of  $h$  there belongs only *one* periodic solution (Whittaker and Watson, 1927, § 23.47). Hence

$$\sum A_{2r} \Gamma(n-2r+1) P_n^{2r}(d) = \lambda \sum A_{2r}^* \Gamma(n-2r+1) P_n^{2r}(d/k'), \quad (14.1)$$

where  $\lambda$  is a constant.



Furthermore, it is easily seen from (5.7) that, the recurrence relations for  $A_{2r}$  and  $A_{2r}^*$  differing only in the sign of  $\beta_{m-2}$ , we may put  $A_{2r}^* = (-)^r A_{2r}$ . Hence instead of (14.1) we obtain the remarkable relation

$$\Sigma A_{2r} \Gamma(n-2r+1) P_n^{2r}(d) = \lambda \Sigma (-)^r A_{2r} \Gamma(n-2r+1) P_n^{2r}(d/k'), \quad (14.2)$$

in which  $\lambda$  is a kind of a characteristic number.

There is no corresponding relation between (9.1) $\rho$  and (9.6) $\rho$ . The extra term  $+1$  in  $\beta_{-1}$  makes the equations (6.4) different in both cases, and hence (9.1) $\rho$  and (9.6) $\rho$  do not belong to the same value of  $h$ . In fact, from the table of § 13 it is seen that they represent functions of different types.

It is of course not surprising at all that a terminating series of  $P_n^{2r}(d)$  should be representable as a terminating series of  $P_n^{2r}(d/k')$ , because all terminating series of  $P_n^{2r}(d)$  have this property. The particular feature of Lamé polynomials is that the coefficients of the two representations differ only in a factor  $(-)^r \lambda$ . I believe that this property characterises Lamé polynomials of first species ( $n$  even) and of second species of form  $d \Pi_{n-1}$  ( $n$  odd) as completely as does the integral equation (Whittaker and Watson, 1927, § 23.6)

$$y(u) = \lambda \int_{-2K}^{2K} P_n \left( \frac{1}{k'} \frac{dn}{du} \right) y(v) dv. \quad (14.3)$$

I do not see, however, any simple way of deducing this integral equation or Lamé's differential equation directly from (14.2).

Similarly we obtain five more relations:

$$\Sigma B_{2r} \Gamma(n-2r+1) P_n^{2r}(s) = \lambda \Sigma (-)^r B_{2r} \Gamma(n-2r+1) P_n^{2r}(ks) \quad (14.4)$$

satisfied by Lamé functions of first species ( $n$  even) or second species with factor  $s$  ( $n$  odd),

$$\Sigma C_{2r} \Gamma(n-2r+1) P_n^{2r}(c) = \lambda \Sigma (-)^r C_{2r} \Gamma(n-2r+1) P_n^{2r}(ick/k') \quad (14.5)$$

for Lamé polynomials of first species ( $n$  even) or second species with factor  $c$  ( $n$  odd),

$$\frac{c}{ks} \Sigma A'_{2r} \Gamma(n-2r+1) P_n^{2r}(d) = \lambda \frac{ks}{c} \Sigma (-)^r A'_{2r} \Gamma(n-2r+1) P_n^{2r}(d/k') \quad (14.6)$$

for Lamé polynomials of third species with factor  $sc$  ( $n$  even) or of fourth species ( $n$  odd),

$$\frac{d}{c} \Sigma B'_{2r} \Gamma(n-2r+1) P_n^{2r}(s) = \lambda \frac{c}{d} \Sigma (-)^r B'_{2r} \Gamma(n-2r+1) P_n^{2r}(ks) \quad (14.7)$$

for Lamé polynomials of third species with factor  $cd$  ( $n$  even) or of fourth species ( $n$  odd), and finally

$$\frac{d}{s} \Sigma C'_{2r} \Gamma(n-2r+1) P_n^{2r}(c) = \lambda \frac{s}{d} \Sigma (-)^r C'_{2r} \Gamma(n-2r+1) P_n^{2r}(ick/k'), \quad (14.8)$$

which is satisfied by Lamé polynomials of third species with factor  $sd$  or by Lamé polynomials of fourth species according as  $n$  is even or odd.

15. A few more remarks may be brought forward regarding Lamé functions of the second kind, *i.e.* a second solution of Lamé's equation. It has been mentioned already that this second solution cannot be periodic (Whittaker and Watson, 1927, § 23.47) when the function of the first kind is a polynomial.

It has been observed in § 7 that in all our series  $P_n^m(z)$  may be replaced by  $P_n^m(-z)$ . This replacement, however, does not yield new solutions in the case of Lamé polynomials, for  $P_n^m(-z) = (-)^{n-m} P_n^m(z)$  if  $m$  and  $n$  are integers and  $0 < m < n$ .

The functions of second kind corresponding to Lamé polynomials are obtained by replacing  $P_n^m(z)$  in all our series by  $Q_n^m(z)$ . The associated Legendre functions of first and second kind satisfying the same differential equation, recurrence relations, etc., the series remain solutions of the differential equation and clearly they are the solutions of second kind.

Proceeding thus, for every function of the second kind six different representations are obtained, and it must be proved that the functions represented by the six different series are constant multiples of each other, *i.e.* that (14.2) and (14.4)–(14.8) are still valid if  $P_n^{2r}(z)$  is replaced everywhere by  $Q_n^{2r}(z)$ . The proof is very easy indeed.

Suppose for instance that the series obtained by replacing  $P_n^m$  by  $Q_n^m$  in (9.1)*e* and (9.6)*e* are not constant multiples of each other. Then they are two linearly independent solutions of (2.1), and hence (9.1)*e*, for instance, a solution of (2.1), must be a linear combination of the two series mentioned. Now,  $Q_n^m(z)$ , and hence any terminating series of these functions, behaves like  $\text{const. } z^{-n-1}$  when  $z$  tends to infinity. Thus a linear combination of two solutions in  $Q$ -series vanishes when  $u = iK'$ , and could not be possibly a polynomial in  $d$ . All six series of  $Q$ -functions represent, apart from a constant factor, the same solution: that one which is known as Lamé function of the second kind (Whittaker and Watson, 1927, § 23.47).

The table of § 13 is also a table of the functions of second kind, giving, *e.g.*, in its first line the series in which ( $n$  being even)  $P_n^m(z)$  has to be replaced by  $Q_n^m(z)$  in order to obtain the Lamé functions of second kind and first species.

### TRANSCENDENTAL LAMÉ FUNCTIONS

16. Besides the  $2n+1$  Lamé polynomials there is an infinity of transcendental Lamé functions for any non-negative integer  $n$  (Ince, 1940 *a*, p. 48). For non-integral values of  $n$  we have only transcendental Lamé functions (*ibid.*, p. 49). We proceed now to the discussion of the expansions of transcendental Lamé functions into series of associated Legendre functions.

Most of what follows is still valid for complex  $n$ . For  $n = -\frac{1}{2} + ip$  ( $p$  real), for instance, we obtain Hobson's functions of the elliptic cone (Hobson, 1892) developed into a series of Mehler's conal harmonics. For the sake of brevity, however, and in order to fix the ideas, we shall assume throughout the rest of the paper that  $n$  is real and that  $k = \sin \theta$  is real and a positive proper fraction. Without further loss of generality we may take  $n$  to be  $\geq -\frac{1}{2}$ .

It is important to bring out the fundamental difference between polynomial and transcendental Lamé functions.

- (i) Lamé polynomials are divided into four species and eight classes (§ 13). No such distinction is possible with transcendental Lamé functions (Ince, 1940 *a*). This was the reason for designing a new notation for Lamé functions of real period by Ince (*ibid.*).
- (ii) Lamé polynomials are doubly periodic functions. Transcendental Lamé functions have *only one primitive period*, this being real, imaginary or complex, as the case may be.

17. (i) Ince (1940 *a*) found every periodic Lamé function with the factor  $\text{dn } u$  to be identical with a form without that factor, and pointed out that therefore, from the analytical point of view, there are only four distinct types of Lamé functions.

Ince's argument is as follows. The solution

$$y = \sum A_{2r} s^{2r} \quad (17.1)$$

should be classified as belonging to the first species. He proves this solution to be convergent for  $|ks| < 1$ , hence certainly in some domain in the vicinity of the real axis of  $u$ . On the other hand,

$$y = d \sum C_{2r} s^{2r}, \quad (17.2)$$

convergent in the same domain, should be classified as belonging to the second species. Now

$$d = (1 - k^2 s^2)^{\frac{1}{2}} = \sum \alpha_r s^{2r}$$

is convergent in the same domain, and hence (17.2) changes to

$$y = \sum \alpha_r s^{2r} \sum C_{2r} s^{2r} = \sum A'_{2r} s^{2r},$$

and this is clearly of the first species, in contradiction to our previous assertion.

A similar remark applies, we may add, to the factor  $\operatorname{sn} u$  when  $u = K + it$  and  $t$  is real. In this case the most suitable representation of functions of the first species will be

$$y = \Sigma B_{2r} d^{2r} \quad (17.3)$$

instead of (17.1). This series is convergent for  $|d| < 1$ , and hence certainly in some domain enclosing the line  $u = K + it$ . Also the series

$$k_s = (1 - d^2)^{\frac{1}{2}} = k \Sigma \beta_r d^{2r}$$

is convergent in the same domain. Hence the solution

$$y = s \Sigma D_{2r} d^{2r},$$

apparently belonging to the second species, is equal to

$$\Sigma \beta_r d^{2r} \Sigma D_{2r} d^{2r} = \Sigma B'_{2r} d^{2r},$$

and hence it belongs to the first species as well, for the last power-series is clearly of the form (17.3).

18. (ii) We shall examine the periodic properties of the power-series representations of Lamé functions as well as that of their Fourier-Jacobi expansions and of their expansions into series of associated Legendre functions.

Let us begin, for instance, with the power-series and take as an example (17.1), convergent in the domain  $|k_s| < 1$ . Clearly (17.1) has the real period  $2K$ , for it is possible to proceed from any point  $u$  inside the domain of convergence to the point  $u + 2K$ , remaining throughout in the domain of convergence. Hence (17.1) as a uniformly convergent series of functions of period  $2K$  has this period.

It is entirely different with the imaginary period. A superficial inspection might suggest that (17.1) has the imaginary period  $2iK'$ , for all its terms have this period. This is, however, not conclusive. We know (Whittaker and Watson, 1927, § 22.34) that

$$\operatorname{sn}(\sigma + ir'K') = (k \operatorname{sn} \sigma)^{-1} \quad (r' = \pm 1, \pm 3, \pm 5, \dots),$$

and if  $\sigma$  is real,  $|\operatorname{sn} \sigma| < 1$ . Hence

$$|k \operatorname{sn}(\sigma + ir'K')| = \frac{1}{|\operatorname{sn} \sigma|} > 1 \quad (r' = \pm 1, \pm 3, \pm 5, \dots).$$

Thus the lines  $u = \sigma + ir'K'$  ( $r' = \pm 1, \pm 3, \dots$ ) lie outside the domain of convergence touching its circumference in the points  $u = rK + ir'K'$  ( $r, r' = \pm 1, \pm 3, \dots$ ). These lines are unsurmountable barriers. There is no way of joining any point  $u$  to the point  $u + 2iK'$  or  $u + 4iK'$  by a path entirely inside the domain of convergence.

Next let us examine the Fourier-Jacobi series of the same solution (17.1), viz.

$$y = \Sigma A_{2r} \cos(2r \operatorname{am} u). \quad (18.1)$$

It is hardly necessary to say that in (9.1), (17.1) and (18.1) the same symbol  $A_{2r}$  has different meanings. (18.1) is convergent in the domain (Ince, 1940b)

$$\exp \{2 | \mathcal{J}(\operatorname{am} u) | \} < \frac{1 + k'}{1 - k'} \quad \text{or} \quad k | \sinh \mathcal{J}(\operatorname{am} u) | < k'$$

where  $\mathcal{J}(z)$  is the imaginary part of  $z$ . The real period follows as in the former case.

Now, on the lines  $u = \sigma + ir'K'$  ( $r' = \pm 1, \pm 3, \dots$ ) we have  $\operatorname{sn} u = \sin \operatorname{am} u = (k \operatorname{sn} \sigma)^{-1}$ , and hence  $\sin \operatorname{am} u$  is real and its modulus is larger than one. Consequently  $\operatorname{am} u = \frac{1}{2}\pi + i\mathcal{J}(\operatorname{am} u)$  along these lines, and

$$k | \sinh \mathcal{J}(\operatorname{am} u) | = k | \cos \operatorname{am} u | = \frac{\operatorname{dn} \sigma}{|\operatorname{sn} \sigma|},$$

and this is certainly  $> k'$ . Hence in this case too the lines  $u = \sigma + ir'K'$  are tangents to the domain of convergence and again barriers preventing any connection between  $u$  and  $u + 2iK'$  inside the domain of convergence.

Finally let us turn to the expansion of (17.1) into a series of Legendre functions. A short consideration shows that we have to take (9.1)e or (9.6)e.

Now, (9.1) $e$  is convergent in the domain (9.4), *i.e.*

$$\left| \frac{1-d}{1+d} \right| < \left( \frac{1+k'}{k} \right)^2. \quad (18.2)$$

The real axis of  $u$  corresponding to the interval  $(k, 1)$  of the  $d$ -plane, and this interval lying in the domain of convergence, the periodicity of this Lamé function modulo  $2K$  can be shown as before,  $\operatorname{dn} u$  having this period.

If there was an imaginary period, it should be  $4iK'$  or a multiple of it,  $4iK'$  being the primitive imaginary period of  $\operatorname{dn} u$ . A short consideration shows that the lines  $u = \sigma + ir'K'$  ( $r' = \pm 1, \pm 3, \dots$ ) are now inside the domain of convergence, but now the lines  $u = \sigma + 2ir'K'$  ( $r' = \pm 1, \pm 3, \dots$ ) are unsurmountable obstacles. We have (Whittaker and Watson, § 22.34)

$$\operatorname{dn}(\sigma + 2ir'K') = -\operatorname{dn} \sigma \quad (r' = \pm 1, \pm 3, \pm 5, \dots),$$

and hence the above-mentioned lines correspond to the interval  $(-1, -k')$  of the  $d$ -plane. This interval is *not* in the domain (18.2), and hence again we cannot prove the periodicity of our solutions modulo  $4iK'$ .

Incidentally it is seen that our series are convergent in parts of the complex  $u$ -plane where neither power-series nor the Fourier-Jacobi series converge.

19. The preceding reasoning does *not* prove the transcendental Lamé functions to be only simply periodic, though it does show where the difficulties appear when one tries to establish the imaginary period of (17.1). Lamé functions of other types present analogous difficulties. Either a real or an imaginary period is readily derived from any of the representations; it seems very likely that it is the *only* primitive period of the solution represented by the respective series.

In fact this is the case. For a doubly periodic function of  $u$  having the primitive periods  $4K$  and  $4iK'$  should be a single-valued function of  $s, c, d$ . Now the point  $u = iK'$ , where  $s, c$  and  $d$  become infinite, is a branch-point of every solution of Lamé's equation unless  $n$  is an integer. Hence the transcendental Lamé functions belonging to non-integer values of  $n$  cannot be doubly periodic. A closer consideration shows that *transcendental* Lamé functions belonging to integer values of  $n$  are not doubly periodic either.

Hence the only doubly periodic solutions of Lamé's equation are the Lamé *polynomials*.

#### LAMÉ FUNCTIONS OF REAL PERIOD

20. Now it will be proved that series of Legendre functions of variable  $\pm d$  or  $\pm d/k'$  represent Lamé functions of real period. At first we shall assume that  $n$  is not an integer.

According to (7.2), the series (9.1) and (10.1) are convergent in the domain

$$\left| \frac{1-d}{1+d} \right| < \frac{1+k'}{1-k'} = \cot^2 \frac{1}{2} \theta, \quad (20.1)$$

*i.e.* outside a circle of the  $d$ -plane, the centre of which is  $d = -\frac{1}{2}(k' + 1/k')$ , the radius being  $\frac{1}{2}(1/k' - k') = k^2/(2k')$ . In the period parallelogram of the  $u$ -plane with vertices  $u = 0, 2K, 2K + 4iK', 4iK'$ , the domain of convergence is the part of the parallelogram outside a certain domain round the line connecting  $2iK'$  and  $2K + 2iK'$ . In particular a strip, including the real axis, is entirely inside the domain of convergence, and hence the real period  $2K$  or  $4K$  according as  $m$  is even or odd is readily established.

In the  $d$ -plane there is a branch-cut along the negative real axis from  $-1$  to  $-\infty$ . The corresponding branch-cuts in the  $u$ -plane are the segment joining  $u = iK'$  to  $u = 3iK'$ , and all congruent segments.

Replacing  $P_n^m(d)$  by  $P_n^m(-d)$  in (9.1) and (10.1) (the originating series will be denoted by (9.1)– and (10.1)– respectively) new Lamé functions are obtained. The domain of convergence in the  $d$ -plane will be obtained by reflection of the domain (20.1) on  $d = 0$ ; correspondingly in the  $u$ -plane by shifting about  $\pm 2iK'$ . Hence these solutions are again Lamé functions of *real* period  $2K$  or  $4K$ . They differ from (9.1) and (10.1) by being valid *not* on and near the real axis, but in a strip enclosing the line  $u = \sigma + 2iK'$  ( $\sigma$  real) and all congruent lines.

The series (9.1)Q and (10.1)Q, *i.e.* the series originating when in (9.1) and (10.1) respectively  $P_n^m$  is replaced by  $Q_n^m$ , represent Lamé functions of the second kind. They are convergent in the domain where both (9.1) and (9.1) – are convergent. Hence in the  $u$ -plane they are neither convergent on the real axis nor on the line  $u = 2iK' + \sigma$ . Their domain of convergence encloses the lines  $u = iK' + \sigma$  and  $u = 3iK' + \sigma$  ( $\sigma$  real). Nevertheless in general (*i.e.* except for integer  $n$ ) they have no period at all because it is not possible to pass from one period parallelogram into another one inside the domain of convergence without crossing branch-cuts.

21. Similarly the domain of convergence in the  $d$ -plane of the series (9.6) and (10.4) is characterised by

$$\left| \frac{k' - d}{k' + d} \right| < \frac{1 + k'}{1 - k'} = \cot^2 \frac{1}{2} \theta. \quad (21.1)$$

It is that part of the  $d$ -plane which is outside the circle with centre  $-\frac{1}{2}(2 - k^2)$  and radius  $\frac{1}{2}k^2$ . The corresponding domain in the period parallelogram excludes the line  $u = 2iK' + \sigma$  and a certain neighbourhood of it, and consists of all the rest of the period parallelogram. There are again branch-cuts joining  $iK'$  to  $3iK'$  and joining  $2iK'$  to  $2K + 2iK'$ , and also all congruent lines.

(14.2) and (14.6) are valid for transcendental Lamé functions too.

The domains of convergence of the corresponding series with  $P_n^m(-d/k')$  and  $Q_n^m(d/k')$  are obtained by the same process of reflecting (in the  $d$ -plane) and shifting (in the  $u$ -plane) respectively as before.

22. In case of an integer  $n \geq 0$ , we have  $m > n$  for the transcendental Lamé functions. In this case the respective domains of convergence do not change at all but all branch-cuts disappear. Therefore in this case we may pass from  $u$  to  $u + 2K$  or  $u + 4K$ , as the case may be, entirely inside the domain of convergence of any of our series. Hence *every* solution of Lamé's equation is a Lamé function in this case, this result being in agreement with the investigations of Ince (1940 *b*, § 5).

#### LAMÉ FUNCTIONS OF IMAGINARY PERIOD

23. Series of Legendre functions of variable  $s$  or  $ks$  represent Lamé functions of imaginary periods  $2iK'$  and  $4iK'$ . The discussion of these series being very similar to those with variable  $d$  and  $d/k'$ , a brief summary of the results will be given only. In fact, the two types of series transform into each other by Jacobi's imaginary transformation

$$\operatorname{dn}(u, k) = k' \operatorname{sn}(K' - iK + iu, k').$$

(11.1) and (11.7) are convergent outside of the circle with centre  $-\frac{1}{2}(k + 1/k)$  and radius  $k^{1/2}/(2k)$  of the  $s$ -plane. The period parallelogram in the  $u$ -plane has the vertices  $0, 4K, 4K + 2iK', 2iK'$ ; the series are convergent in this parallelogram except a certain domain round the line joining  $3K$  to  $3K + 2iK'$ . The branch-cuts are: in the  $s$ -plane the negative real axis from  $-1$  to  $-\infty$ , in the  $u$ -plane the two lines joining  $3K$  to  $3K + 2iK'$  and  $2K + iK'$  to  $4K + iK'$  respectively, and also all congruent lines. For instance, in the strip  $0 < \Re(u) < 2K$  an imaginary period is easily established being  $2iK'$  if  $m$  is even and  $4iK'$  if  $m$  is odd.

The domain of convergence of (11.1) – and (11.7) – is obtained by reflection on  $s = 0$  in the  $s$ -plane and by shifting parallel to the real axis about  $2K$  in the  $u$ -plane. These series too have an imaginary period, established for instance in the strip  $2K < \Re(u) < 4K$  and in congruent strips.

The series (11.1)Q and (11.7)Q represent Lamé functions of second kind and are convergent in the domain where *both* (11.1) and (11.1) – converge. Except for integer  $n$  and  $m > n$ , they have no period at all.

24. The series (11.2) and (11.8) are convergent outside of the circle with centre  $-\frac{1}{2}(1 + 1/k^2)$  and radius  $k^{1/2}/(2k^2)$  of the  $s$ -plane. The period parallelogram is the same as in § 23; the series are convergent in it, except in a certain domain round the line joining  $3K$  to  $3K + 2iK'$ . The branch-cuts in the  $u$ -plane are the line joining  $2K + iK'$  to  $4K + iK'$ , and all congruent lines.

The remarks of the preceding section regarding periods, series with  $P_n^m(-ks)$  and  $Q_n^m(ks)$  apply to this case too.

## LIMITING CASES

25. There are a few remarkable limiting cases in which Lamé's equation reduces to some simpler equation.

Let  $k$  approach zero; (2.1) becomes

$$\frac{d^2 y}{du^2} + ky = 0,$$

having the solutions  $\sin h^{\frac{1}{2}}u$  and  $\cos h^{\frac{1}{2}}u$ . It is interesting to see how the series of Legendre functions represent these solutions.

Take for instance (9.1). For small values of  $k$  we have

$$d = (1 - k^2 s^2)^{\frac{1}{2}} \sim 1 - \frac{1}{2} k^2 s^2, \quad (25.1)$$

and hence from the expression of  $P_n^m(z)$  by a hypergeometric series in § 4,

$$P_n^m(d) \sim P_n^m(1 - \frac{1}{2} k^2 s^2) \sim (-\frac{1}{2} k)^m \frac{\Gamma(n+m+1)s^m}{\Gamma(n-m+1)\Gamma(m+1)}. \quad (25.2)$$

Hence if  $k$  is very small, (9.1) is approximately

$$\sum_{m=0}^{\infty} (-\frac{1}{2} k)^m A_m \frac{\Gamma(n+m+1)}{\Gamma(m+1)} s^m = \sum_{m=0}^{\infty} \frac{\mathcal{A}_m s^m}{\Gamma(m+1)}. \quad (25.3)$$

Since in this case  $\lim k^2 \beta_{m-2}$  ( $k \rightarrow 0$ ) is finite, we have in the limit  $k \rightarrow 0$  for

$$\mathcal{A}_m = (-\frac{1}{2} k)^m A_m \Gamma(n+m+1)$$

the recurrence formula

$$\mathcal{A}_{m+2}/\mathcal{A}_m = m^2 - k = 4(\frac{1}{2}m - \frac{1}{2}\sqrt{h})(\frac{1}{2}m + \frac{1}{2}\sqrt{h}).$$

Hence

$$\frac{\mathcal{A}_{2m}}{\Gamma(2m+1)} = \frac{(-\frac{1}{2}\sqrt{h})_m (\frac{1}{2}\sqrt{h})_m}{(\frac{1}{2})_m m!} \mathcal{A}_0, \quad \frac{\mathcal{A}_{2m+1}}{\Gamma(2m+2)} = \frac{(\frac{1}{2} - \frac{1}{2}\sqrt{h})_m (\frac{1}{2} + \frac{1}{2}\sqrt{h})_m}{(\frac{3}{2})_m m!} \mathcal{A}_1, \quad (25.4)$$

and we have the two solutions

$$y_1 = \sum_{m=0}^{\infty} \frac{\mathcal{A}_{2m} s^{2m}}{\Gamma(2m+1)} = \mathcal{A}_0 F(-\frac{1}{2}\sqrt{h}, \frac{1}{2}\sqrt{h}; \frac{1}{2}; s^2)$$

and

$$y_2 = \sum_{m=0}^{\infty} \frac{\mathcal{A}_{2m+1} s^{2m+1}}{\Gamma(2m+2)} = \mathcal{A}_1 s F(\frac{1}{2} - \frac{1}{2}\sqrt{h}, \frac{1}{2} + \frac{1}{2}\sqrt{h}; \frac{3}{2}; s^2).$$

Now, since  $k=0$ , we have  $s = \sin u$  and (Gauss, 1866, p. 127, XX and XVI)

$$y_1 = \mathcal{A}_0 \cos h^{\frac{1}{2}}u \quad \text{and} \quad y_2 = \mathcal{A}_1 \sin h^{\frac{1}{2}}u. \quad (25.5)$$

The characteristic numbers belonging to periodic solutions are obviously  $h=m^2$ .

26. Another important limiting case is  $k \rightarrow 1$ . In this case we have to use the series of § 11 and shall choose (11.1). The corresponding limiting processes with the other series of § 11 are similar.

For  $k \rightarrow 1$  we have

$$s \rightarrow \tanh u \quad \text{and} \quad P_n^m(s) \rightarrow P_n^m(\tanh u).$$

Hence (11.1) changes to

$$\Sigma B_m \Gamma(n-m+1) P_n^m(\tanh u),$$

and the recurrence formulæ for the  $B_m$  degenerate into

$$\{m^2 - n(n+1) + h\} B_m = 0 \quad (m=0, 1, 2, \dots).$$

Hence all the  $B_m$ , with the possible exception of a single one,  $B_M$  say, are to vanish, and it

must be  $h = n(n+1) - M^2$ , the corresponding periodic solution (of imaginary period only) being  $P_n^M(\tanh u)$ . This result is, except in the notation, in agreement with Dr Ince's investigations (Ince, 1940 a, § 9).

26. Another limiting case of interest is when  $h$  tends to zero and at the same time  $n$  to infinity, so that  $\lim n(n+1)k^2$  remains finite. We shall put  $h = 2(-\theta)^{\frac{1}{2}}/n$  and make  $n$  tend to infinity,  $\theta$  being kept fixed. We have

$$d = (1 - k^2 s^2)^{\frac{1}{2}} \sim 1 - \frac{1}{2}k^2 s^2 \sim 1 + \frac{2\theta}{n^2} \sin^2 u \quad (n \rightarrow \infty),$$

and hence (Whittaker and Watson, 1927, § 17.4)

$$P_n^m(d) \sim P_n^m\left(1 + \frac{2\theta}{n^2} \sin^2 u\right) \sim n^m J_m(2i\sqrt{\theta} \sin u).$$

Thus Lamé's equation tends to Mathieu's equation

$$\frac{d^2 y}{du^2} + (h + 4\theta \sin^2 u)y = 0,$$

and the series (9.1) for instance to a series of the form

$$\Sigma A_m J_m(2i\theta^{\frac{1}{2}} \sin u),$$

which is Heine's expansion of the functions of the elliptic cylinder (Heine, 1878, p. 413) mentioned in § 2.

There is another similar limiting process:  $n$  tends to infinity so that  $h = n(n+1)$  and  $n(n+1)(1-k^2)$  tend to finite limits. This limiting process again yields series of Bessel functions, this time from the series of § 10, and so need not be dealt with *in extenso*.

#### OTHER EXPANSIONS

27. Besides the expansions dealt with already, there are some other expansions of Lamé functions by series of Legendre functions worthy of mention.

Applying Whipple's transformation [Hobson, 1931, p. 245, equation (92), and p. 247, equation (93)],

$$P_n^m(z) = \left(\frac{1}{2}\pi\right)^{\frac{1}{2}} e^{(n+\frac{1}{2})\pi i} \frac{(z^2-1)^{-\frac{1}{4}}}{\Gamma(-m-n)} Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right) \quad [\Re(z) > 0], \quad (27.1)$$

$$Q_n^m(z) = \left(\frac{1}{2}\pi\right)^{\frac{1}{2}} e^{m\pi i} \frac{\Gamma(m+n+1)}{(z^2-1)^{\frac{1}{4}}} P_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right) \quad [\Re(z) > 0], \quad (27.2)$$

to the associated Legendre functions occurring in our series, new expansions are readily obtained. These new expansions, representing the same functions and having the same domains of convergence, are entirely equivalent to, in fact only another way of writing of, the series already studied and need not be considered more closely.

Ince's Fourier-Jacobi expansions of Lamé functions lead to another type of series. We have, for instance,

$$\cos m\omega = \left(\frac{1}{2}\pi \sin \omega\right)^{\frac{1}{2}} P_{m-\frac{1}{2}}^{\frac{1}{2}}(\cos \omega), \quad (27.3)$$

and therefore Ince's series (2.2) may be written alternatively

$$y = d\left(\frac{1}{2}\pi s\right)^{\frac{1}{2}} \sum_{r=0}^{\infty} C_{2r} P_{2r-\frac{1}{2}}^{\frac{1}{2}}(c), \quad (27.4)$$

where  $C_{2r}$  denote Ince's coefficients, not those denoted by this letter in § 12.

It is hardly necessary to mention that this latter series, when convergent, is better suited for numerical calculations than the series of §§ 9-12. But it seems that there is no similar representation of the corresponding Lamé function of second kind.

# SUMMARY

28. This paper contains the investigation of certain properties of periodic solutions of Lamé's differential equation by means of representation of these solutions by (in general infinite) series of associated Legendre functions. *Terminating* series of associated Legendre functions representing Lamé *polynomials* have been used by E. Heine and G. H. Darwin. The latter used them also for numerical computation of Lamé polynomials. It appears that infinite series of Legendre functions representing transcendental Lamé functions have not been discussed previously. In two respects these series seem to be superior to the generally used power-series and Fourier-Jacobi series. (i) They are convergent in some parts of the complex plane of the variable where both power-series and Fourier-Jacobi series diverge. (ii) They permit by simply replacing Legendre functions of first kind by those of second kind, to deal with Lamé functions of second kind as well as Lamé functions of first kind (§ 15).

In §§ 2 and 8 of the present paper the series are heuristically deduced from the integral equations satisfied by periodic Lamé functions. Inserting the series found heuristically, with unknown coefficients, into Lamé's differential equation, recurrence relations for the coefficients are obtained (§§ 9-12). These recurrence relations yield the (in general transcendental) equations in form of (in general infinite) continued fractions for the determination of the characteristic numbers. The convergence of the series can be discussed completely.

There are altogether forty-eight different series. Every one of the eight types of Lamé polynomials is represented by six different series (see table in § 13). There are interesting relations (§ 14) between series representing the same function.

Next infinite series representing transcendental Lamé functions are discussed. It is seen that *transcendental* Lamé functions are only *simply-periodic* (§§ 18, 19). Lamé functions of real (§§ 20-22) and imaginary (§§ 23-24) period are represented by series of Legendre functions the variables of which are different in both cases.

The paper concludes with a brief discussion of the most important limiting cases, and a short mention of other types of series of Legendre functions representing Lamé functions.

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XXVIII.—The Discriminant of a Certain Ternary Quartic. By W. L. Edge,  
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1. By the *discriminant*  $D$  of a homogeneous polynomial  $\phi$  is, in accordance with the general custom, to be understood that function of its coefficients whose vanishing is the necessary and sufficient condition for the locus  $\phi=0$  to have a node. It is the resultant, or eliminant, of the set of equations obtained by equating all the first partial derivatives of  $\phi$  simultaneously to zero. If  $\phi$  contains  $n$  variables and is of order  $p$ , the degree of  $D$  in the coefficients of  $\phi$  is  $n(p-1)^{n-1}$ .

Suppose, now and henceforward, that  $\phi$  is a ternary quartic  $\phi(x_1, x_2, x_3)$ . Then  $D$  is of degree 27, and there is a procedure, attributed by Klein (1890, p. 56) to Gordan, for writing down a certain determinant which is a constant multiple of  $D$  without any extraneous factors; this determinant has fifteen rows, of which nine consist of elements linear in the coefficients of  $\phi$ , and the remaining six consist of elements of the third degree in these coefficients. Gordan's procedure is as follows. He remarks that if  $(x_1, x_2, x_3)$  is a node of  $\phi=0$ , it is common to the polar curves  $\phi_1=0$ ,  $\phi_2=0$ ,  $\phi_3=0$ , where  $\phi_i$  denotes  $\partial\phi/\partial x_i$ ; the nine quartic polynomials

$$x_1\phi_1, \quad x_1\phi_2, \quad x_1\phi_3, \quad x_2\phi_1, \quad x_2\phi_2, \quad x_2\phi_3, \quad x_3\phi_1, \quad x_3\phi_2, \quad x_3\phi_3,$$

whose coefficients are linear in those of  $\phi$ , therefore all vanish at the node. He adds six further quartic polynomials, with coefficients of the third degree in those of  $\phi$ , which also vanish at the node. The determinantal form of  $D$  follows instantly. These six latter polynomials are obtained by using a certain concomitant of  $\phi$  encountered by Dersch (1874, p. 510) in the theory of the bitangents, namely

$$_3T \equiv \sum (abc)^2 a_x^2 b_x^2 c_x^2 - x b_y^2 - y c_y^2 - r,$$

where the summation is over all non-negative integers  $p, q, r$  whose sum is 4 yet none of which exceeds 2,  $a, b, c$  being Aronhold symbols for  $\phi \equiv a_x^4 \equiv b_x^4 \equiv c_x^4$ . Gordan remarks that  $T$  must vanish identically in  $y$  if  $(x_1, x_2, x_3)$  is a node of  $\phi=0$ ; thus, if

$$T \equiv T_{11}\mathcal{Y}_1^2 + T_{22}\mathcal{Y}_2^2 + T_{33}\mathcal{Y}_3^2 + 2T_{23}\mathcal{Y}_2\mathcal{Y}_3 + 2T_{31}\mathcal{Y}_3\mathcal{Y}_1 + 2T_{12}\mathcal{Y}_1\mathcal{Y}_2,$$

the co-ordinates of the node cause each of the six expressions  $T_{ij}$  to vanish, and these expressions are quartic polynomials whose coefficients are of the third degree in those of  $\phi$ .

This strikingly elegant and convincing fashion of writing down a determinantal form for  $D$  is not at all well known; it has remained inviolate in its adornment of Klein's paper, and escaped the notice both of writers of textbooks and of compilers of the standard articles on plane algebraic curves. The object of these few pages is to furnish for the first time an illustration of Gordan's procedure by the actual working out of an example.

2. We choose for this purpose the specialised quartic

$$\phi(x_1, x_2, x_3) \equiv ax_1^4 + bx_2^4 + cx_3^4 + 6fx_2^2x_3^2 + 6gx_3^2x_1^2 + 6hx_1^2x_2^2,$$

and it is this that is henceforward signified by the symbol  $\phi$ . This specialised form is used by Salmon (1879, pp. 269-274) at the end of the chapter on quartic curves in his treatise, and he calculates, undaunted by the far from inconsiderable labour involved, several of its invariants and covariants. It may be the burden of carrying through these sesquipedalian calculations that is the cause of Salmon's surprising omission to notice what the discriminant of  $\phi$  actually is; although he alludes (p. 274) expressly to it he conveys no hint that he has detected its true

structure. Yet the detection is simple enough. For  $\phi=0$  becomes a pair of conics, and so has four nodes, if

$$\Delta \equiv \begin{vmatrix} a & 3h & 3g \\ 3h & b & 3f \\ 3g & 3f & c \end{vmatrix} = 0,$$

has a node at a vertex of the triangle of reference if any one of  $a, b, c$  vanishes, and has a pair of nodes on a side of the triangle of reference if any one of  $A, B, C$  (by which are signified the cofactors of  $a, b, c$  in  $\Delta$ ) vanishes. Thus it is to be expected that the factors  $abcA^2B^2C^2\Delta^4$  occur in  $D$ , and since they make up the requisite degree there can be no further factor except a numerical multiplier. The application of Gordan's procedure to  $\phi$  gives a result in agreement with this.

3. The working out of the determinantal form for  $D$  proves to be quite feasible. For it transpires that, with this special form of  $\phi$ ,  $T_{23}$  contains only three terms whose coefficients do not vanish, namely  $x_1^2x_2x_3, x_2^2x_1x_3, x_3^2x_1x_2$ . The same is clearly true of  $x_3\phi_2$  and  $x_2\phi_3$ . There is thus an isolated block of nine elements at the intersections of those three rows of  $D$  associated with  $T_{23}, x_3\phi_2, x_2\phi_3$ , and those three columns of  $D$  associated with  $x_1^2x_2x_3, x_2^2x_1x_3, x_3^2x_1x_2$ ; "isolated", that is, in the sense that all other elements of these rows are zero. A second such isolated set of nine elements occurs in the rows associated with  $T_{31}, x_1\phi_3, x_3\phi_1$ , and a third in the rows associated with  $T_{12}, x_2\phi_1, x_1\phi_2$ .  $D$  is therefore the product of the three three-rowed determinants (each of degree five in the coefficients of  $\phi$ ) arising from these isolated blocks and of an outstanding six-rowed determinant. The three-rowed determinants will be found to be numerical multiples of  $A\Delta, B\Delta, C\Delta$ , and the six-rowed determinant a numerical multiple of  $abcABC\Delta$ .

4. Dersch's concomitant is

$$3T \equiv (abc)^2\{a_y^2b_x^2c_x^2 + a_x^2b_y^2c_x^2 + a_x^2b_x^2c_y^2 + a_x^2b_xb_yc_xc_y + a_xa_yb_x^2c_xc_y + a_xa_yb_xb_yc_x^2\},$$

which, since  $a, b, c$  are equivalent symbols, gives

$$T \equiv (abc)^2\{a_y^2b_x^2c_x^2 + a_x^2b_xb_yc_xc_y\}.$$

Every symbol, whether it be  $a, b$  or  $c$ , appears in every term to degree four and, because of the special form of  $\phi$ , all such combinations of symbols vanish identically except

$$\begin{aligned} a_1^4 &= b_1^4 = c_1^4 = a, & a_2^4 &= b_2^4 = c_2^4 = b, & a_3^4 &= b_3^4 = c_3^4 = c, \\ a_2^2a_3^2 &= b_2^2b_3^2 = c_2^2c_3^2 = f, & a_3^2a_1^2 &= b_3^2b_1^2 = c_3^2c_1^2 = g, & a_1^2a_2^2 &= b_1^2b_2^2 = c_1^2c_2^2 = h. \end{aligned}$$

5. We proceed now to the calculation of

$$2T_{23} \equiv (abc)^2\{2a_2a_3b_x^2c_x^2 + (b_2c_3 + b_3c_2)a_x^2b_xc_x\}.$$

Any non-vanishing contribution from  $2(abc)^2a_2a_3b_x^2c_x^2$  must contain the product  $a_2^2a_3^2$  and so arise from

$$\begin{aligned} &4a_2^2a_3^2(b_3c_1 - b_1c_3)(b_1c_2 - b_2c_1)b_x^2c_x^2 \\ &= 4f(b_3b_1c_1c_2 - b_2b_3c_1^2 - b_1^2c_3c_2 + b_1b_3c_3c_1)b_x^2c_x^2 \\ &= 8f(b_3b_1c_1c_2 - b_2b_3c_1^2)b_x^2c_x^2 \end{aligned}$$

since the symbols  $b$  and  $c$  are equivalent. Rejecting, next, terms which vanish because the symbols  $b$  do not appear in the requisite combinations, and then omitting, from the terms retained, those which vanish because the symbols  $c$  do not appear in the requisite combinations, there remains

$$\begin{aligned} &8f(2b_3^2b_1^2c_1c_2x_3x_1 - 2b_2^2b_3^2c_1^2x_2x_3)c_x^2 \\ &= 16f(gc_1c_3c_x^2x_3x_1 - f c_1^2c_x^2x_2x_3) \\ &= 16f(2gc_1^2c_2^2x_1^2x_2x_3 - f[c_1^4x_1^4 + c_1^2c_2^2x_2^2 + c_1^2c_3^2x_3^2]x_2x_3) \\ &= 16f\{(2gh - af)x_1^4 - hf x_2^2 - fg x_3^2\}x_2x_3. \end{aligned} \tag{5.1}$$

In order to obtain  $2T_{23}$  we have to add to this the contribution from  $(b_2c_3 + b_3c_2)(abc)^2 a_x^2 b_x c_x$  which, since  $b$  and  $c$  are equivalent, is double the contribution from  $b_2c_3(abc)^2 a_x^2 b_x c_x$ . This is

$$a_x^2(b_2b_1x_1 + b_3^2x_2 + b_2b_3x_3)(c_1c_3x_1 + c_2c_3x_2 + c_3^2x_3)(abc)^2. \quad (5.2)$$

Now the two trinomials in brackets, when multiplied together, give a sum consisting of nine terms. Of these nine terms one is  $b_3^2c_3^2x_3x_3$ , and this gives rise, when multiplied by  $(abc)^2$ , to six non-vanishing contributions; in fact

$$\begin{aligned} & a_x^2 b_3^2 c_3^2 (abc)^2 \\ &= a_x^2 b_3^2 c_3^2 (a_1^2 b_1^2 c_1^2 + a_1^2 b_3^2 c_2^2 + a_2^2 b_1^2 c_3^2 + a_2^2 b_3^2 c_1^2 + a_3^2 b_1^2 c_2^2 + a_3^2 b_2^2 c_1^2) \\ &= a_x^2 \{ (bc + f^2) a_1^2 + (ch + fg) a_2^2 + (hf + bg) a_3^2 \} \\ &= (bc + f^2)(ax_1^2 + hx_2^2 + gx_3^2) + (ch + fg)(hx_1^2 + bx_2^2 + fx_3^2) + (hf + bg)(gx_1^2 + fx_2^2 + cx_3^2). \end{aligned} \quad (5.3)$$

Each of the remaining eight terms arising from the product of the two trinomials in (5.2) gives, it is found, only one non-vanishing contribution when multiplied by  $(abc)^2$ . The term  $b_2^2c_1c_3x_1x_2$ , for example, must be coupled with terms of  $(abc)^2$  that contain the product  $c_1c_3$ ; but in this squared determinant every term contains each suffix twice and twice only, so that no term which contains  $c_1c_3$  can contain either  $b_1^2$  or  $b_3^2$ . Thus, in order to provide a non-vanishing contribution,  $b_2^2c_1c_3x_1x_2$  must be multiplied by a term of  $(abc)^2$  which contains  $b_3^2c_1c_3$ , and the only such term is  $-2a_3a_1b_3^2c_1c_3$ . The same mode of reasoning shows that the term  $b_3^2c_2c_3x_2^2$  in the product of the trinomials must, in order that it should provide a non-vanishing contribution, be multiplied by the term  $-2a_3a_2b_3^2c_1c_3$  of  $(abc)^2$ . Considerations of this kind show that, taking the eight terms other than  $b_3^2c_3^2x_3x_3$  from the trinomial product,

$$\begin{aligned} & a_x^2(b_2^2c_1c_3x_1x_2 + b_3^2c_2c_3x_2^2 + b_1b_2c_3^2x_3x_1 + b_2b_3c_3^2x_3^2 + b_1b_2c_1c_3x_1^2 + b_2b_3c_1c_3x_3x_1 + b_1b_2c_2c_3x_1x_2 \\ & \quad + b_2b_3c_2c_3x_2x_3)(abc)^2 \\ &= -4bg^2x_1^2x_3x_3 - 4hbf^2x_3^2x_3 - 4ch^2x_1^2x_3x_3 - 4f^2gx_3x_3^3 + 4fghx_1^2x_3x_3 + 4fghx_1^2x_2x_3 + 4fghx_1^2x_2x_3 \\ & \quad - 2f^2(ax_1^2 + hx_2^2 + gx_3^2)x_2x_3. \end{aligned}$$

Adding this to the product of  $x_2x_3$  and the expression (5.3) we find, as the contribution of  $b_2c_3(abc)^2 a_x^2 b_x c_x$  to  $2T_{23}$ , the product of  $x_2x_3$  and

$$(abc + 14fgh - af^2 - 3bg^2 - 3ch^2)x_1^2 + 2(bch + bfg - 2hf^2)x_2^2 + 2(bcg + chf - 2f^2g)x_3^2.$$

This result, combined with (5.1), gives finally

$$T_{23} \equiv \{(abc + 3fgh - 9af^2 - 3bg^2 - 3ch^2)x_1^2 + 2(bch + bfg - 6hf^2)x_2^2 + 2(bcg + chf - 6f^2g)x_3^2\}x_2x_3,$$

thus showing, as foretold in section 3, that  $T_{23}$  contains only three of the fifteen terms that can be present in a homogeneous quartic polynomial in three variables.

6. Since

$$\frac{1}{4}x_3\phi_2 \equiv (3hx_1^2 + bx_2^2 + 3fx_3^2)x_2x_3$$

and

$$\frac{1}{4}x_2\phi_3 \equiv (3gx_1^2 + 3fx_2^2 + cx_3^2)x_2x_3,$$

the isolated determinant that is associated with the triad of polynomials  $T_{23}$ ,  $x_3\phi_2$ ,  $x_2\phi_3$  is a numerical multiple of

$$\begin{vmatrix} abc + 3fgh - 9af^2 - 3bg^2 - 3ch^2 & 3h & 3g \\ 2(bch + bfg - 6hf^2) & b & 3f \\ 2(bcg + chf - 6f^2g) & 3f & c \end{vmatrix},$$

the determinant being written in this rather than in the transposed form to economise horizontal space. The last two columns are the same as the last two columns of  $\Delta$ . Now add to the first column the product of  $ch - 5fg$  and the second column as well as the product of  $bg - 5hf$

and the third column. This adjustment does not change the value of the determinant, and it turns its first column into the product of  $bc - 9f^2$ , which is  $A$ , and the first column of  $\Delta$ . So that the value of the determinant is  $A\Delta$ .

Since the simultaneous cyclic interchanges of the suffixes 1, 2, 3, of the coefficients  $a, b, c$  and of the coefficients  $f, g, h$  turn  $T_{23}, x_3\phi_2, x_2\phi_3$  into  $T_{31}, x_1\phi_3, x_3\phi_1$ , respectively, the isolated determinant associated with this latter triad of polynomials is a multiple of  $B\Delta$ . And that associated with the triad  $T_{12}, x_2\phi_1, x_1\phi_2$  is a multiple of  $C\Delta$ .

7. The three isolated three-rowed determinants having now been accounted for, it only remains to find the outstanding six-rowed determinant. This is constituted by elements that occupy those of the fifteen rows and columns which are not involved in any of the three determinants that have been dealt with; it is thus associated with the six polynomials

$$T_{11}, \quad T_{22}, \quad T_{33}, \quad x_1\phi_1, \quad x_2\phi_2, \quad x_3\phi_3,$$

but only with those of their terms wherein occur the combinations

$$x_1^4, \quad x_2^4, \quad x_3^4, \quad x_2^2x_3^2, \quad x_3^2x_1^2, \quad x_1^2x_2^2.$$

Now

$$T_{11} \equiv (abc)^2 \{a_1^2 b_x^2 c_x^2 + a_x^2 b_x c_x b_1 c_1\},$$

and it will be agreed that the details of the evaluation may be passed over since enough explanation has already been given in the calculation of  $T_{23}$  to enable a reader to carry through the necessary manipulations. The upshot is that  $\frac{1}{2}T_{11}$  contains the terms

$$6aghx_1^4 + b(af + gh)x_2^4 + c(af + gh)x_3^4 + (abc + 6fgh - 3af^2)x_2^2x_3^2 + 3(ach + afg - 2g^2h)x_3^2x_1^2 + 3(abg + ahf - 2gh^2)x_1^2x_2^2,$$

while the corresponding terms in  $\frac{1}{2}T_{22}$  and  $\frac{1}{2}T_{33}$  are of course derived from these by imposing cyclic interchanges on 1, 2, 3, on  $a, b, c$  and on  $f, g, h$ . Hence, since

$$\begin{aligned} \frac{1}{4}x_1\phi_1 &\equiv ax_1^4 + 3hx_1^2x_2^2 + 3gx_1^2x_3^2, \\ \frac{1}{4}x_2\phi_2 &\equiv bx_2^4 + 3fx_2^2x_3^2 + 3hx_2^2x_1^2, \\ \frac{1}{4}x_3\phi_3 &\equiv cx_3^4 + 3gx_3^2x_1^2 + 3fx_3^2x_2^2, \end{aligned}$$

the six-rowed determinant is a numerical multiple of

$$\begin{vmatrix} 6agh & b(af + gh) & c(af + gh) & abc + 6fgh - 3af^2 & 3(ach + afg - 2g^2h) & 3(abg + ahf - 2gh^2) \\ a(bg + hf) & 6bhf & c(bg + hf) & 3(bch + bfg - 2hf^2) & abc + 6fgh - 3bg^2 & 3(abf + bgh - 2h^2f) \\ a(ch + fg) & b(ch + fg) & 6cfg & 3(bcg + chf - 2f^2g) & 3(caf + cgh - 2fg^2) & abc + 6fgh - 3ch^2 \\ a & . & . & . & 3g & 3h \\ . & b & . & 3f & . & 3h \\ . & . & c & 3f & 3g & . \end{vmatrix}$$

The factors  $a, b, c$  can be removed one from each of the first three columns of this determinant. Having removed them, we then modify the fourth column by subtracting from it the product of  $3f$  and the sum of the second and third columns; similarly we subtract from the fifth the product of  $3g$  and the sum of the third and first columns, and from the sixth the product of  $3h$  and the sum of the first and second columns. This produces a block of nine zeros in the right-hand bottom corner of the determinant, while the nine elements in the right-hand top corner constitute the determinant

$$\begin{vmatrix} a(bc - 9f^2) & 3h(ca - 9g^2) & 3g(ab - 9h^2) \\ 3h(bc - 9f^2) & b(ca - 9g^2) & 3f(ab - 9h^2) \\ 3g(bc - 9f^2) & 3f(ca - 9g^2) & c(ab - 9h^2) \end{vmatrix},$$

which is  $ABCA$ . The six-rowed determinant is thus a numerical multiple of  $abcABCA$ .

This is all the information that we require, and Gordan's procedure has now established the discriminant of  $\phi$  to be

$$D = abcA^2B^2C^2\Delta^4,$$

no numerical multiplier being necessary if it is stipulated that the term  $a^9b^9c^9$  has coefficient +1.

8. Before closing, a word may perhaps be said concerning the more special quartic

$$\psi \equiv \lambda(x_1^4 + x_2^4 + x_3^4) + 6\mu(x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_2^2),$$

for which

$$a = b = c = \lambda,$$

$$A = B = C = \lambda^2 - 9\mu^2 = (\lambda + 3\mu)(\lambda - 3\mu),$$

$$\Delta = \lambda^3 - 27\lambda\mu^2 + 54\mu^3 = (\lambda + 6\mu)(\lambda - 3\mu)^2;$$

$$D = abcA^2B^2C^2\Delta^4 = \lambda^3(\lambda + 3\mu)^6(\lambda + 6\mu)^4(\lambda - 3\mu)^{14}.$$

If  $\lambda = 0$ , the curve  $\psi = 0$  is trinodal, having a node at each vertex of the triangle of reference.

If  $\lambda = -3\mu$ , the curve consists of four lines and so has six nodes; the quadrilateral  $q$  formed by the lines has the triangle of reference for diagonal triangle.

If  $\lambda = -6\mu$ , the curve is a pair of conics and so has four nodes.

If  $\lambda = 3\mu$ , then not only does  $\Delta$  vanish but its rank sinks to 1, and all its first minors, including  $A$ ,  $B$ ,  $C$ , also vanish;  $\psi = 0$  is a *repeated conic*  $S$ .

As  $\lambda : \mu$  varies,  $\psi = 0$  describes a pencil of quartic curves which all touch one another at the eight intersections of  $S$  with the sides of  $q$ . Among the members of a pencil of plane quartics there are, in general, 27 nodal curves; they correspond to those values of the parameter for which the discriminant vanishes. In the above pencil these 27 curves are all accounted for by only four distinct curves, these four curves being reckoned with respective multiplicities 3, 4, 6, 14. No other curve of the pencil can possess a node. That the repeated conic contributes 14 to the total of 27 nodal curves is noteworthy.

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XXIX.—On a Problem in Correlated Errors. By A. C. Aitken, D.Sc., F.R.S.,  
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(MS. received January 16, 1946. Read May 6, 1946)

I. INTRODUCTORY

THE problem with which this paper is concerned arose in the discussion of a series of chronometric observations, but it is of more general application, and is capable of wide extension. Pairs of readings  $(x_i, y_i)$  were taken at times  $t_i, i=1, 2, \dots, n$ . These readings were known to be affected by respective errors  $(\xi_i, \eta_i)$  from sources different but possessing some common part. It was important to have an estimate of the consequent correlation and to assess its precision. The assumptions made in the particular experiment were that  $x$  and  $y$  were both linear in  $t$ , representable by  $x=a_0+a_1t, y=b_0+b_1t$ , and that the distributions of error in  $x$  and  $y$  were normal. The parameters  $a_0$  and  $a_1, b_0$  and  $b_1$  were therefore obtained from two separate sets of normal equations, and the unknown correlation was then estimated from the sum of products of corresponding residuals  $u_i, v_i$ , one from each set. In the corresponding situation in  $n$  samples  $(x_i, y_i)$  from a bivariate normal distribution the mean value of  $\sum(x_i - \bar{x})(y_i - \bar{y})$  is  $(n-1)\rho\sigma_1\sigma_2$ , where  $\sigma_1^2, \sigma_2^2$  are the variances of  $x$  and  $y$  and  $\rho\sigma_1\sigma_2$  is their product moment. One might therefore anticipate, by analogy, that in the present case the mean value of  $\sum u_i v_i$  would be  $(n-2)\rho\sigma_1\sigma_2$ . So indeed it proves to be, and the sampling variance of  $\sum u_i v_i$  conforms likewise with standard results; but it is desirable, by an extension of the problem, both to see why this is so and to take notice of cases where the analogy fails to hold.

2. GENERALIZATION OF THE PROBLEM

We shall extend the problem, adopting notations suited to a matrix formulation. Let there be  $n$  independent pairs of observations  $(x_i + \xi_i, y_i + \eta_i)$ , where  $\xi_i, \eta_i$  are errors normally distributed with variances  $\sigma_1^2, \sigma_2^2$  and product moment  $\rho\sigma_1\sigma_2$ . Let  $x$  and  $y$  be represented by

$$x = a_0 + a_1 p_1(t) + a_2 p_2(t) + \dots + a_{k-1} p_{k-1}(t), \quad (1)$$

$$y = b_0 + b_1 p_1(t) + b_2 p_2(t) + \dots + b_{k-1} p_{k-1}(t), \quad (2)$$

where the  $p_j(t)$  are values of a prescribed basis of functions, linearly independent over the range of  $t$ . These might commonly be polynomial or harmonic, but might even be arbitrary. Let us write

$$x = \{x_1 x_2 \dots x_n\}, \quad y = \{y_1 y_2 \dots y_n\}; \quad (3)$$

$$a = \{a_0 a_1 \dots a_{k-1}\}, \quad b = \{b_0 b_1 \dots b_{k-1}\}; \quad (4)$$

and so on for all vectors concerned, and let  $P = [p_j(t_i)]$  be the  $n \times k$  matrix of the functional values. Then the two sets of observational equations are  $Pa = x, Pb = y$ ; the normal equations are  $P'Pa = P'x, P'Pb = P'y$ , and so the solutions for  $a$  and  $b$  are

$$a = (P'P)^{-1}P'x, \quad b = (P'P)^{-1}P'y. \quad (5)$$

The vectors of residuals are therefore

$$u = Mx, \quad v = My, \quad \text{where } M = I - P(P'P)^{-1}P', \quad (6)$$

and so the variance matrices of the two separate sets of residuals are  $\sigma_1^2 M, \sigma_2^2 M$ , since  $MM' = M$ . In view of the linear independence of the  $k$  functions  $p_j(t)$  we see that  $P$  is of rank  $k$ , and so  $P(P'P)^{-1}P'$  is of rank  $k$ , and  $M$  is of rank  $n - k$ . Again, since  $M^2 = M$ , it follows that  $M$  is idempotent with  $n - k$  latent roots equal to 1, and  $k$  roots equal to 0. It is therefore reducible

by orthogonal transformation  $HMH'$  to a diagonal matrix in which the leading submatrix of order  $n-k$  is a unit matrix, all other elements being null.

The variance matrix of the partitioned vector  $z = \{x; y\}$  is

$$V = \begin{bmatrix} \sigma_1^2 I & \rho \sigma_1 \sigma_2 I \\ \rho \sigma_1 \sigma_2 I & \sigma_2^2 I \end{bmatrix}, \quad (7)$$

a matrix of order  $2n \times 2n$ , quadripartite as above. The multiple normal probability differential of  $z$  is thus, apart from a constant factor,  $\exp(-\frac{1}{2}z'V^{-1}z)dz$ , where

$$dz = dx_1 dx_2 \dots dx_n dy_1 \dots dy_n.$$

Let us write  $\Sigma u_i v_i$ , namely

$$u'v = x'M'My = x'My, \quad (8)$$

as a quadratic form in  $z$ ,

$$\frac{1}{2}z' \begin{bmatrix} \cdot & M \\ M & \cdot \end{bmatrix} z = \frac{1}{2}z' Lz. \quad (9)$$

We construct a moment-generating function (m.g.f.)  $G(a)$  for  $u'v$  by evaluating the  $2n$ -fold integral of

$$\text{const. exp}(\frac{1}{2}az'Lz) \exp(-\frac{1}{2}z'V^{-1}z) \quad (10)$$

over the range  $-\infty$  to  $\infty$  in all variables. By the standard and well-known result, since a continuous range of values of  $a$  can be found such that the quadratic form in the exponent is negative definite over the whole range, we obtain

$$G(a) = |I - aVL|^{-\frac{1}{2}}, \quad (11)$$

the constant factor being fixed by the fact that the term independent of  $a$  must be the moment of zero order, namely unity. Writing this result in partitioned form, we have

$$\begin{aligned} G(a) &= \begin{vmatrix} I - a\rho\sigma_1\sigma_2M & a\sigma_1^2M \\ a\sigma_2^2M & I - a\rho\sigma_1\sigma_2M \end{vmatrix}^{-\frac{1}{2}} \\ &= |I - 2a\rho\sigma_1\sigma_2M - a^2(1-\rho^2)\sigma_1^2\sigma_2^2M|^{-\frac{1}{2}}, \end{aligned} \quad (12)$$

since the submatrices are commutative in multiplication, and  $M^2=M$ . The orthogonal transformation  $H(\dots)H'$  can now be applied to the matrix enframed in this determinant, thus producing a purely diagonal matrix having units for its last  $k$  diagonal elements. So finally, expanding by the product of the first  $n-k$  diagonal elements, we have

$$G(a) = \{1 - 2\rho\sigma_1\sigma_2a - (1-\rho^2)\sigma_1^2\sigma_2^2a^2\}^{-\frac{1}{2}(n-k)}. \quad (13)$$

The coefficient of  $a$  in the expansion of the above gives the mean value of  $\Sigma u_i v_i$  as  $(n-k)\rho\sigma_1\sigma_2$ . The coefficient of  $a^2/2!$  gives the mean square, and this, when adjusted so as to refer to the mean as origin, yields the sampling variance  $(n-k)(1+\rho^2)\sigma_1^2\sigma_2^2$ .

### 3. ANALOGY WITH STANDARD RESULTS

The form of these results is not unfamiliar. The m.g.f. of the product-moment estimate from a sample of  $n$  bivariate normally correlated observations can be derived (cf. Aitken, 1931) by a similar procedure, and is the same as  $G(a)$  in § 2 (13), but with  $n-1$  instead of  $n-k$ . It is indeed the special case in which the representation of  $x$  and  $y$  is by single constants. By a different approach Wishart (1928) found in this case the values of the sampling variance of the product moment, as well as many higher moments. We shall now trace the analogy to its source, and shall also show under what conditions it fails to hold.

Let us make an orthogonal transformation from  $x$  to  $Hx$ ,  $y$  to  $Hy$ , where  $H$  is such that  $HH'=I$  and  $HMH'=J_{n-k}$ , this last matrix being that particular diagonal canonical form of  $M$  which has the first  $n-k$  diagonal elements equal to unity and the remainder null. We have then  $Hu=J_{n-k}Hx$ ,  $Hv=J_{n-k}Hy$ , while  $u'v=u'H'Hv$ ,  $\xi'\eta=\xi'H'H\eta$ . Further, since the

errors  $\xi$  are uncorrelated, so are their transforms  $H\xi$ ; likewise  $H\eta$ ; and the respective variances are unchanged, since  $\xi'\xi = \xi'H'H\xi$ ,  $\eta'\eta = \eta'H'H\eta$ .

Thus the problem of the paper may be transferred entire, but in a simpler version, to the equivalent vectors  $Hx$ ,  $Hy$ ,  $Hu$ ,  $Hv$ ,  $H\xi$ ,  $H\eta$ ; and since  $Hu$  now consists of the first  $n-k$  elements of  $Hx$ , and  $Hv$  of the first  $n-k$  elements of  $Hy$ , we have therefore  $n-k$  linearly independent transformed residuals  $Hu$  correlated with a similar set  $Hv$ . In the bivariate sampling problem to which allusion has been made we have a similar pairing of sets of  $n-1$  linearly independent transformed residuals, the loss of a unit of rank ("degree of freedom") being due in that case to the fact that the original residuals were deviations from the means of sample. Thus  $n-1$  in the earlier problem plays the part of  $n-k$  in the present one. Indeed the derivation of  $G(\alpha)$  in the earlier problem (Aitken, 1931) might have been simplified in a number of respects by the methods of the present paper.

Naturally the question of the adequacy of the representation of  $x$  and  $y$  will in every case be the subject of examination *a posteriori*, by the usual division of the sum of squared deviations from the mean into two parts, namely the part used up in fitting the  $k$ -terms and the part that may properly be called the sum of squared residuals. The principles of this "analysis of variance", as applied (Fisher and Yates, 1943) to the fitting of orthogonal terms, are well known.

#### 4. CASE OF DIFFERENT REPRESENTATIONS

For its tractable handling the problem clearly depends on the condition that the basis of functions chosen for the representation of  $x$  and  $y$  shall be the same for each. If a different basis is chosen for each, expressed by matrices  $P$  and  $T$ , say, then the transforming matrices for the residuals,  $M = I - P(P'P)^{-1}P'$  and  $N = I - T(T'T)^{-1}T'$ , will in general be non-commutative. For example, if  $x$  is to be represented by a basis of polynomial terms, and  $y$  by one of harmonic terms, the evaluation of the determinantal expression for  $G(\alpha)$  will become more complicated. There is, however, one intermediate case in which the commutative rule still holds. Let it be supposed that by suitable linear combination we have replaced the basis  $P$  by an equivalent orthonormal basis  $Q$ . This can always be done. Let  $x$  be expressed in terms of  $Q_h$  and  $y$  in terms of  $Q_k$ , where  $Q_k$  is obtained from  $Q_h$  by deleting the last  $h-k$  columns. Thus  $x$  is expressed by  $h$  functions, and  $y$  by a subset of  $k$  of those functions, all from one orthogonal set. Denoting  $QQ'$ , the "graduating matrix" (Aitken, 1945), by  $G$ , we have (*loc. cit.*)  $G_h G_k = G_k G_h = G_k$ . Further, if we denote the vectors of residuals by  $u = M_h x$ ,  $v = M_k y$ , we have

$$M_h M_k = M_k M_h = (I - G_h)(I - G_k) = I - G_h = M_h. \quad (1)$$

The procedure adopted in § 2 then leads to

$$u'v = \frac{1}{2} z' \begin{bmatrix} \cdot & M_h \\ M_k & \cdot \end{bmatrix} z, \quad (2)$$

and so the partitioned determinantal form for  $G(\alpha)$  becomes

$$G(\alpha) = \begin{vmatrix} I - \alpha \rho \sigma_1 \sigma_2 M_h & \alpha \sigma_1^2 M_h \\ \alpha \sigma_2^2 M_k & I - \alpha \rho \sigma_1 \sigma_2 M_k \end{vmatrix}^{-\frac{1}{2}}. \quad (3)$$

This again, since the submatrices are commutative, becomes

$$| I - \alpha \rho \sigma_1 \sigma_2 (M_h + M_k) - \alpha^2 (1 - \rho^2) \sigma_1^2 \sigma_2^2 M_h |^{-\frac{1}{2}}. \quad (4)$$

Now in this case  $M_h$  and  $M_k$  can be simultaneously transformed to diagonal canonical form by one and the same orthogonal matrix. For let us extend the orthonormal basis until it includes  $n$  functions  $q_j(x)$ ,  $j=0, 1, 2, \dots, n-1$ . Its matrix  $Q$  is then a non-singular orthogonal matrix of order  $n \times n$ . Evidently we have

$$Q'(Q_h Q_h')Q = \begin{bmatrix} I \\ \cdot \end{bmatrix} [I \cdot] = \begin{bmatrix} I & \cdot \\ \cdot & \cdot \end{bmatrix} = J_h, \quad (5)$$

where the submatrices denoted by  $I$  are of order  $h \times h$ . This means that the canonical form



of the complementary matrix  $M_h = I - G_h$  has its *last*  $n - h$  diagonal elements equal to unity and the rest null; but for our purpose it does not matter whether the units are in leading positions or last positions. Transforming therefore the matrix enframed in (4) above by  $Q'(\dots)Q$ , and expanding by diagonal elements (of three different kinds), we obtain  $G(\alpha)$  as

$$\begin{vmatrix} 1 - 2\rho\sigma_1\sigma_2\alpha - (1 - \rho^2)\sigma_1^2\sigma_2^2\alpha^2 & \dots & -\frac{1}{2}(n-h) \\ \vdots & & \vdots \\ 1 - \rho\sigma_1\sigma_2\alpha - (1 - \rho^2)\sigma_1^2\sigma_2^2\alpha^2 & \dots & -\frac{1}{2}(h-k) \end{vmatrix} \quad (6)$$

The result is not so simple as before. The coefficient of  $\alpha$  gives the mean of  $\sum u_i v_i$  as  $\frac{1}{2}(2n - h - k)\rho\sigma_1\sigma_2$ . The coefficient of  $\alpha^2/2!$  gives the mean square, and this, adjusted so as to refer to the mean as origin, gives the sampling variance

$$\{(n - k)(1 + \rho^2) - \frac{3}{2}(h - k)\rho^2\}\sigma_1^2\sigma_2^2, \quad (7)$$

again less simple than before.

The sampling probability distribution of the estimate of product moment,  $\sum u_i v_i / (n - \frac{1}{2}h - \frac{1}{2}k)$ , is also different in this case from that of the estimate  $\sum u_i v_i / (n - k)$  derived from § 2. It is known (*cf.* Wishart and Bartlett, 1932, 1933) that in the normal bivariate sampling problem the estimate of product moment is distributed according to a certain Bessel function of the second kind with imaginary argument. For the case of § 2 we have merely to take the Bessel function, as given in Wishart and Bartlett, and change  $n - 1$  into  $n - k$ . In the same way the distribution of  $r$ , the estimate of  $\rho$  obtained from the mean squares and mean product of § 2, is Fisher's distribution of  $r$  (Fisher, 1915) with  $n - k$  degrees of freedom. But these no longer hold in the present section; the distribution of estimate of product moment, for example, will involve at least a series of Bessel functions, and the distribution of the estimate of  $\rho$  will be more complicated than that of Fisher.

The simultaneous orthogonal transformation of  $M_h$  and  $M_k$  can be given a further extension. Let us suppose that  $x$  is represented in terms of  $h$  functions or columns from  $Q$ , and  $y$  in terms of  $k$  columns, neither of these sets being necessarily from consecutive columns; and suppose that these sets have  $s$  columns in common. If we denote the matrices of these three sets, namely the first two and their "intersection", by  $Q_h, Q_k, Q_s$ , it follows easily under simultaneous orthogonal transformation by the whole matrix  $Q$  that  $G_h G_k = G_k G_h = G_s$ , and that the diagonal units in the canonical forms of  $G_h, G_k$  and  $G_s$  are in those respective columns that characterize  $Q_h, Q_k$  and  $Q_s$ . The complementary canonical forms of  $M_h$  and  $M_k$  have a community or intersection involving  $n - h - k + s$  unit diagonal elements, and another of  $s$  zero elements, the remaining  $h + k - 2s$  units appearing singly among them. We have therefore, for  $G(\alpha)$ ,

$$\begin{vmatrix} 1 - 2\rho\sigma_1\sigma_2\alpha - (1 - \rho^2)\sigma_1^2\sigma_2^2\alpha^2 & \dots & -\frac{1}{2}(n-h-k+s) \\ \vdots & & \vdots \\ 1 - \rho\sigma_1\sigma_2\alpha - (1 - \rho^2)\sigma_1^2\sigma_2^2\alpha^2 & \dots & -\frac{1}{2}(h+k-2s) \end{vmatrix} \quad (8)$$

from which the mean of  $\sum u_i v_i$ , namely  $\frac{1}{2}(2n - h - k)\rho\sigma_1\sigma_2$ , and the variance

$$\{(n - s)(1 + \rho^2) - \frac{3}{2}(h + k - 2s)\rho^2\}\sigma_1^2\sigma_2^2, \quad (9)$$

are derived.

The extension considered here is by no means far-fetched. It might easily be the case, for example, that  $x$  was represented by a set of odd functions,  $y$  by a set of even functions, taken from one basis  $Q$ .

## 5. CASE OF DIFFERENT BASIC REPRESENTATIONS

We have next the case in which the respective bases for  $x$  and  $y$  belong to different sets of orthogonal functions. There is then in general no such relation as  $MN = NM$ , where we use  $M$  and  $N$  for the transforming matrices for residuals, nor is there any simultaneous transformation of  $M$  and  $N$  to diagonal form. If  $x$  is polynomial and  $y$  harmonic, for example, we cannot combine the two bases and orthonormalize them, for that would have the effect of making both  $x$  and  $y$  a mixture of polynomial and harmonic functions. However, we reach without difficulty the stage of § 4 (3). Here the first check occurs; but by linear operations upon "rows" of submatrices,

$$\text{row}_1 + \rho \frac{\sigma_1}{\sigma_2} \text{row}_2, \quad \text{row}_2 - \alpha \sigma_1^2 M, \quad (1)$$

we obtain

$$G(\alpha) = \begin{vmatrix} I - \alpha \rho \sigma_1 \sigma_2 (M + N) - \alpha^2 (1 - \rho^2) \sigma_1^2 \sigma_2^2 MN & \dots \\ \vdots & \ddots \end{vmatrix} - \frac{1}{2}, \quad (2)$$

the relation of which to the earlier forms of  $G(\alpha)$  is readily perceived. Since we cannot

simultaneously transform  $M$  and  $N$  into diagonal shape, the most that we can do is to record the mean value, the coefficient of  $\alpha$  in the expansion. This is the trace of  $\frac{1}{2}\rho\sigma_1\sigma_2(M+N)$ , namely  $\frac{1}{2}(2n-h-k)\rho\sigma_1\sigma_2$ , since the separate traces of  $M$  and  $N$  are  $n-h$ ,  $n-k$ . For example, if  $x$  is represented by a quadratic polynomial and  $y$  by 5 terms of a Fourier series, the mean value of  $\sum u_i v_i$  will be  $(n-4)\rho\sigma_1\sigma_2$ .

The sampling variance of  $\sum u_i v_i$ , on the other hand, is in this case difficult to evaluate, though quite explicit. The coefficient of  $\alpha^2/2!$  in the expansion of  $G(\alpha)$  involves the diagonal elements of  $MN$ , as well as *all principal minors of the second order* in  $M+N$ , and though these might be found and summed in any particular case, the process will be laborious if  $n$  is large, for there are  $\frac{1}{2}n(n-1)$  such minors. The fact is that there are no convenient general formulæ for the traces of compounds of a matrix sum  $A+B$ . Therefore we leave  $G(\alpha)$  in determinantal form.

## 6. EXTENSION TO MULTIVARIATE CASE

Finally, as might be anticipated, the result of § 2 (13) admits of extension to the case of  $s$  variates ( $x, y, z, \dots$ ) observed at a succession of times  $t$ . Let us assume that the vector of  $s$  errors  $\epsilon_i$  at each observation obeys a multivariate normal law characterized by  $\epsilon' V^{-1} \epsilon$ . Then, provided always that the representations of  $x, y, z, \dots$  are in terms of the same basis of orthonormal functions, a joint m.g.f. for all sums of squares and binary products of corresponding residuals may be constructed by assigning an indeterminate  $\alpha_{ji}$  to carry the moments of each; and one arrives, through the commutativity of all submatrices in a partitioned determinant, at the following function of the elements of the matrix  $A = [\alpha_{ji}]$ ,

$$G(A) = |I - VA|^{-\frac{1}{2}(n-k)}. \quad (1)$$

Corresponding expressions, involving many factors like the above, could be derived for the case where  $x, y, z, \dots$  are represented by different selections, with varying community of columns, from the same orthonormal basis given by  $Q$ . The question is one of the intersections of complementary sets of given sets; and a combinatory diagrammatic rule can be found for the several exponents. These wide generalizations are remote from application. One can do well enough by estimating variances for one variate at a time, and product moments according to pairs of variates.

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[*Corrigendum*.—The opportunity is taken of making a correction in a previous paper, "On the Independence of Linear and Quadratic Forms in Samples of Normally Distributed Variates", *Proc. Roy. Soc. Edin.*, LX, 1940, 40-46. On p. 45, the coefficient of the middle term in the bracket on the left of (3) should be +2, not -2. The rest of the section is to be deleted except for the sentence, "This is not independent of  $\beta$ ".]

XXX.—On Hill's Problems with Complex Parameters and a Real Periodic Function. By M. J. O. Strutt. *Communicated by* Sir EDMUND WHITTAKER, F.R.S. (With Five Text-figures.)

(MS. received November 1, 1945. Read May 6, 1946)

#### SUMMARY

HILL's differential equation (1.1) derives its importance from being the prototype of the different equations of Lamé and of the equation of Mathieu, which are connected with wave and potential problems in mathematical physics. Besides this, numerous instances of its occurrence in problems of elasticity and of dynamical or statical stability are known. In the present treatment, conditions are reversed with respect to most of the older publications, since the characteristic multiplier  $\sigma$  of equations (1.2) is not sought as a function of the given parameters  $\lambda$  and  $\gamma$  of equation (1.1), but  $\sigma$  is supposed given and the corresponding values of  $\lambda$  and  $\gamma$  are regarded as unknown. Thus a linear homogeneous boundary value problem of the second order and of non-self-adjoint type ensues, the values of  $\sigma$  and of  $\lambda$ ,  $\gamma$  being in general *complex*. On this latter point the present paper considerably enlarges the scope of some previous papers published by the author during the war along somewhat similar lines but for *real* characteristic values (Nos. 13–18 of the references at the end).

In § 1 the problems are stated together with a number of recently found facts pertaining to the case of real characteristic values. In § 2 Green's function, on which the present treatment is largely based, is calculated in terms of two linearly independent solutions for general values of  $\sigma$  by solving eight linear equations with eight unknown constants. From Green's function some lower bounds for the characteristic values of smallest modulus are derived. These bounds are illustrated by figs. 1–5 in some cases which are of interest from the point of view of applied mathematics. The asymptotic solutions of equations (1.1) and (2.2) are dealt with in § 3 considering the two cases according as  $1 + \phi\Phi(z)$  has zeros or not. In the former case, the occurrence of Stoke's phenomenon entails elaborate expressions for the asymptotic solutions under different conditions, based on publications by R. E. Langer. The equations determining the characteristic values are given in both cases, including two theorems on the occurrence of real and complex characteristic values in the latter case. The discussion of the equations determining the characteristic values is continued in § 4. It is shown that the characteristic values corresponding to  $|\sigma| = 1$  are spaced at regular intervals in the latter case. By the examination of eighteen sub-cases the corresponding characteristic values are found to be clustered in very narrow intervals, spaced regularly in the former case. An estimate of the width of these intervals is given. Attention is drawn to the similarity of this situation to the results of several previous investigations, the present result including the latter as particular instances. Section 5 is devoted to the application of A. L. Cauchy's integral theorem in obtaining a convergent infinite series expansion of a generalized Green's function. The second case ( $1 + \phi\Phi(z)$  having no zeros), mentioned above, is dealt with in the first place, the said series being derived by application of the corresponding asymptotic solutions. In § 6 the first case ( $1 + \phi\Phi(z)$  having zeros) is examined, leading to twenty-four sub-cases, which may be reduced to twelve. The fundamental formulæ for their discussion are given, and the discussion itself is recorded in one of these twelve cases. The results obtained in the remaining eleven cases are quoted. In order to complete the said series expansion, the residues are calculated in § 7. Hereupon the absolute and uniform convergence of the series obtained is proved in § 8, for the two cases mentioned, by application of the asymptotic solutions of § 3. The series are then applied in § 9 to the expansion of arbitrary functions as well as to the solution of Hill's linear inhomogeneous problems. Finally in § 10, by the iteration of Green's

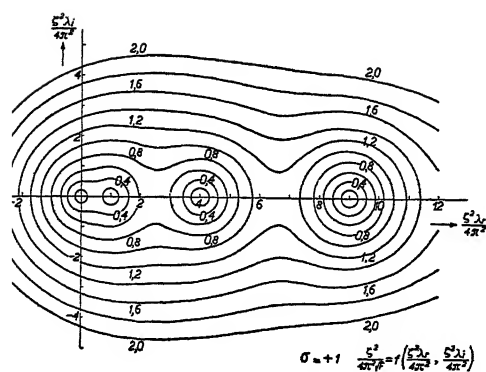


FIG. 1.

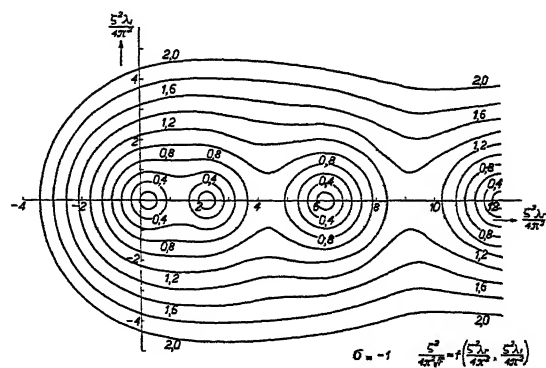


FIG. 2.

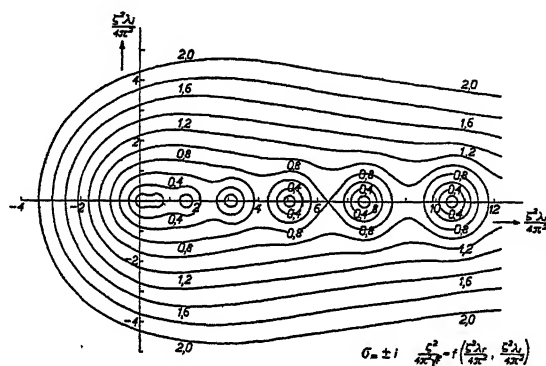


FIG. 3.

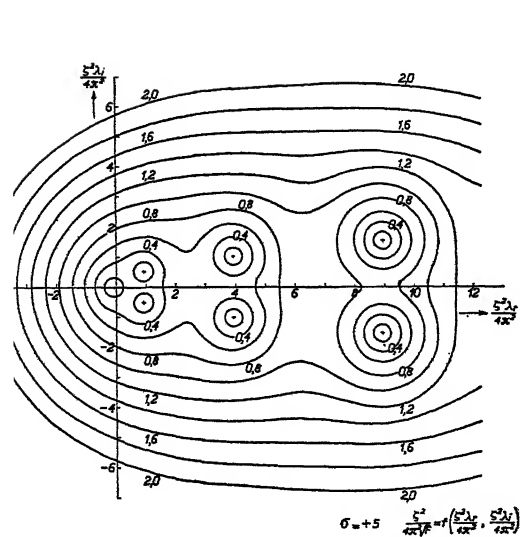


FIG. 4.

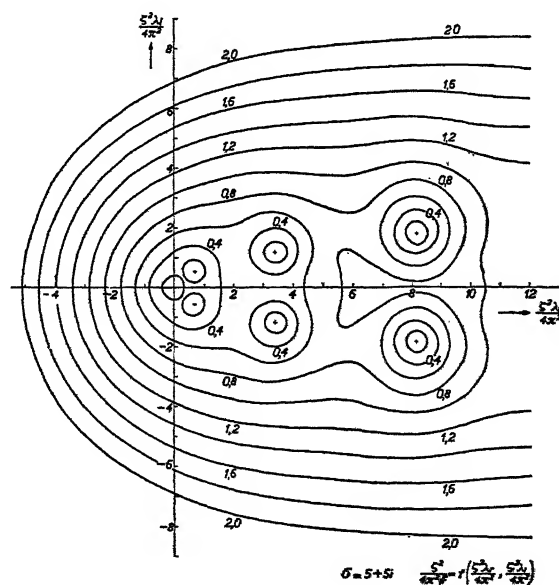


FIG. 5.

functions, expressions are obtained for the characteristic values as limits of certain integral operations. The characteristic functions are obtained by similar processes, and the procedure is recorded in the cases of suffixes 1 and 2. The formulæ thus obtained are thought to be of considerable practicable importance.

# 1. STATEMENT OF PROBLEM AND OF SOME PREVIOUS RESULTS

We shall consider solutions of the differential equation

$$\frac{d^2 w}{dz^2} + w(z)\{\lambda + \gamma\Phi(z)\} = 0, \quad (1.1)$$

where  $z$  is a real variable,  $\lambda$  and  $\gamma$  are real or complex parameters, and  $\Phi(z)$  is a real function of  $z$ , periodic with a fundamental real period  $\zeta$ , satisfying Dirichlet's conditions and not persistently zero for any finite interval of  $z$ . The solutions  $w$  are required to satisfy the linear homogeneous conditions:

$$\left. \begin{aligned} w(z + \zeta) &= \sigma w(z); \\ w'(z + \zeta) &= \sigma w'(z), \end{aligned} \right\} \quad (1.2)$$

accents denoting differentiation with respect to  $z$ . The parameter  $\sigma$ , indicated as "characteristic multiplier", may be real or complex. If  $\sigma$  is given a definite value, the parameters  $\lambda$  and  $\gamma$  together with the function  $w$  being regarded as unknown, we have a linear homogeneous boundary problem,  $\lambda$  and  $\gamma$  assuming "characteristic values" for any solution  $w$ . The latter will be assumed continuous in itself as well as regards its first derivative with respect to  $z$ , and not persistently zero throughout an entire period. The problem thus formulated will be called Hill's boundary problem, whilst equation (1.1) is Hill's equation [22].† Mathieu's equation and the different forms of Lamé's equation pertaining to potential as well as to wave-problems may be considered as particular cases of equation (1.1) under certain conditions [19].† This boundary problem is not self-adjoint (equation (1.1) being so, but equations (1.2) not). By a simple transformation the boundary conditions could be made self-adjoint, the resulting differential equation then being not self-adjoint. Besides Hill's problem as formulated in equations (1.1) and (1.2) we shall also consider the adjoint problem, in which  $w^*$  satisfies equation (1.1) with  $w^*$  instead of  $w$  but different boundary conditions, adjoint to (1.2):

$$\left. \begin{aligned} \sigma w^*(z + \zeta) &= w^*(z); \\ \sigma w^{*'}(z + \zeta) &= w^{*'}(z). \end{aligned} \right\} \quad (1.3)$$

The solutions  $w$  and  $w^*$  of these two adjoint problems, owing to their homogeneous character, each still contain an arbitrary multiplier.

If a pair of parameters  $\lambda, \gamma$  together with a particular function  $w(z)$  satisfy the equations (1.1) and (1.2), the same pair of parameters, together with a function  $w^*(z)$  (in general different from  $w$ ), satisfy the equations (1.1) and (1.3). Thus it may be stated that Hill's problem and its adjoint one have the same characteristic values of  $\lambda$  and  $\gamma$ . The functions  $w$  and  $w^*$  corresponding to different pairs of parameters indicated by  $\lambda_a, \gamma_a$  and  $\lambda_b, \gamma_b$  by Green's integral theorem satisfy the bi-orthogonal relation

$$\int_z^{z+\zeta} \{\lambda_a - \lambda_b + (\gamma_a - \gamma_b)\Phi(z)\} w(z) w^*(z) dz = 0. \quad (1.4)$$

If  $|\sigma| = 1$  and  $\sigma$  is real, the functions  $w$  and  $w^*$  are periodic ( $\sigma = +1$ ) or half-periodic ( $\sigma = -1$ ). If  $|\sigma| = 1$  and  $\sigma$  is non-real, the functions  $w$  and  $w^*$  are almost-periodic in the sense of H. Bohr. If  $|\sigma| \geq 1$ , the functions  $w$  and  $w^*$  are neither periodic nor almost-periodic but will be called pseudo-periodic. All three cases have been treated in previous publications, but most of the published results pertain to the former two cases. It was shown [17, 18] that all characteristic values of  $\gamma$  are then real under the above assumptions, if  $\lambda$  is real, and conversely, and they are then situated on a set of continuous discrete curves in the  $\lambda, \gamma$ -plane, of infinite

† These numbers refer to the list at the end.

order,  $d\lambda/d\gamma$  varying continuously along each curve [15]. Any two separate curves can only intersect in the finite part of the  $\lambda$ -,  $\gamma$ -plane if  $\sigma = \pm 1$ , and no curve can ever intersect itself here,  $\lambda$  being an integral function of  $\gamma$  along each curve if  $|\sigma| = 1$ . A number of theorems, determining general and asymptotic (for  $\zeta^2 |\lambda| + \zeta^2 |\gamma| \gg 1$ ) properties of these curves, has been derived [15, 16].

In the present general case, however, the characteristic values are not always real. It is easy to prove that  $\lambda$  is an analytic function of  $\gamma$  in any case. If  $w_I$  and  $w_{II}$  are two independent solutions of equation (1.1) satisfying the conditions  $w_I(z_0) = 1$ ,  $w_I'(z_0) = 0$ ,  $w_{II}(z_0) = 0$ ,  $w_{II}'(z_0) = 1$ , and if  $z_1 = z_0 + \zeta$ , we have by equations (1.1) and (1.2) [19]:

$$\sigma^2 - \sigma\{w_I(z_1) + w_{II}'(z_1)\} + 1 = 0. \quad (1.5)$$

As  $w_I$  and  $w_{II}$  are integral functions of  $\lambda$  and  $\gamma$ , an analytic relation exists, by equation (1.5), between these parameters for any definite value of  $\sigma$ .

## 2. THE GREEN'S FUNCTION OF HILL'S PROBLEM

In equation (1.1) we may introduce

$$\lambda - \Lambda_0 = \Lambda, \quad \gamma - \Gamma_0 = \Gamma,$$

and obtain

$$\frac{d^2 w}{dz^2} + w(z)\{\Lambda_0 + \Gamma_0 \Phi(z)\} + w(z)\{\Lambda + \Gamma \Phi(z)\} = 0. \quad (2.1)$$

If  $\Gamma/\Lambda = \phi$  and  $\Lambda_0, \Gamma_0, \phi$  are regarded as fixed, we thus obtain a one-parametric problem:

$$L[w(z)] + \Lambda w(z)\{1 + \phi \Phi(z)\} = 0, \quad (2.2)$$

where  $L[w]$  is an abbreviation for the sum of the first two terms of (2.1). In case  $\Lambda$  in equation (2.1) is constantly zero, equation (2.2) has obviously to be replaced by

$$L[w] + \Gamma w \Phi(z) = 0. \quad (2.3)$$

In the case of real parameters  $\lambda, \gamma$  the parameters  $\Lambda$  and/or  $\Gamma$  are connected with straight lines in the  $\lambda$ -,  $\gamma$ -plane, including the point  $\Lambda_0, \Gamma_0$  [15].

We now consider a solution  $G$  (Green's function) of Hill's problem, satisfying the conditions: (a) it is a continuous function of two variables  $z, t$ ; (b)  $L[G] = 0$ , except at  $z = t$ , with either  $z$  or  $t$  as independent variable in the expression  $L$  (being a linear differential operator connected with the problem in hand); (c)  $G$  satisfies equations (1.2); (d)  $G'$  is discontinuous at  $z = t$  according to the condition:

$$\lim_{\epsilon \rightarrow 0} [G'(z, t)]_{z=t-\epsilon}^{z=t+\epsilon} = -1. \quad (2.4)$$

By

$$\begin{aligned} G(z, t) &= A_1 w_I(t) + A_2 w_{II}(t) & \text{if } z_0 \leq z \leq t, \\ G(z, t) &= B_1 w_I(z) + B_2 w_{II}(z) & \text{if } t \leq z \leq z_1 = z_0 + \zeta; \end{aligned}$$

and

$$\begin{aligned} A_1 &= A_{11} w_I(z) + A_{12} w_{II}(z), \\ A_2 &= A_{21} w_I(z) + A_{22} w_{II}(z), \\ B_1 &= B_{11} w_I(t) + B_{12} w_{II}(t), \\ B_2 &= B_{21} w_I(t) + B_{22} w_{II}(t), \end{aligned}$$

the functions  $w_I$  and  $w_{II}$  being independent solutions of  $L[w] = 0$  satisfying the conditions:  $w_I(z_0) = 1$ ,  $w_I'(z_0) = 0$ ,  $w_{II}(z_0) = 0$ ,  $w_{II}'(z_0) = 1$ , Green's function results from the solution of eight linear equations for the eight constants  $A_{11}, A_{12}$ , etc., obtained by applying the above conditions to  $G$  and varying  $t$ . The result is:

$$\left. \begin{aligned} G(z, t) &= \frac{\{\sigma w_I(z + \zeta) - w_I(z)\} w_{II}(t) - \{\sigma w_{II}(z + \zeta) - w_{II}(z)\} w_I(t)}{\sigma^2 - \sigma\{w_I(z_1) + w_{II}'(z_1)\} + 1} & \text{if } z \leq t, \\ G(z, t) &= \sigma \frac{\{w_I(t + \zeta) - \sigma w_I(t)\} w_{II}(z) - \{w_{II}(t + \zeta) - \sigma w_{II}(t)\} w_I(z)}{\sigma^2 - \sigma\{w_I(z_1) + w_{II}'(z_1)\} + 1} & \text{if } t \leq z. \end{aligned} \right\} \quad (2.5)$$

These equations may be easily verified by again applying the said conditions to  $G$ . Obviously, if the denominator is zero, equations (2.5) are invalid, and we then have to fall back on the modified Green's function [2]. Equation (1.5) corresponds to the latter case. If the conditions (1.3) are substituted for (1.2) we obtain the adjoint Green's function  $G^*(z, t)$ , satisfying the relation

$$G^*(z, t) = G(t, z). \quad (2.6)$$

The generally non-self-adjoint character of Hill's problem, if  $\sigma$  does not coincide with some special values, expresses itself by the asymmetry of  $G(z, t)$  with respect to  $z$  and  $t$ .

With the aid of Green's function several bounds for characteristic values may be obtained. Considering the problem (1.1), (1.2) we have:

$$\frac{1}{|\gamma_1|^2} \leq \int_z^{z+\zeta} \int_z^{z+\zeta} \Phi^2(t) |G^2(z, t)| dz dt, \quad (2.7)$$

$\gamma_1$  being the (or a) characteristic value of smallest modulus, and the operator  $L$  being

$$L[w] \equiv \frac{d^2 w}{dz^2} + \lambda w.$$

Similarly, in the case (2.2), (1.2) we obtain:

$$\frac{1}{|\Lambda_1|^2} \leq \int_z^{z+\zeta} \int_z^{z+\zeta} \{1 + \phi\Phi(t)\}^2 G^2(z, t) |dz dt, \quad (2.8)$$

$\Lambda_1$  again being the (or a) characteristic value of smallest modulus, and

$$L[w] \equiv \frac{d^2 w}{dz^2} + w\{\Lambda_0 + \Gamma_0\Phi(z)\}.$$

As an application of equation (2.7), let the maximum value of  $\Phi^2(z)$  be  $\Phi_0^2$ . Then

$$\frac{1}{|\gamma_1|^2} \leq \Phi_0^2 F; \quad F = \int_z^{z+\zeta} \int_z^{z+\zeta} |G^2(z, t)| dz dt. \quad (2.9)$$

We obtain, inserting  $w_I = \cos\{\sqrt{\lambda}(z - z_0)\}$ ,  $w_{II} = \lambda^{-\frac{1}{2}} \sin\{\sqrt{\lambda}(z - z_0)\}$  in equation (2.5):

$$\left. \begin{aligned} \frac{\zeta^4}{F} &= \frac{F_1}{F_2} \\ F_1 &= (\alpha_1^2 + \alpha_2^2) \{(\rho + 1)^2 + 4\sigma_2 \rho \sin \alpha_1 \operatorname{sh} \alpha_2 - 4\sigma_1(\rho + 2) \cos \alpha_1 \operatorname{ch} \alpha_2 \\ &\quad + 2(\rho + 1)(\operatorname{ch} 2\alpha_2 + \cos 2\alpha_1) + 2(\sigma_1^2 - \sigma_2^2) + 1\}, \\ F_2 &= \rho^2 \left( \frac{\operatorname{sh}^2 \alpha_2}{4\alpha_2^2} - \frac{\sin^2 \alpha_1}{4\alpha_1^2} \right) + (\rho + 1) \left( \frac{\operatorname{sh} 2\alpha_2}{2\alpha_2} - \frac{\sin 2\alpha_1}{2\alpha_1} \right) - \\ &\quad - \sigma_1(\rho + 2) \left( \cos \alpha_1 \frac{\operatorname{sh} \alpha_2}{2\alpha_2} - \frac{\sin \alpha_1}{2\alpha_1} \operatorname{ch} \alpha_2 \right) + \\ &\quad + \sigma_2 \rho \left( \sin \alpha_1 \frac{\operatorname{ch} \alpha_2}{2\alpha_2} + \frac{\cos \alpha_1}{2\alpha_1} \operatorname{sh} \alpha_2 \right) - \\ &\quad - 2\sigma_2 \rho \left( \frac{1}{4\alpha_1^2} + \frac{1}{4\alpha_2^2} \right) \sin \alpha_1 \operatorname{sh} \alpha_2, \\ \sigma &= \sigma_1 + i\sigma_2, \quad i = \sqrt{-1}, \quad \zeta\sqrt{\lambda} = \alpha_1 + i\alpha_2, \quad \rho = \sigma_1^2 + \sigma_2^2 - 1, \\ |\gamma_1|^2 \zeta^4 \Phi_0^2 &\geq \frac{\zeta^4}{F} \quad \text{or} \quad \zeta^2 |\gamma_1 \Phi_0| \geq \frac{\zeta^2}{\sqrt{|F|}} = \left| \frac{\sqrt{F_1}}{\sqrt{F_2}} \right|. \end{aligned} \right\} \quad (2.10)$$

In some cases, pertaining to  $\sigma = \pm 1$ ,  $\sigma = \pm i$ ,  $\sigma = 5$ ,  $\sigma = 5 + 5i$ , the expression  $\zeta^2/4\pi^2 |\sqrt{F}|$  has been calculated numerically and is shown in figs. 1-5, in which  $\lambda = \lambda_r + i\lambda_i$ . These results may be directly applied to the equations of Mathieu, Lamé, and to related equations, which occur in connection with problems of elastic stability.

Other and closer bounds for the characteristic parameter values of smallest moduli will be given in the course of this paper (§ 10).

### 3. ASYMPTOTIC SOLUTIONS AND CHARACTERISTIC VALUES

The asymptotic solutions of Hill's problem are fundamental for our further argument. Equation (2.2) will be used in obtaining these asymptotic solutions, the related equation (2.3) being only a special case. If  $|\sigma| = 1$  and  $\phi$  is real, all the characteristic values  $\Lambda$  are real, but if  $|\sigma| \neq 1$  this is not always true, as will be shown presently. Two different cases will be considered: (a)  $1 + \phi\Phi$  has no zeros; (b)  $1 + \phi\Phi$  has a (finite) number of zeros within each period  $\zeta$ . In the case (a)  $\phi$  may be real or complex, but in the case (b) it must be real as  $\Phi(z)$  is real.

We turn our attention to case (a). Fractional powers of complex quantities will, by convention, be made determinate as follows:—

$$q^a = \{ |q| \exp(i \arg q) \}^a = |q|^a \exp(ia \arg q).$$

The argument of real quantities is taken as zero.

Then, assuming

$$w(z) = \{1 + \phi\Phi(z)\}^{\frac{1}{2}} v(z), \quad z = \int_{z_0}^z \{1 + \phi\Phi(z)\}^{\frac{1}{2}} dz,$$

equation (2.2) becomes

$$\frac{d^2 w(z)}{dz^2} + \Psi_0(z) w(z) + \Lambda w(z) = 0, \quad (3.1)$$

where the function  $\Psi_0$  may be calculated from  $\Phi$  and is, by our assumption, bounded in modulus,  $z_0$  being a constant of integration. Asymptotically, if  $|\Lambda \zeta^2| \rightarrow \infty$  the solutions of (3.1) are:

$$w(z) \approx \Lambda \exp(\pm iz\sqrt{\Lambda}) \left\{ 1 + O\left(\frac{1}{|\zeta\sqrt{\Lambda}|}\right) \right\}, \quad (3.2)$$

$\Lambda$  denoting an arbitrary multiplier and the sign  $O$  having the usual sense. These solutions are valid for values of  $z$  and  $\Lambda$ , for which the imaginary part of  $z\sqrt{\Lambda}$  is either invariably below or invariably above a constant. Combining equations (3.2) and (1.2) we obtain:

$$\exp \left\{ \pm i\sqrt{\Lambda} \int_z^{z+\zeta} \sqrt{1 + \phi\Phi(z)} dz \right\} = \sigma \equiv \exp(\mu\zeta).$$

Hence

$$i\sqrt{\Lambda} \mu \approx \pm \frac{\mu\zeta \pm 2m\pi i}{\int_z^{z+\zeta} \{1 + \phi\Phi(z)\}^{\frac{1}{2}} dz}, \quad m = 0, 1, 2, \dots \quad (3.3)$$

The value  $\mu$  is sometimes called the *characteristic exponent* of Hill's problem. Equation (3.3) determines the asymptotic characteristic values of  $\Lambda$  and settles the question of their being real or complex. Thus, if  $|\sigma| = 1$ ,  $\mu$  is purely imaginary, and hence the characteristic values are asymptotically real and positive if  $\phi$  is real and  $1 + \phi\Phi(z) > 0$ , and are real and negative if  $1 + \phi\Phi(z) < 0$ . Two theorems result from equation (3.3):

**Theorem 2.1.** If  $|\sigma| \neq 1$  in equations (1.2), and  $1 + \phi\Phi$  in equation (2.2) has no zeros, the number of real characteristic values of  $\Lambda$  is finite, the number of complex characteristic values being enumerably infinite.

**Theorem 2.2.** If  $|\sigma| = 1$  in equations (1.2), and  $1 + \phi\Phi(z)$  in equation (2.2) has no zeros, the number of complex characteristic values of  $\Lambda$  is finite, the number of real characteristic values being enumerably infinite.

Turning our attention to the case (b) ( $1 + \phi\Phi$  having zeros), the occurrence of Stokes's phenomenon complicates the discussion considerably. In order to keep it within reasonable bounds, some further assumptions will be introduced. The function  $1 + \phi\Phi(z)$  of equation (2.2) (or  $\Phi(z)$  of equation (2.3)) is supposed to have two simple zeros within each period, situated at  $a_1$  and  $a_2$  if the said period is taken from  $z_0$  to  $z_1$ , and  $z_0 < a_1 < a_2 < z_1$ . Homologous adjacent zeros outside this interval are:  $a_0 < z_0$  and  $a_5 > a_4 > a_3 > z_1 = z_0 + \zeta$ . It will further



be assumed that  $1 + \phi\Phi(z)$  is negative if  $a_0 < z < a_1$  (interval I), positive if  $a_1 < z < a_2$  (interval II), negative if  $a_2 < z < a_3$  (interval III), positive if  $a_3 < z < a_4$  (interval IV), and negative if  $a_4 < z < a_5$  (interval V). The zeros  $a_0, a_1, a_2, a_3, a_4, a_5$  are excluded from the said intervals. The asymptotic solutions, as applied in this and in subsequent paragraphs, are simple deductions of formulæ, given by R. E. Langer [8, 9, 10]. Only a few principal steps of this derivation will be quoted. The remainder terms, approaching zero if  $|\Lambda \zeta^2| \rightarrow \infty$ , will be omitted for sake of brevity, as they may be found in the references given. The notation  $\rho = (\Lambda)^{\frac{1}{2}}$  is introduced in the sense mentioned. The general asymptotic solutions in the intervals I to V are:

$$\left. \begin{aligned} \text{Intervals I and II:} \\ w &\sim \Psi \xi^{-\frac{1}{2}} (C_1 c_{11} e^{i\xi} + C_1 c_{12} e^{-i\xi} + C_2 c_{21} e^{i\xi} + C_2 c_{22} e^{-i\xi}). \\ \text{Intervals II and III:} \\ w &\sim \Psi \xi^{-\frac{1}{2}} (D_1 c_{11} e^{i\xi} + D_1 c_{12} e^{-i\xi} + D_2 c_{21} e^{i\xi} + D_2 c_{22} e^{-i\xi}). \\ \text{Intervals III and IV:} \\ w &\sim \Psi \xi^{-\frac{1}{2}} (E_1 c_{11} e^{i\xi} + E_1 c_{12} e^{-i\xi} + E_2 c_{21} e^{i\xi} + E_2 c_{22} e^{-i\xi}). \\ \text{Intervals IV and V:} \\ w &\sim \Psi \xi^{-\frac{1}{2}} (F_1 c_{11} e^{i\xi} + F_1 c_{12} e^{-i\xi} + F_2 c_{21} e^{i\xi} + F_2 c_{22} e^{-i\xi}). \end{aligned} \right\} \quad (3.4)$$

In these formulæ,  $C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2$  are arbitrary constants to be determined by boundary conditions, imposed on the solutions. The values of  $\Psi, \xi, c_{11}, c_{12}, c_{21}, c_{22}$  are fixed in the two consecutive intervals corresponding to each asymptotic solution. Thus we have:

Interval I:

$$\Psi = \frac{\left\{ \int_z^{a_1} |1 + \phi\Phi|^{-\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{|1 + \phi\Phi|^{-\frac{1}{2}}}, \quad \xi = -i\rho \int_z^{a_1} |1 + \phi\Phi|^{-\frac{1}{2}} dz, \quad \text{constants } C,$$

$$c_{11} = e^{\frac{3\pi i}{12}}, \quad c_{12} = e^{\frac{5\pi i}{12}}, \quad c_{21} = e^{\frac{15\pi i}{12}}, \quad c_{22} = e^{\frac{\pi i}{12}}, \quad \text{if } 0 < \arg \rho \leq \pi.$$

Interval II:

$$\Psi = \frac{\left\{ \int_{a_1}^z (1 + \phi\Phi)^{\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{(1 + \phi\Phi)^{\frac{1}{2}}}, \quad \xi = \rho \int_{a_1}^z (1 + \phi\Phi)^{\frac{1}{2}} dz, \quad \text{constants } C,$$

$$\Psi = \frac{\left\{ \int_z^{a_2} (1 + \phi\Phi)^{\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{(1 + \phi\Phi)^{\frac{1}{2}}}, \quad \xi = \rho \int_z^{a_2} (1 + \phi\Phi)^{\frac{1}{2}} dz, \quad \text{constants } D,$$

$$c_{11} = e^{-\frac{\pi i}{12}}, \quad c_{12} = e^{\frac{\pi i}{12}}, \quad c_{21} = e^{-\frac{5\pi i}{12}}, \quad c_{22} = e^{\frac{5\pi i}{12}}, \quad \text{if } 0 < \arg \rho \leq \frac{\pi}{2},$$

$$c_{11} = e^{\frac{3\pi i}{12}}, \quad c_{12} = e^{\frac{\pi i}{12}}, \quad c_{21} = e^{\frac{15\pi i}{12}}, \quad c_{22} = e^{\frac{5\pi i}{12}}, \quad \text{if } \frac{\pi}{2} < \arg \rho \leq \pi.$$

Interval III:

$$\Psi = \frac{\left\{ \int_{a_2}^z |1 + \phi\Phi|^{-\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{|1 + \phi\Phi|^{-\frac{1}{2}}}, \quad \xi = -i\rho \int_{a_2}^z |1 + \phi\Phi|^{-\frac{1}{2}} dz, \quad \text{constants } D,$$

$$\Psi = \frac{\left\{ \int_z^{a_3} |1 + \phi\Phi|^{-\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{|1 + \phi\Phi|^{-\frac{1}{2}}}, \quad \xi = -i\rho \int_z^{a_3} |1 + \phi\Phi|^{-\frac{1}{2}} dz, \quad \text{constants } E,$$

$c_{11}, c_{12}, c_{21}, c_{22}$  being as in the interval I.

Interval IV:

$$\Psi = \frac{\left\{ \int_{a_2}^z (1 + \phi\Phi)^{\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{(1 + \phi\Phi)^{\frac{1}{2}}}, \quad \xi = \rho \int_{a_2}^z (1 + \phi\Phi)^{\frac{1}{2}} dz, \quad \text{constants E,}$$

$$\Psi = \frac{\left\{ \int_z^{a_4} (1 + \phi\Phi)^{\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{(1 + \phi\Phi)^{\frac{1}{2}}}, \quad \xi = \rho \int_z^{a_4} (1 + \phi\Phi)^{\frac{1}{2}} dz, \quad \text{constants F,}$$

$c_{11}, c_{12}, c_{21}, c_{22}$  being as in the interval II.

Interval V:

$$\Psi = \frac{\left\{ \int_{a_4}^z |1 + \phi\Phi|^{\frac{1}{2}} dz \right\}^{\frac{1}{2}}}{|1 + \phi\Phi|^{\frac{1}{2}}}, \quad \xi = -i\rho \int_{a_4}^z |1 + \phi\Phi|^{\frac{1}{2}} dz, \quad \text{constants F,}$$

$c_{11}, c_{12}, c_{21}, c_{22}$  being as in the interval I.

The value of  $\Psi$  is real in any interval,  $\xi$  being either real or purely imaginary in consecutive intervals.

From these asymptotic solutions of Hill's equation under the present conditions the characteristic values of  $\Lambda$  pertaining to Hill's problem (2.2), (1.2) may be determined by means of equation (1.5). We have to determine the solutions  $w_I$  and  $w_{II}$  occurring in this equation, i.e. to determine the constants C, D, and E from the boundary conditions imposed on these solutions. We shall not put the somewhat cumbersome steps, leading to the final formulæ for  $w_I(z_1)$  and  $w_{II}(z_1)$  (see equation (1.5)), on record and only state these results:

$$\left. \begin{aligned} 2w_I(z_1) &= e^{i\rho\beta_1 + \rho\beta_2} + e^{-i\rho\beta_1 + \rho\beta_2} - ie^{-i\rho\beta_1 - 2\rho\beta_2 + \rho\beta_2} + ie^{-i\rho\beta_1 + 2\rho\beta_2 - \rho\beta_2} + e^{-i\rho\beta_1 - \rho\beta_2}, \text{ if } 0 < \arg \rho \leq \frac{\pi}{2}, \\ 2w_I(z_1) &= e^{-i\rho\beta_1 + \rho\beta_2} - ie^{-i\rho\beta_1 - 2\rho\beta_2 + \rho\beta_2} + e^{i\rho\beta_1 - \rho\beta_2} + ie^{-i\rho\beta_1 + 2\rho\beta_2 - \rho\beta_2} + e^{-i\rho\beta_1 - \rho\beta_2}, \text{ if } \frac{\pi}{2} < \arg \rho \leq \pi, \\ 2w'_{II}(z_1) &= e^{i\rho\beta_1 + \rho\beta_2} + e^{-i\rho\beta_1 + \rho\beta_2} + ie^{-i\rho\beta_1 - 2\rho\beta_2 + \rho\beta_2} - ie^{-i\rho\beta_1 - \rho\beta_2 + 2\rho\beta_2} + e^{-i\rho\beta_1 - \rho\beta_2}, \text{ if } 0 < \arg \rho \leq \frac{\pi}{2}, \\ 2w'_{II}(z_1) &= e^{-i\rho\beta_1 + \rho\beta_2} + ie^{-i\rho\beta_1 + \rho\beta_2 - 2\rho\beta_2} + e^{i\rho\beta_1 - \rho\beta_2} - ie^{-i\rho\beta_1 + 2\rho\beta_2 - \rho\beta_2} + e^{-i\rho\beta_1 - \rho\beta_2}, \text{ if } \frac{\pi}{2} < \arg \rho \leq \pi. \end{aligned} \right\} \quad (3.5)$$

In these equations the abbreviations

$$\beta_1 = \int_{a_1}^{a_2} |1 + \phi\Phi(z)|^{\frac{1}{2}} dz, \quad \beta_2 = \int_{a_2}^{a_3} (1 + \phi\Phi(z))^{\frac{1}{2}} dz, \quad \beta_3 = \int_{z_0}^{a_1} |1 + \phi\Phi(z)|^{\frac{1}{2}} dz \quad (3.6)$$

have been used. From equations (3.5) it is simple to obtain the expression:

$$w_I(z_1) + w'_{II}(z_1) = W = 2 \operatorname{ch} \mu \zeta \quad (3.7)$$

determining the characteristic values of  $\Lambda$ . Our results are:

$$\left. \begin{aligned} W &\approx e^{i\rho\beta_1 + \rho\beta_2} + e^{-i\rho\beta_1 + \rho\beta_2} + e^{-i\rho\beta_1 - \rho\beta_2}, \text{ if } 0 < \arg \rho \leq \frac{\pi}{2}, \\ W &\approx e^{-i\rho\beta_1 + \rho\beta_2} + e^{i\rho\beta_1 - \rho\beta_2} + e^{-i\rho\beta_1 - \rho\beta_2}, \text{ if } \frac{\pi}{2} < \arg \rho \leq \pi. \end{aligned} \right\} \quad (3.8)$$

#### 4. DISCUSSION OF CHARACTERISTIC VALUES DERIVED FROM EQUATIONS (3.3), (3.7) AND (3.8)

The discussion of these three equations will reveal some interesting facts on the asymptotic situation of complex characteristic values, similar to those proved previously in the case of real

parameters. We shall first consider equation (3.3) and the preceding equation. Using the notation

$$\rho^2 = \Lambda, \quad \rho = \rho_1 + i\rho_2, \quad \text{and} \quad \int_z^{z+\zeta} \sqrt{1 + \phi\Phi(z)} dz = \rho_1 + i\rho_2,$$

we obtain:

$$\exp \{ \pm i(\rho_1 \rho_1 - \rho_2 \rho_2) \mp (\rho_1 \rho_2 + \rho_2 \rho_1) \} = \sigma.$$

Hence, if  $|\sigma| = 1$ ,

$$\frac{\rho_1}{\rho_2} = -\frac{\rho_1'}{\rho_2'}, \quad \exp(\pm i|\rho\rho|) = \sigma. \quad (4.1)$$

These characteristic values thus correspond to a fixed argument of  $\rho$  and hence of  $\Lambda$ . If  $|\rho|$  increases, the characteristic values corresponding to a particular value of  $\sigma$ , ( $|\sigma| = 1$ ), are situated at regular intervals. The characteristic values corresponding to  $\sigma = +1$  as well as to  $\sigma = -1$  are each double, and  $|\rho\rho| = n\pi$ ,  $n = 0, \pm 1, \pm 2, \pm 3, \dots$  in this case. This situation is similar to the case when  $1 + \phi\Phi(z)$  has no zeros and  $\phi$  is real. Two consecutive completely periodic characteristic values as well as two consecutive half-periodic characteristic values coincide asymptotically.

If  $\Phi$  has two simple zeros within each period  $\zeta$ , this will also be true for  $\lambda + \gamma\Phi$  and for  $1 + \phi\Phi$ , if  $|\gamma| \gg |\lambda|$ . We are thus automatically led to the second case, considered in § 3. Hence the situation of the characteristic values may asymptotically be obtained from equations

(3.7) and (3.8) under these conditions. It will be assumed at first that  $0 < \arg \rho \leq \frac{\pi}{2}$ . Let  $\rho = \rho_1 + i\rho_2$ , then  $\rho_1$  and  $\rho_2$  are both positive. Hence:

$$2 \operatorname{ch} \mu\zeta = W \approx e^{\rho_1\beta_2 - \rho_2\beta_1} e^{i(\rho_1\beta_1 + \rho_2\beta_2)} + e^{-\rho_1\beta_2 + \rho_2\beta_1} e^{-i(\rho_1\beta_1 + \rho_2\beta_2)} + e^{\rho_1\beta_1 + \rho_2\beta_2} e^{i(-\rho_1\beta_1 + \rho_2\beta_2)}. \quad (4.2)$$

If  $\beta_1$  is comparable in magnitude to  $\beta_2$ , we may consider three sub-cases: (a)  $\rho_1 \ll \rho_2$ , (b)  $\rho_2 \ll \rho_1$ , and (c)  $\rho_1$  comparable to  $\rho_2$ . In the sub-case (a) ( $\rho_1 \ll \rho_2$ ), the first term of  $W$  is negligible in comparison to the other two terms. Hence:

$$2 \operatorname{ch} \mu\zeta = W \approx 2e^{\rho_2\beta_2(1+i)} \cos \{ \rho_2\beta_2(1+\delta) \}, \quad |\epsilon| \ll 1, \quad |\delta| \ll 1. \quad (4.2a)$$

The characteristic values  $\rho_2$  pertaining to characteristic exponents  $\mu$  such that  $\operatorname{ch} \mu\zeta$  is comparable with unity (*i.e.*  $|\sigma|$  approximately or exactly unity) are, by the condition  $\rho_2\beta_1 \gg 1$ , approximately determined by  $\rho_2\beta_2(1+\delta)$  being near to an odd multiple of  $\pi/2$ . Considering one particular multiple of  $\pi/2$ , all the characteristic values corresponding to varying values of  $\sigma$  are very close together, and the more so the larger  $\rho_2\beta_1$  is. Asymptotically, if  $\rho_2\beta_1 \rightarrow \infty$ , they approach the characteristic values connected with  $W=0$  and pertaining to  $\sigma = \pm i$ , the period of the corresponding solutions being  $4\zeta$ , *i.e.* four times the fundamental period (as  $i^4 = +1$ ). The distance  $\Delta\rho_2$  between the two values of  $\rho_2$  corresponding to adjacent characteristic values for  $\sigma = \pm 1$  or  $\operatorname{ch} \zeta\mu = 1$  is given by:

$$\beta_2 \Delta\rho_2 \approx 2e^{-\rho_2\beta_1},$$

which is very small compared with unity. This is also true for  $|\zeta\Delta\rho|$ .

In the sub-case (b) the second term of  $W$  in equation (4.2) is approximately negligible, and we obtain:

$$2 \operatorname{ch} \mu\zeta = W \approx e^{\rho_1\beta_2(1+i)} \cos \{ \rho_1\beta_1(1+\delta) \}, \quad |\epsilon| \ll 1, \quad |\delta| \ll 1. \quad (4.2b)$$

Hence similar conclusions may be drawn as in sub-case (a).

In the sub-case (c) a new argument arises from the first two terms of  $W$  in equation (4.2) being negligible compared with the third one. In this case no characteristic values corresponding to  $|\sigma|$  exactly or approximately equal to unity occur at all.

As a second case, we may assume  $\beta_1 \ll \beta_2$  and again obtain three sub-cases as above: (a)  $\rho_1 \ll \rho_2$ , (b)  $\rho_1 \gg \rho_2$ , and (c)  $\rho_1$  comparable with  $\rho_2$ . As a third case, we have  $\beta_1 \gg \beta_2$ , again with the same three sub-cases. The results of discussing these cases and sub-cases are similar to those obtained in the first case mentioned.

An exactly similar discussion has been applied to  $\pi/2 < \arg \rho \leq \pi$ , corresponding to the second equation (3.8). The results are again similar to those of the previous cases. As a result of these discussions (eighteen sub-cases in all) we may formulate the

**Theorem 4.1.** The characteristic values, resulting from equations (3.7) and (3.8) corresponding to  $|\sigma|$  nearly or exactly equal to unity, are clustered in intervals  $|\zeta \Delta \rho|$  of very small width compared with unity, the width decreasing exponentially with increasing  $|\rho|$ .

From the above discussion the fact may be derived that the moduli of the characteristic values  $\Lambda$  corresponding to any particular value of  $\sigma$  asymptotically increase proportionally to the squares of entire numbers in both the cases that  $1 + \phi\Phi(z)$  has or has no zeros. This is an important item in the proof of the uniform convergence of some series expansions in § 8.

These interesting facts about the asymptotic situation of characteristic values have been found for the Mathieu and Lamé equations with real parameters by E. L. Ince [4, 5, 6, 7, 3], and empirically for the Mathieu equation with one purely imaginary parameter by H. P. Mulholland and S. Goldstein [11]. The author has found them for E. Meissner's equation [19] with real and with complex parameters (the latter case still unpublished), and has proved them under general conditions for Hill's equation with real parameters [15, 16]. Now this property has been shown to be valid for Hill's equation with complex parameters under specified conditions. Thus this sequence may perhaps be said to have met its completion, comprising the particular cases considered hitherto.

## 5. SERIES EXPANSIONS IN TERMS OF CHARACTERISTIC FUNCTIONS

The series expansion as considered here is based on A. L. Cauchy's integral theorem and has been applied by H. Poincaré [12]. A more general application was given by J. Tamarkine [21]. The case, in which  $1 + \phi\Phi$  has zeros, is, however, probably beyond the existing applications.

We consider a function  $K(z, t)$  of two variables satisfying the conditions: (a)  $K$  satisfies equation (2.2) with respect to  $z$  and to  $t$  as independent variables except at  $t = z$ ; (b)  $K$  is a continuous function of both variables; (c)  $K$  satisfies equation (1.2); (d)  $K'$  is discontinuous at  $z = t$  according to:

$$\lim_{\epsilon \rightarrow 0} \left[ K'(z, t) \right]_{z=t-\epsilon}^{z=t+\epsilon} = -1.$$

Hence  $K$  is similar to the Green's function  $G$  of § 2, the differential equation being, however, slightly extended. This function  $K$  is a meromorphic function of  $\Lambda$ , having, as we shall see below, generally simple poles at the characteristic values of this parameter. We shall consider a simple closed curve in the complex  $\Lambda$ -plane, not touching any characteristic value of  $\Lambda$ . Let these characteristic values be arranged according to the order of their moduli and be thus numbered. Values with equal moduli may be numbered according to increasing arguments. A particular curve of the kind considered surrounds the characteristic values of suffix up to  $n$  (any integral number) and is therefore indicated as  $C_n$ . According to A. L. Cauchy's theorem we have:

$$\frac{1}{2\pi i} \oint_{C_n} \frac{K(z, t, \kappa)}{\kappa - \Lambda} d\kappa = K(z, t, \Lambda) + \sum_{m=1}^n \frac{R_m(z, t)}{\Lambda_m - \Lambda}. \quad (5.1)$$

Here the contour-integral is taken along the entire curve  $C_n$  in the complex  $\kappa$ -plane, whilst the value  $\kappa$  is inserted for  $\Lambda$  in the function  $K(z, t, \kappa)$  under the integral sign. The expression  $K(z, t, \Lambda)$  represents the residue of the integrand at  $\kappa = \Lambda$ , and  $R_m(z, t)$  is the residue of the integrand at  $\kappa = \Lambda_m$ , a characteristic value of  $\Lambda$  inside  $C_n$ . The value  $\Lambda$  is also inside this

contour. From equation (5.1) a series-expansion of  $K(z, t, \Lambda)$  may be derived:

$$K(z, t, \Lambda) = \sum_{m=1}^n \frac{-R_m(z, t)}{\Lambda_m - \Lambda} + \frac{1}{2\pi i} \oint_{C_n} \frac{K(z, t, \kappa)}{\kappa - \Lambda} d\kappa. \quad (5.2)$$

We shall now expand the said closed curve so as to enclose more and more characteristic values  $\Lambda_m$ . The shapes of the curves of this sequence may be chosen so as to yield simple analytical results. It will be shown that the contour-integral approaches zero if  $n \rightarrow \infty$ . Thus a convergent infinite series expansion for  $K(z, t)$  is obtained if the residues  $R_m(z, t)$  are known. The function  $K(z, t)$  may evidently be represented by equations exactly similar to (2.5),  $w_I$  and  $w_{II}$  being solutions of equation (2.2) with  $\kappa$  instead of  $\Lambda$ , satisfying the conditions at  $z = z_0$  that were stated previously. From equations (2.5) we may conclude that the numerator as well as the denominator of  $K(z, t, \kappa)$  are integral transcendental functions of  $\kappa$ ,  $w_I$  and  $w_{II}$  being such functions by a well-known theorem of H. Poincaré. In the finite part of the complex  $\kappa$ -plane the function  $K$  hence has (generally simple) poles at  $\kappa = \Lambda_m$  corresponding to the zeros of the denominator.

From equation (3.2) we obtain asymptotically, if  $|\kappa \zeta^2| \rightarrow \infty$ , subject to the condition that the imaginary part of  $z\sqrt{\kappa}$  remains either above or below a fixed value if  $z$  is varied:

$$w_I(z) \approx \left\{ \frac{1 + \phi\Phi(z_0)}{1 + \phi\Phi(z)} \right\}^{\frac{1}{2}} \cos(z\sqrt{\kappa}), \quad w_{II}(z) \approx \frac{\sin(z\sqrt{\kappa})}{\sqrt{\kappa} \{1 + \phi\Phi(z)\}^{\frac{1}{2}} \{1 + \phi\Phi(z_0)\}^{\frac{1}{2}}}, \quad (5.3)$$

remainder terms approaching zero asymptotically having been omitted. Furthermore,

$$\left. \begin{aligned} w_I(z + \zeta) &\approx \left\{ \frac{1 + \phi\Phi(z_0)}{1 + \phi\Phi(z)} \right\}^{\frac{1}{2}} \cos\{\sqrt{\kappa}(z + z_1)\}, \\ w_{II}(z + \zeta) &\approx \frac{\sin\{\sqrt{\kappa}(z + z_1)\}}{\{1 + \phi\Phi(z)\}^{\frac{1}{2}} \{1 + \phi\Phi(z_0)\}^{\frac{1}{2}} \sqrt{\kappa}}, \\ w_I(z_1) &\approx w'_{II}(z_1) \approx \cos(\sqrt{\kappa}z_1), \\ z_1 &= \int_{z_0}^{z_0 + \zeta} \{1 + \phi\Phi(z)\}^{\frac{1}{2}} dz. \end{aligned} \right\} \quad (5.4)$$

If  $z$  is replaced by  $t$ , the variable  $z$  will be assumed to change into  $t$ . By this remark and (5.3), (5.4), equations (2.5) yield in the case  $|\kappa \zeta^2| \rightarrow \infty$  the equations:

$$\left. \begin{aligned} K(z, t, \kappa) &= \frac{\sin\{\sqrt{\kappa}(z - t)\} - \sigma \sin\{\sqrt{\kappa}(z_1 + z - t)\}}{\sqrt{\kappa} \{\sigma^2 - 2\sigma \cos(\sqrt{\kappa}z_1) + 1\} \{1 + \phi\Phi(z)\}^{\frac{1}{2}} \{1 + \phi\Phi(t)\}^{\frac{1}{2}}}, \\ &\quad \text{if } z \geq t \text{ and hence } |z| \geq |t|; \\ K(z, t, \kappa) &= \sigma \frac{\sigma \sin\{\sqrt{\kappa}(t - z)\} - \sin\{\sqrt{\kappa}(z_1 - z + t)\}}{\sqrt{\kappa} \{\sigma^2 - 2\sigma \cos(\sqrt{\kappa}z_1) + 1\} \{1 + \phi\Phi(z)\}^{\frac{1}{2}} \{1 + \phi\Phi(t)\}^{\frac{1}{2}}}, \\ &\quad \text{if } z \leq t \text{ and hence } |z| \leq |t|. \end{aligned} \right\} \quad (5.5)$$

If  $\kappa$  varies along a contour in the complex  $\kappa$ -plane, not touching any characteristic value  $\kappa = \Lambda_m$ , the expression  $|\sqrt{\kappa}K|$  remains below a fixed finite positive bound  $M$ , if  $|\kappa \zeta^2| \rightarrow \infty$ . This follows from  $|t - z| \leq |z_1|$  and  $|z_1 + z - t| \leq |t_1|$  if  $|z| \leq |t|$ , and from  $|t - z| \leq |z_1|$  and  $|z_1 + t - z| \leq |z_1|$  if  $|z| \geq |t|$ , and from the properties of  $\Phi(z)$ , if

$z$  and  $t$  are real. The exponential functions with positive real part of the argument are always smaller in the numerator of  $K$  than in the denominator. For real values of  $\kappa \neq \Lambda_m$  the expression  $|\sqrt{\kappa}K|$  remains also below a suitable fixed positive finite bound  $M$ , as may be concluded from (5.5). Hence it is obvious that the contour-integral of (5.2), assuming the contours, *e.g.*, to be circles of radius  $|\kappa|$ , approaches zero if  $|\kappa\zeta^2| \rightarrow \infty$ . Varying  $\kappa$  along the said contour, we have  $0 < \arg \sqrt{\kappa} \leq \pi$ , and this complies exactly with the condition imposed on the asymptotic solutions (3.2), (5.3) and (5.4).

## 6. ASYMPTOTIC EVALUATION OF $K(z, t, \kappa)$ IF $1 + \phi\Phi(z)$ HAS ZEROS

The above evaluation of  $K$  and of the contour-integral in equation (5.2), in the simple case of  $1 + \phi\Phi(z)$  having no zeros, must be extended to the case in which the latter condition is not satisfied. We shall show that twenty-four sub-cases have to be examined in the latter case, reducing to twelve sub-cases upon proper handling. The function  $K(z, t, \kappa)$  being given by the two equations (2.5), two cases arise according as  $z$  is either smaller or larger than  $t$  (equality included in each one as a boundary case). If we vary  $\kappa$  along a circular contour as quoted above,  $\arg \kappa$  varies from 0 to  $2\pi$ . If  $\rho^2 = \kappa$ , hence  $\arg \rho$  varies from 0 to  $\pi$ . In this interval of  $\arg \rho$  the asymptotic solutions, according to § 3, are in general different, if  $0 < \arg \rho \leq \pi/2$  and if  $\pi/2 < \arg \rho \leq \pi$ . Furthermore, different asymptotic solutions arise, according to equation (2.5), if  $z$  or  $t$  is in either one of the intervals I, II and III. Hence, in the case that  $z \leq t$  we have twelve sub-cases as follows:—

Sub-case No.	$\arg \rho$	Interval of $z$	Interval of $t$
1	$0 < \arg \rho \leq \frac{\pi}{2}$	I	I
2	"	I	II
3	"	I	III
4	"	II	II
5	"	II	III
6	"	III	III
7	$\frac{\pi}{2} < \arg \rho < \pi$	I	I
8	"	I	II
9	"	I	III
10	"	II	II
11	"	II	III
12	"	III	III

Here the intervals IV and V of § 3 come into use. If, *e.g.*,  $z$  is in the interval III,  $z + \zeta$  is in V, etc.

Twelve exactly similar sub-cases arise if  $z \geq t$ . From equations (2.5) it is seen, however, that the two expressions for  $K(z, t, \kappa)$ , if  $z$  is either smaller or larger than  $t$ , differ only in different coefficients  $\sigma$ . If  $\sigma$  is finite, these coefficients cannot materially alter the way in which  $K(z, t, \kappa)$  approaches zero if  $|\kappa\zeta^2| \rightarrow \infty$ . Hence the discussion may be limited to the twelve sub-cases quoted above. If it is proved that  $|K(z, t, \kappa)\sqrt{\kappa}|$  is bounded asymptotically in these twelve sub-cases, the same conclusion directly applies to the remaining twelve sub-cases corresponding to  $z \geq t$ . The complete formulæ pertaining to the said twelve sub-cases cover 62 pages in small handwriting of a large-size writing-book in the author's possession. My thanks are due to Mr N. S. Markus for assistance in their discussion. It would of course be far too tedious even to attempt to render an abstract of these formulæ. It might be sufficient to record the handling of one sub-case in an abbreviated form and to quote the results of the examination of the remaining eleven sub-cases.

Considering the sub-case No. 1 above, the numerator, or  $K(z, t, \kappa)$ , calls for an evaluation of  $w_I(z + \zeta)$ ,  $w_I(z)$ ,  $w_{II}(t)$ ,  $w_{II}(z + \zeta)$ ,  $w_{II}(z)$  and  $w_I(t)$ , the intervals being I for  $z, t$ , and III for  $z + \zeta, t + \zeta$ . From the equations of § 3 we obtain:

$$\begin{aligned}
 w_I(z) &= \frac{1}{2} \left| \frac{1 + \phi\Phi(z_0)}{1 + \phi\Phi(z)} \right|^{\frac{1}{2}} \left( e^{\rho \int_{z_0}^z |1 + \phi\Phi(z)|^{\frac{1}{2}} dz} + e^{-\rho \int_{z_0}^z |1 + \phi\Phi(z)|^{\frac{1}{2}} dz} \right), \\
 &\quad \text{if } z \text{ is in the interval I;} \\
 w_I(z) &= \frac{1}{2} \left| \frac{1 + \phi\Phi(z_0)}{1 + \phi\Phi(z)} \right|^{\frac{1}{2}} \left( e^{i\rho\beta_1 + \rho\beta_2 + \rho \int_{a_2}^z |1 + \phi\Phi|^{\frac{1}{2}} dz} + \right. \\
 &\quad + e^{-i\rho\beta_1 + \rho\beta_2 + \rho \int_{a_2}^z |1 + \phi\Phi|^{\frac{1}{2}} dz} \\
 &\quad - i e^{-i\rho\beta_1 - \rho\beta_2 + \rho \int_{a_2}^z |1 + \phi\Phi|^{\frac{1}{2}} dz} + i e^{-i\rho\beta_1 + \rho\beta_2 - \rho \int_{a_2}^z |1 + \phi\Phi|^{\frac{1}{2}} dz} \\
 &\quad \left. + i e^{-i\rho\beta_1 - \rho\beta_2 - \rho \int_{a_2}^z |1 + \phi\Phi|^{\frac{1}{2}} dz} \right), \\
 &\quad \text{if } z \text{ is in the interval III;} \\
 w_{II}(z) &= \frac{1}{2\rho} \left| \frac{1}{\{1 + \phi\Phi(z_0)\}\{1 + \phi\Phi(z)\}} \right|^{\frac{1}{2}} (e^{\rho\beta_2 - \rho[za_1]} - e^{-\rho\beta_2 + \rho[za_1]}), \\
 &\quad \text{if } z \text{ is in the interval I, } [za_1] = \int_z^{a_1} |1 + \phi\Phi(z)|^{\frac{1}{2}} dz; \\
 w_{III}(z) &= \frac{1}{2\rho} \left| \frac{1}{\{1 + \phi\Phi(z_0)\}\{1 + \phi\Phi(z)\}} \right|^{\frac{1}{2}} (e^{\rho\beta_2 + i\rho\beta_1 + \rho[a_2z]} + \\
 &\quad + e^{\rho\beta_2 - i\rho\beta_1 + \rho[a_2z]} + i e^{-\rho\beta_2 + i\rho\beta_1 - \rho[a_2z]} + i e^{\rho\beta_2 - i\rho\beta_1 - \rho[a_2z]} \\
 &\quad - e^{-\rho\beta_2 - i\rho\beta_1 - \rho[a_2z]}), \\
 &\quad \text{if } z \text{ is in the interval III, the abbreviation } [a_2z] \text{ being used for} \\
 &\quad [a_2z] = \int_{a_2}^z |1 + \phi\Phi(z)|^{\frac{1}{2}} dz.
 \end{aligned} \tag{6.1}$$

The other expressions are obtained from (6.1) by substitution of  $t$  for  $z$ . Thus the numerator  $N$  of  $K(z, t, \kappa)$  in the sub-case No. 1, using the abbreviations,

$$\begin{aligned}
 [a_0z] &= \int_{a_0}^z |1 + \phi\Phi(z)|^{\frac{1}{2}} dz, \\
 [ta_1] &= \int_t^{a_1} |1 + \phi\Phi(z)|^{\frac{1}{2}} dz,
 \end{aligned}$$

becomes :

$$\begin{aligned}
 N &= \frac{1}{2\rho} \left| \frac{1}{\{1 + \phi\Phi(z)\}\{1 + \phi\Phi(t)\}} \right|^{\frac{1}{2}} \left( -\sigma e^{i\rho\beta_1 + \rho[a_0z] + \rho[ta_1]} - \sigma e^{-i\rho\beta_1 + \rho[a_0z] + \rho[ta_1]} + \sigma e^{-i\rho\beta_1 - \rho[a_0z] - \rho[ta_1]} \right. \\
 &\quad \left. - \sigma i e^{-i\rho\beta_1 + \rho[a_0z] - \rho[ta_1]} - \sigma i e^{-i\rho\beta_1 - \rho[a_0z] + \rho[ta_1]} + e^{-\rho[za_1] + \rho[ta_1]} - e^{\rho[za_1] - \rho[ta_1]} \right).
 \end{aligned}$$

The denominator  $D$  of  $K(z, t, \kappa)$  is given by

$$D = \sigma^2 - \sigma W + 1,$$

$W$  being the first expression (3.8). Writing  $\rho = \rho_r + i\rho_i$ , we have  $\rho_r \geq 0$  and  $\rho_i > 0$  in the present sub-case No. 1. Assuming  $\rho_r > 0$  at first, the modulus of the numerator  $|N|$  if  $|\rho\zeta| \rightarrow \infty$  approaches either

$$\left| \frac{1}{2\rho} \right| \left| \frac{1}{\{1 + \phi\Phi(z)\}\{1 + \phi\Phi(t)\}} \right|^{\frac{1}{2}} | \sigma | e^{\rho_i\beta_1 + \rho_r[a_0z] + \rho_r[ta_1]}$$

or

$$\left| \frac{1}{2\rho} \right| \left| \frac{1}{\{1 + \phi\Phi(z)\}\{1 + \phi\Phi(t)\}} \right|^{\frac{1}{2}} e^{\rho_r[2a_1] - \rho_r[2a_1]}.$$

Under these conditions  $|D|$  approaches

$$|\sigma| e^{\rho_i\beta_1 + \rho_r\beta_2}.$$

Hence  $|\rho K(z, t, \kappa)|$  in both cases is smaller than a positive upper bound  $A$ , which satisfies

$$A \geq \frac{1}{2} \left| \frac{1}{\{1 + \phi\Phi(z)\}\{1 + \phi\Phi(t)\}} \right|^{\frac{1}{2}} = B.$$

If  $\rho_r = 0$  and  $|\rho\zeta| \rightarrow \infty$ , the numerator  $N$  approaches

$$-\frac{B}{\rho} 2\sigma i e^{\rho_i\beta_1} \{\sin(\rho_i[a_0z]) + \cos(\rho_i[a_0z])\} \times \{\sin(\rho_i[ta_1]) + \cos(\rho_i[ta_1])\},$$

and  $D$  approaches

$$-2\sigma e^{\rho_i\beta_1} \cos(\rho_i\beta_2).$$

Hence, the characteristic values corresponding to  $\rho_i\beta_2$  equal to a multiple of  $\pi/2$  being excluded, we again find  $|\rho K(z, t, \kappa)|$  to be smaller than a finite positive upper bound, if  $|\rho\zeta| \rightarrow \infty$ .

The examination of the other eleven sub-cases has been carried out along exactly similar lines. The result of this discussion is that  $|\rho K(z, t, \kappa)|$  is always bounded if  $|\rho\zeta| \rightarrow \infty$  in all twenty-four sub-cases corresponding to the condition that  $1 + \phi\Phi(z)$  has two simple zeros in each period  $\zeta$ . If  $z$  and  $t$  are complex and no zeros exist, the above formulæ may be applied too.

## 7. EVALUATION OF THE RESIDUES $R_m(z, t)$ IN EQUATION (5.2)

In the vicinity of a pole  $\kappa = \Lambda_m$  the function  $K(z, t, \kappa)$  is approximately represented by the expression:

$$\frac{R_m(z, t)}{\kappa - \Lambda_m}.$$

As  $K(z, t, \kappa)$  satisfies equation (2.2) with  $\kappa = \Lambda_m$ , this is also true for  $R_m(z, t)$  for all values of  $z$  and  $t$  except  $z = t$ . As  $K(z, t, \kappa)$  satisfies equations (1.2) with respect to  $z$  ( $K$  being substituted for  $w$ ) and equations (1.3) with respect to  $t$  ( $K$  being substituted for  $w^*$  and  $t$  for  $z$ ), the function  $R_m(z, t)$  satisfies these same conditions. It may be proved that the solution of equation (2.2) with either the equations (1.2) or (1.3) is unique but for an arbitrary multiplier, if  $\Lambda$  coincides with a simple characteristic value  $\Lambda_m$ . Denoting a solution of (2.2) and (1.2) with  $\Lambda = \Lambda_m$  by  $w_m(z)$ , and a solution of (2.2) and (1.3) with  $\Lambda = \Lambda_m$  and  $t$  instead of  $z$  by  $w_m^*(t)$ , these being characteristic functions of the said problems, corresponding to the characteristic value  $\Lambda_m$ , we hence have, in this case,

$$R_m = -r_m w_m(z) w_m^*(t), \quad (7.1)$$

$r_m$  indicating a multiplier independent of  $z$  or  $t$  to be determined below. In the case of a double characteristic value  $\Lambda_m$ , two corresponding and linearly independent characteristic functions of each problem exist. This case can only arise if  $\sigma = \pm 1$ , as was stated in § 1, and the corresponding two problems are hence self-adjoint, entailing that  $w(z) = w^*(z)$ . In this case the characteristic values  $\Lambda_m$  are all real if  $1 + \phi\Phi$  has zeros and the expansion problem is considerably simplified, so as to justify its dismissal from the present discussion, referring to previous publications [17, 18].

We now proceed with the determination of  $r_m$ . To effect this, Green's function  $G(z, t)$ , resulting from  $K(z, t, \Lambda)$  in the case that  $\Lambda = 0$  is considered. This function  $G(z, t)$  satisfies the same conditions as  $K$ , but with  $\Lambda = 0$ . Hence its convergent infinite series expansion may



be derived from equation (5.2), taking  $\Lambda = 0$ :

$$G(z, t) = \sum_{m=1}^{\infty} -\frac{R_m(z, t)}{\Lambda_m} = \sum_{m=1}^{\infty} \frac{r_m w_m(z) w_m^*(t)}{\Lambda_m}. \quad (7.2)$$

It will be assumed that this series is uniformly convergent. This proposition will be proved below. Multiplying both sides of (7.2) by  $w_m(t)\{1 + \phi\Phi(t)\}$  and integrating with respect to  $t$  from  $t$  to  $t + \zeta$ , taking into account the bi-orthogonality resulting from (1.4):

$$\int_z^{z+\zeta} w_{m_1}(t) w_{m_2}^*(t) \{1 + \phi\Phi(t)\} dt = 0, \quad \text{if } m_1 \neq m_2, \quad (7.3)$$

we obtain:

$$\int_z^{z+\zeta} G(z, t) w_m(t) \{1 + \phi\Phi(t)\} dt = \frac{r_m w_m(z) \int_z^{z+\zeta} w_m(t) w_m^*(t) \{1 + \phi\Phi(t)\} dt}{\Lambda_m}. \quad (7.4)$$

The left side of (7.4) is equal to  $w_m(z)/\Lambda_m$  by virtue of the linear homogeneous integral equation, equivalent to Hill's problem (2.2), (1.2). Hence the expression:

$$\frac{1}{r_m} = \int_z^{z+\zeta} w_m(t) w_m^*(t) \{1 + \phi\Phi(t)\} dt \quad (7.5)$$

results. With the aid of this expression the convergent infinite series expansion, resulting from (5.2), may be completed:

$$K(z, t, \Lambda) = \sum_{m=1}^{\infty} \frac{r_m w_m(z) w_m^*(t)}{\Lambda_m - \Lambda}. \quad (7.6)$$

The equations (7.2), (7.6) and (7.5) will prove of great value in the solution of several sub-problems, derived from Hill's problem.

## 8. UNIFORM AND ABSOLUTE CONVERGENCE OF THE SERIES EXPANSIONS OBTAINED

As a first step to the application of equations (7.2), (7.6) and (7.5) it is necessary to prove the uniform convergence of the infinite series involved within each period  $z_0 \leq z$ ,  $t \leq z_0 + \zeta$ , taking into account the conditions of Hill's problem. We shall discuss this convergence in the two cases: (a)  $1 + \phi\Phi(z)$  has no zeros, (b)  $1 + \phi\Phi(z)$  has zeros.

In the case (a), the characteristic values  $\Lambda_m$  and the corresponding characteristic functions are given in equations (3.3) and (3.2). We shall show that the transformed characteristic function  $|w_m(z)|$ , apart from its arbitrary multiplier  $A$ , and hence the corresponding characteristic function  $|w_m(z)|$  too, are finite for any suffix  $m$  within the said interval of  $z$  and of  $z$ , if  $\sigma$  and hence  $\mu$  is finite. This property becomes obvious if we insert (3.3) in (3.2) and thereupon apply the transformation necessary in order to obtain  $w_m(z)$ . An upper bound may be indicated for  $|w_m(z)|$  within the said interval of  $z$ , this bound being independent of the suffix  $m$ . The multiplier  $A$  need not be considered further, as  $A$  occurs equally in both the numerator and the denominator of each term of the infinite series (7.2) and (7.6). Hence only the value of  $r_m$  remains to be considered. Making use of the transformation of  $w_m$  to  $w_m$  and of  $z$  to  $z$ , equation (7.5) may be written (omitting the multipliers  $A$  as stated above):

$$\frac{1}{r_m} = \int_{z_0}^{z_1} w_m(z) w_m^*(z) dz = \int_{z_0}^{z_1} dz = z_1 - z_0. \quad (8.1)$$

This value is also independent of  $m$ . If  $\Lambda$  in equation (7.6) is finite (this assumption being essential) and  $m \rightarrow \infty$ , the absolute values of the consecutive terms are, from a certain fixed value of  $m$  onwards, smaller than  $M/m^2$ , where  $M$  is a positive finite bound, independent of  $m$ . Thus it is shown that this series is absolutely and uniformly convergent in the present case (a).

In the case (b) we may conclude from the discussion of characteristic values by equation (4.2) that  $|\Lambda_m|$  is approximately proportional to  $m^2$  for any finite value of  $\sigma$  (and hence of  $\mu$ ) from a certain fixed value of  $m$  onwards. We proceed to discuss the value of  $|w_m(z)|$  if

$m \rightarrow \infty$ . As no single representation of solutions of equation (2.2) is possible in this case throughout one period, the discussion has to be applied to the different intervals I to III quoted in § 3. We have represented a characteristic function  $w_m(z)$  by a linear combination of  $w_I(z)$  and  $w_{II}(z)$ , inserting  $\Lambda_m$  for  $\rho^2$ , with two constant multipliers. The ratio of these multipliers has been determined from the conditions (1.2), the resulting two linear homogeneous equations for these multipliers being consistent on account of the value of  $\rho^2$  inserted. In this way expressions with one still arbitrary multiplier in front were obtained for  $w_m(z)$  and for  $w_m^*(z)$ , which, on account of  $w_I(z)$  and  $w_{II}(z)$  being known throughout the said intervals I, II and III (see, e.g., eqs. (6.1)), are known there too. Apart from the said arbitrary multiplier (denoted by  $B_1$ ), these expressions for  $|w_m(z)|$  and  $|w_m^*(z)|$  are bounded throughout the said intervals for any finite value of  $\sigma$  independently of  $m$ . Considering equations (7.2) and (7.6), the multiplier  $B_1$  occurs equally in both the denominator and the numerator and hence cancels out. Disregarding it further, the value of  $r_m$  has been shown by insertion of the said expressions for  $w_m(z)$  and  $w_m^*(z)$  to be bounded independently of  $m$ . Hence the same conclusion applies to the series (7.2) and (7.6) in the present case ( $\delta$ ) as in the case ( $\alpha$ ) above. Thus the assumption made in § 7 in order to obtain equation (7.5) is justified. The equations (7.6) and (7.2) may be combined so as to obtain an equation, similar to a well-known equation in the case of real characteristic values and self-adjoint problems:

$$K(z, t, \Lambda) = G(z, t) + \Lambda \sum_{m=1}^{\infty} \frac{r_m w_m(z) w_m^*(t)}{\Lambda_m (\Lambda_m - \Lambda)}. \quad (8.2)$$

The series on the right may be indicated as the meromorphic part of  $K(z, t, \Lambda)$  with respect to  $\Lambda$ , and is absolutely and uniformly convergent within the said interval.

## 9. SERIES EXPANSIONS OF ARBITRARY FUNCTIONS AND SOLUTION OF HILL'S INHOMOGENEOUS PROBLEMS

We assume  $g(t)$  to be an integrable real function of the real variable  $t$  and obtain  $f(z)$  by the operation:

$$f(z) = \int_z^{z+\xi} G(z, t) \{1 + \phi\Phi(t)\} g(t) dt. \quad (9.1)$$

This function  $f(z)$  satisfies the conditions (1.2) as  $G(z, t)$  does so with respect to  $z$ . Substitution of the series expansion (7.2) for  $G(z, t)$  in (9.1) yields:

$$\left. \begin{aligned} f(z) &= \sum_{m=1}^{\infty} c_m w_m(z), \\ \Lambda_m c_m &= \int_z^{z+\xi} w_m^*(t) \{1 + \phi\Phi(t)\} g(t) dt. \end{aligned} \right\} \quad (9.2)$$

We have hence obtained an absolutely and uniformly convergent series expansion of a function  $f(z)$ , representable by (9.1) but otherwise arbitrary, as is  $g(t)$  (apart from being integrable). In consequence of the uniform convergence of (9.2) we obtain the relation:

$$\lim_{n \rightarrow \infty} \int_z^{z+\xi} |f(z) - \sum_{m=1}^n c_m w_m(z)|^2 dz \rightarrow 0. \quad (9.3)$$

This relation testifies to the *closed* and *complete* character of the infinite bi-orthogonal set of characteristic functions  $w_m(z)$  and of their adjoint characteristic functions  $w_m^*(z)$ . This property has previously been proved in the case of real parameters and  $|\sigma| = 1$  [17, 18].

The above series expansion of an arbitrary function may be applied to the solution of Hill's inhomogeneous problem, consisting of the differential equation,

$$\frac{d^2 w}{dz^2} + w\{\Lambda_0 + \Gamma_0 \Phi(z)\} + w\Lambda\{1 + \phi\Phi(z)\} = f(z)\{1 + \phi\Phi(z)\}, \quad (9.4)$$

or of the equation, arising from (2.3), with this same expression on the right, and of the



$\Lambda_1$  denoting the characteristic value of smallest modulus and  $k_1$  the multiplicity of this value. Hence:

$$\left| \frac{k_1}{\Lambda_1} \right| \geq \lim_{n \rightarrow \infty} \left| \frac{F_n^{\alpha_n}}{\Lambda_n^{\alpha_n}} \right|. \quad (10.2)$$

Thus we have obtained an upper bound for  $\Lambda_1$  by iteration. By the asymptotic formulæ for the characteristic values  $k_1$  is always finite. We have:

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{\infty} \frac{1}{\Lambda_m^{\alpha_n}} = \sum_{m=1}^{k_1} \frac{1}{\Lambda_m^{\alpha_n}}. \quad (10.3)$$

Besides the iterated functions  $G_n$  we consider:

$$H_n(z, t) = \int_z^{z+\xi} G_n(t, t_1) G^*(t, t_1) \{1 + \phi \Phi(t_1)\} dt_1,$$

$$F_{n,0} = \int_z^{z+\xi} \int_z^{z+\xi} G_n(z, t) G^*(z, t) \{1 + \phi \Phi(z)\} \{1 + \phi \Phi(t)\} dz dt = \sum_{m=1}^{\infty} \frac{1}{\Lambda_m^{\alpha_n+1}}.$$

Now

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} G_n(z, t) &= \sum_{m=1}^{k_1} \frac{r_m w_m(z) w_m^*(t)}{\Lambda_m^{\alpha_n}}, \\ \lim_{n \rightarrow \infty} H_n(z, t) &= \sum_{m=1}^{k_1} \frac{r_m w_m(z) w_m^*(t)}{\Lambda_m^{\alpha_n+1}}, \\ &\text{except at a zero of } w_m \text{ or of } w_m^*. \end{aligned} \right\} \quad (10.4)$$

Hence, if  $k_1 = 1$ ,

$$\frac{1}{\Lambda_1} = \lim_{n \rightarrow \infty} \frac{H_n(z, t)}{G_n(z, t)} = \lim_{n \rightarrow \infty} \frac{F_{n,0}}{F_n}. \quad (10.5)$$

Thus we obtain the first complex characteristic value in this case. Multiplication by  $\Lambda_1^{\alpha_n}$  or  $\Lambda_1^{\alpha_n+1}$  of the first or the second equation (10.4) then yields the corresponding complex characteristic functions too.

In this case,  $k_1 = 1$ , we now consider:

$${}_1G(z, t) = G(z, t) - \frac{w_1(z) w_1^*(t)}{\Lambda_1}.$$

With the aid of  ${}_1G(z, t)$  the processes of iteration as described above may be carried out in the same way as with  $G(z, t)$ . The resulting formulæ yield the second characteristic value and the corresponding functions if its multiplicity  $k_2 = 1$ :

$$\left. \begin{aligned} \frac{1}{\Lambda_2} &= \lim_{n \rightarrow \infty} \frac{{}_1H_n(z, t)}{{}_1G_n(z, t)} = \lim_{n \rightarrow \infty} \frac{{}_1F_{n,0}}{{}_1F_n}, \\ \lim_{n \rightarrow \infty} {}_1G_n(z, t) &= \sum_{m=2}^{k_2} \frac{r_m w_m(z) w_m^*(t)}{\Lambda_m^{\alpha_n}}. \end{aligned} \right\} \quad (10.6)$$

This process may be continued in order to obtain the third characteristic value, etc.

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XXXI.—Thermal Diffusion in some Aqueous Solutions. By Archibald C. Docherty and Mowbray Ritchie, Chemistry Department, University of Edinburgh. (With Five Text-figures.)

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THE study of thermal diffusion in gases has recently been so far developed that a satisfactory theory can in general be applied to explain and predict separations in gaseous mixtures. The results have been of value from the practical point of view of simple separation; at the same time the method can be used in the elucidation of the laws of interaction between colliding gas molecules and of the departures from the postulates of simple kinetic theory. On the other hand, thermal diffusion theory as applied to liquid mixtures is much more unsatisfactory, the complexity of the liquid system being such as to render uncertain the prediction of sign and magnitude of separation.

When equilibrium conditions are considered, an appreciable separation of two liquids involves generally large changes in viscosity, density, etc., such changes making theoretical interpretation difficult. The present paper gives some measurements of initial rates of separation as near as possible to zero time, when the concentrations and other properties of the solution are definitely known. The experimental method was an adaptation of the "cascade" method originally applied to gases by Clusius and Dickel (1938), and extended to liquids by, *inter alia*, Korsching and Wirtz (1940). The work was begun in the first instance with a view to the possible separation of sugars in aqueous solution. This class of substance was regarded as possessing properties which would facilitate such a study, viz. the non-electrolytic nature, the "normality" of osmotic pressure values for dilute solution, the general similarity throughout the group of possible hydroxyl and hydrogen bonding with water, the range of solubility, the non-volatility, and the availability of such data as density and viscosity. The results given refer to the solutes glucose, sucrose, xylose, raffinose; glycerol and acetone were included for comparison.

#### EXPERIMENTAL PROCEDURE

*The Thermal Diffusion Apparatus.*—The thermal diffusion column consisted essentially of two concentric thin-glass tubes approx. 150 cm. long, of approx. 1 cm. diameter, so sealed together at the lower end that the annular space between the tubes gave an average radial gap of 0.68 mm. Initial attempts were made to ensure an exactly constant gap along the tube length by suitable spacing materials; this, however, was found difficult practically, more particularly as it was necessary to use as thin glass as possible to ensure that the temperature gradient applied was not largely lost in the glass walls themselves. The use of spacing material was finally dispensed with and reliance placed on the selection of suitable tubes correctly sealed into positions which would not alter from experiment to experiment. To facilitate filling and to avoid breakage in operation the upper ends of the tubes were not sealed but held in position by rubber. This arrangement gave satisfactorily reproducible results.

The solution to be investigated was introduced into the annular space by a fine capillary and stopcock at the extreme lower end of the column, this system being also used for the withdrawal of samples for analysis. The capillary was as short as possible and was sealed on in such a way as to leave unaltered the radial separation between the tubes. An outer jacket of copper lagged by asbestos was so attached as to cover as much as possible of the column, particularly at the lower end; the capillary projected immediately below the heating jacket. The temperature gradient was applied by passing cold water of known temperature rapidly through the inner tube, and steam or hot water of known temperature rapidly through the copper jacket. The difference in temperature for any one stream between entrance and exit was never more than a few degrees. As a matter of convenience, both heating and cooling streams entered at the column top.

It was of importance to determine how accurately the effective mean temperature of the solution in the annular space could be estimated from the measured mean temperatures of the heating and cooling streams, and how closely the measured temperature gradient corresponded to the actual gradient across the solution. To this end a balancing vertical column was temporarily attached to the capillary stopcock and the annular space and this column filled with water to a fixed level just above the top of the heating jacket. By means of the relationship  $\rho_1 h_1 = \rho_2 h_2$ , the mean density of the water in the diffusion column was determined from the difference in column lengths for different applied temperature gradients. Results are given in Table I.

TABLE I

Exp.	Measured Temperatures °C.						Mean Temp.	Diff. in h. (cm.)	$\rho$ Calc.	Calc. Mean Temp.
	Inner Tube			Outer Tube						
	Top	Bottom	Av.	Top	Bottom	Av.				
1	7.0	7.0	7.0	7.0	7.0	7.0	7.0	0.00	0.99993	7.0
2	7.0	7.0	7.0	23.0	23.0	23.0	15.0	0.15	0.99865	17.8
3	7.0	7.0	7.0	29.0	29.0	29.0	18.0	0.25	0.99798	21.2
4	7.0	7.5	7.3	44.0	44.0	44.0	25.7	0.35	0.99720	24.5
5	7.0	7.5	7.3	51.0	51.0	51.0	29.2	0.45	0.99643	27.4
6	7.0	8.0	7.5	60.0	60.0	60.0	33.8	0.65	0.99488	32.5
7	7.0	8.5	7.8	88.0	83.0	85.5	46.7	1.40	0.98913	40.8
8	7.0	9.0	8.0	100.0	100.0	100.0	54.0	2.20	0.98307	60.0

It was concluded from columns 8 and 11 that for the present purposes the mean solution temperature could be satisfactorily obtained from the mean temperatures of the heating and cooling streams. The divergences from these values as shown by the density method above were connected with the relative thicknesses of the glass tube walls and with the rates of flow, and could not with the present experimental arrangements be made appreciably smaller.

The main separation between the tubes was determined by running water from a burette into the dry diffusion column and noting the volume required to fill a measured height of annular space. The mean separation was thus 0.68 mm. The total volume of water or solution corresponding to a "full" tube was approx. 28 ml.

*Methods of Analysis.*—In all cases the change in concentration at the foot of the tube was determined from a 0.5 ml. sample withdrawn from the column after the rejection of a similar volume corresponding to the volume of the capillary tube and the portion of the diffusion column not directly subjected to the temperature gradient. Preliminary tests on successive samples at successive time intervals showed a straight line concentration-time relationship for the early time intervals; later results were calculated from two sets of data only. In all cases the time of diffusion was taken as small as was consistent with accurate sample analysis.

In any experiment, the cooling stream was first turned on and the solution in question then introduced. At zero time the heating stream was applied; as judged by the expansion of the column, a few seconds only were required to reach temperature equilibrium. The sample was removed at noted time intervals under these steady conditions, the temperatures being noted. Normally the duration of such intervals was of the order 5 to 30 minutes. Changes in concentration were expressed in gm. molecules per litre (at 15° C.) per min. and were obtained by the following methods of analysis. In all cases the methods were checked on control samples; further control experiments in which solutions were maintained at 100° C. for times similar to those involved in actual diffusion runs showed no detectable concentration changes such as might be expected by sugar hydrolysis.

Solutions containing one solute, *e.g.* raffinose, xylose, sucrose, glucose, glycerol, were examined by a constant temperature Pulfrich refractometer on the basis of previous calibration in sodium light of solutions of known concentrations. In the case of acetone the method used was that of Munro (1936), involving the quantitative oxidation of acetone by alkaline hypiodite, acidification, and titration of unused iodine by thiosulphate.

For solutions containing both sucrose and glucose, a Hilger polarimeter with mercury green light was used to give a direct reading of a suitably diluted sample, which was then inverted by concentrated hydrochloric acid, the change of rotation being proportional to the sucrose present. The glucose was then determined by difference.

For solutions containing both sucrose and glycerol the method was based on that of Fulmer, Hickey and Underkofler (1940), which involves the determination of glucose by the copper titration procedure of Shaffer and Somogyi (1933), in conjunction with the oxidation of a second similar sample by ceric sulphate. Both glucose and glycerol are oxidised, the glycerol being calculated from the amount of ceric sulphate used by the glucose. In order to adapt the method to the determination of sucrose, the sucrose was first inverted at 70° C. by sulphuric acid.

In solutions containing both acetone and glycerol, acetone was determined by the alkaline hypiodite method above, control experiments showing that glycerol at the concentrations in question did not interfere with the determination. Ceric sulphate was used to oxidise both glycerol and acetone, the glycerol being determined by difference.

#### EXPERIMENTAL RESULTS AND DISCUSSION

Debye (1939), in a theoretical discussion of the thermal diffusion apparatus of two parallel vertical plates with a temperature difference of  $T^\circ$ , at a distance apart  $a$  and of height  $h$ , found the value of  $a$  for which maximum separation at equilibrium should be obtained, by the relation  $a^3 = 600\eta D/B\rho gT$ , where  $\eta$  is the solvent viscosity,  $D$  the coefficient of (ordinary) diffusion,  $B$  the coefficient of cubic expansion,  $\rho$  the density, and  $g$  the acceleration due to gravity. With appropriate values for aqueous solutions at 60° C.,  $a$  is approx. 0.1 mm. For such a maximum separation condition Debye further considered the separation in a time

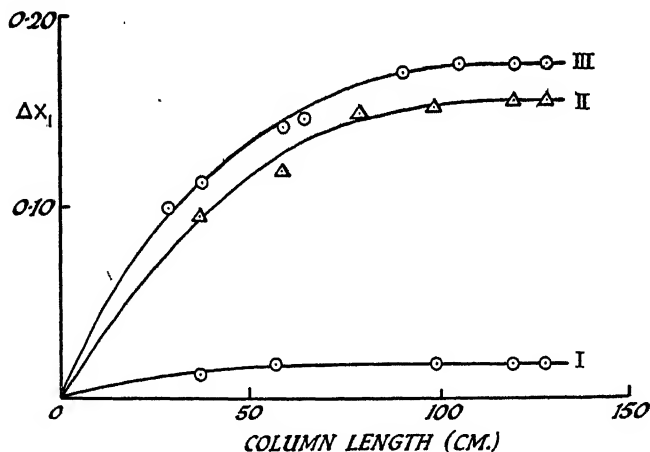


FIG. 1.

shorter than that required to reach equilibrium; when the heated volume is very much greater than the volume of the reservoir, the reservoir comprising all liquid not subjected to the temperature gradient, the separation is given by a relation into which the height of the column does not enter. The separation is then independent of the column length. Since this holds, strictly speaking, for the above value of  $a = 0.1$  mm., it was of interest first to ascertain if such independence also held for the  $a$  value of 0.68 mm. of the experimental arrangement.

The change in concentration of a glucose solution of 1.726 gm. mol. per litre was measured for various lengths of column liquid for various mean temperatures and gradients. The time of each experiment was 30 minutes. Results are shown graphically in fig. 1, where the increase in glucose concentration at the column foot is plotted against the length of column liquid.

Curve I refers to gradient 50° C., mean temperature 30° C., curve II to gradient 51° C., mean temperature 75° C., and curve III to gradient 95° C., mean temperature 53° C. It will be observed that for these conditions, representing the maximum variation which could be



practically applied, the rate of separation is independent of the column length for lengths greater than 100 cm. Thus under the general experimental conditions of a full column of 130 cm. the rate of separation will not be directly affected by the small reduction in length due to the removal of samples for analysis; further, all experiments are comparable in that they lie in a region for which the column length is immaterial. As will be indicated later (p. 301), the relative positions of these curves are in general agreement with the effects of alteration of mean temperature and temperature gradient.

*Single-Solute Solutions.*—Fig. 2 shows the effect of variation of concentration of a series of single solutes, the operating conditions being otherwise the same throughout, viz. mean temperature 60° C., temperature gradient 80° C., total solution volume 28.2 ml.

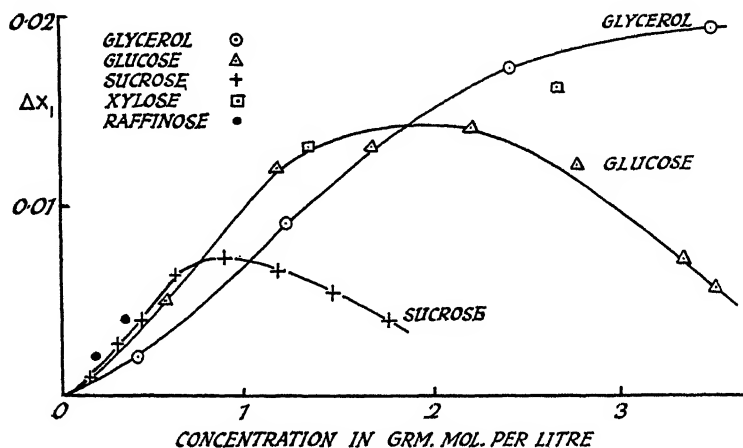


FIG. 2.

The initial separation rate  $\Delta X_1$  in gm. mol. per litre per min. is plotted against the initial concentration. In all cases the concentration increased at the column foot. All the curves show a maximum at an intermediate concentration: the greater the solute molecular weight the less the molar concentration at which the maximum occurs. At low concentrations it would appear that the greater the molecular weight the greater is the initial rate of separation, although for such conditions the curves for glucose and sucrose differ only very slightly. The relative position of the curves in general suggested that for some concentrations partial solute separation might be practicable and some two-solute solutions were thus examined.

*Two-Solute Solutions.*—(a) *Sucrose-Glucose.*—Results are given for a solution containing 1.00 gm. mol. of each solute. Operating conditions were as before. In Table II  $F_g$  and  $F_s$  represent the factors for glucose and sucrose increases in molecular concentrations. The relative separation is very small though in a direction according to expectations based on fig. 2.

TABLE II

Time (min.)	$F_g$	$F_s$	$F_s/F_g$
0	1.00	1.00	1.00
2	1.02	1.00	0.98
5	1.03	1.01	0.98
10	1.05	1.03	0.98
20	1.08	1.03	0.95
30	1.10	1.05	0.95

The experiments were repeated for different concentrations of glucose and sucrose;  $F_s/F_g$  ratios are given in Table III for the various times of experiment.

Again the separations are in general small and in certain instances contrary to expectation. It is obvious that single-solute solution results cannot be used to predict such separations; certain other properties of the solution must be considered.

(b) *Sucrose-Glycerol*.—A solution containing per litre 0.90 gm. mol. sucrose and 0.97 gm. mol. glycerol was subjected to the same conditions of thermal diffusion for 30 min. At the end of that time the concentration of sucrose had increased by the factor 1.22, while that of the glycerol increased by the factor 1.16; relative separation was again only of the order 5 per cent.

TABLE III

Glucose Conc.	Sucrose Conc.	Time (min.)				
		0	5	10	20	120
2.00	1.00	1.00	0.99	1.00	0.99	0.99
2.00	0.50	1.00	1.03	1.05	..	1.03
3.00	0.50	1.00	1.15	1.12	1.21	1.15
0.50	1.00	1.00	1.00	0.93	..	0.96

## TEMPERATURE VARIATION

(a) *Variation of Mean Temperature*.—The temperature gradient was kept constant at 40° C. and the mean temperature altered for single-solute solutions of glucose, sucrose, glycerol, all of 30 gm. per 100 ml. Results are shown graphically in fig. 3.

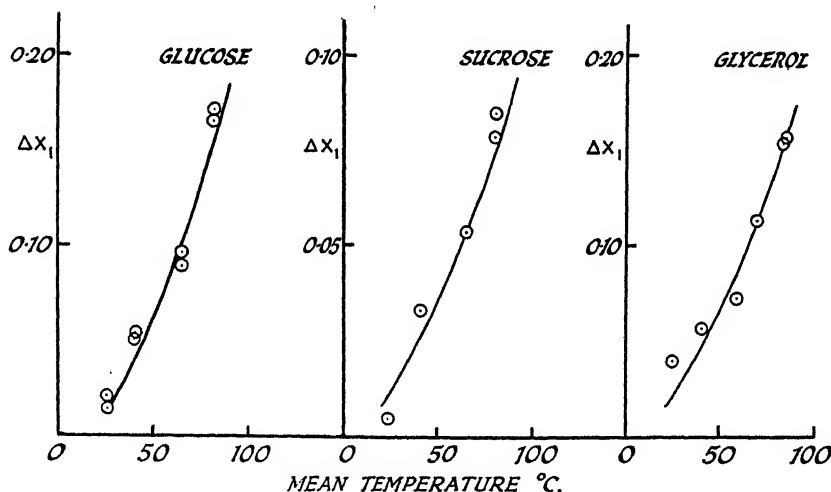


FIG. 3.

A rapid increase in initial rate appears for increased mean temperature; the continuous curves have been drawn on the basis of a general formula derived later (see following paper).

(b) *Variation of Temperature Gradient*.—A glucose solution of 30 gm. per 100 ml. was here employed, and the temperature gradient varied for constant mean temperatures. Practical adjustment of such temperatures was somewhat difficult, but results are shown in fig. 4, where the numbers at each experimental point refer to the actual mean temperature, and the continuous curves I, II and III are calculated curves for mean temperatures 40°, 60° and 80° C. respectively. Again increase in temperature gradient increases the separation rate; the relationship is not a linear one.

These results are in general correspondence with the curves of fig. 1. In curve I of fig. 1 gradient and mean temperature are both low; in curve II the gradient is the same, and the marked increase in rate is due to the increased mean temperature; in curve III the reduction of mean temperature is compensated by the increase in temperature gradient.

*The Sign of the Separation*.—In all the previous experiments with raffinose, sucrose, glucose, xylose and glycerol, the separations have been positive in the sense that an increase in concentration was found at the lower end of the column. The literature (see Korsching and Wirtz, 1940; Gillespie and Breck, 1941) shows that there is a general correspondence between

the sign of the separation and the density of the solution as compared with the solvent density, although exceptions appear to occur. All the above give solutions of density greater than water itself, and the results are thus in agreement with this generalisation. The case of acetone was then considered; this substance will be different in its relation to water by the absence of appreciable hydroxyl or hydrogen bonding as compared with the sugars, but may be compared with glycerol in that the density of water is almost intermediate between the two, and the apparent molecular volumes as calculated directly from the densities and the molecular weights are approximately the same.

Because of the low boiling-point of acetone a lower operating mean temperature and gradient were necessary. The relation between initial rates of separation and concentration was determined for a mean temperature of 30° and a gradient of 40° C. Results are shown in fig. 5; in all experiments a decrease in concentration was observed at the column foot.

Apart from the change in sign, the curve is of the same general type as before, although the separation at low concentrations appears abnormally small.

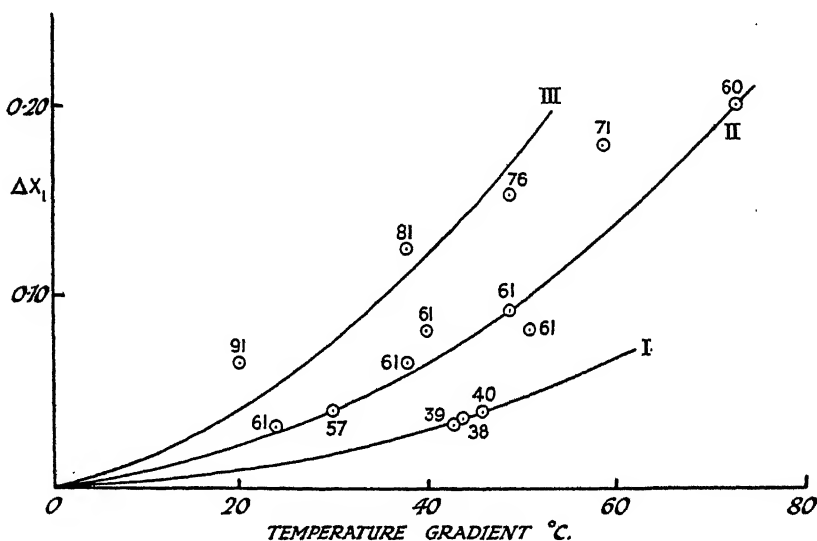


FIG. 4.

It was first necessary to inquire whether the experimental method involving convection did not introduce a factor independent of the actual thermal diffusion. In such convection as determined by gravity the tendency will always be for the liquid of less density to accumulate at the top. There was therefore the possibility that the real movement of acetone was to the colder region as with glycerol, but the resultant density change in the measured time of experiment might cause a reversal of convection.

Early experiments by Wereide (1914) by the Soret method involving no "cascade" convection gave a separation to the cold region of aqueous solutions of glycerol, but no observed separation of acetone or of ethyl and methyl alcohols, all of which latter give solutions of less density than water. This might thus be the result of two competing influences, thermal diffusion to the cold region balanced by convection in the system. An attempt to repeat such a Soret-type experiment was made by means of an inverted V-tube containing 50 per cent. by volume of acetone. One limb was kept at 50° and the other at 5° C.; after 66 hours, examination by a Zeiss interferometer indicated a very slight migration of acetone to the hot region.

A return was made to the convection column with three methods of approach:

(a) The time of diffusion was reduced to a minimum consistent with measurement of concentration change, so that any density change would be very small. The interferometer, however, showed that even after only 30 seconds with a 10 per cent. volume solution some acetone had migrated from the foot of the column.

(b) The acetone concentration was reduced as far as possible. The rate of separation

then was very small, but a 1 per cent. volume solution after 30 seconds still showed a detectable reduction of acetone concentration at the foot of the column.

(c) A ternary mixture of water, glycerol and acetone was examined. If the acetone does in fact move to the cold region, the simultaneous migration of glycerol would bring the density up to a value such that normal convection would take place. For a solution containing 0.2965 gm. mol. glycerol and 0.0962 gm. mol. acetone, the concentrations after 10 min. at the column foot were 0.405 and 0.0935 respectively; for solutions containing 0.165 and 0.1375 the resultant concentrations were 0.295 and 0.126. It was thus concluded that for the concentration and temperature conditions in question, acetone does migrate to the hot region apart from any convection complications. This is then in agreement with recent work of Wirtz (1943), although De Groot (1945) reports a separation to the cold region for certain unspecified conditions.

From the practical point of view, the previous results indicate that an apparatus of the present simplicity cannot conveniently be employed to separate dissolved sugar solutes in such aqueous solutions. An attempt to correlate and explain such results theoretically is

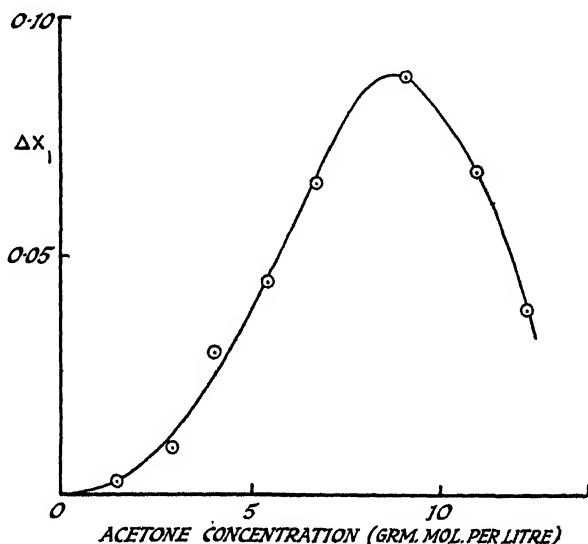


FIG. 5.

presented in the following paper. While the experimental results were satisfactorily reproducible, it may be emphasised here that the present arrangement with its glass construction and irregular annular space may be expected to complicate the already difficult problem of such thermal diffusion, more particularly from the aspects of turbulence in convection and of correct mean temperature and temperature gradient determination. Further refinement of the experimental arrangement is therefore necessary.

The authors desire to express their thanks to the Carnegie Trustees for the award of a Scholarship to one of them (A. C. D.), during the tenure of which part of the above work was carried out.

#### SUMMARY

Thermal diffusion in aqueous solutions of raffinose, sucrose, glucose, xylose, glycerol and acetone was studied in respect of variation of concentration and of temperature. Initial rates of separation were determined as produced by an apparatus consisting of two vertical concentric glass tubes of length 130 cm. and mean annular separation 0.68 mm.

With a solution of glucose of 1.7 gm. mol. per litre, the initial rate of separation for the extremes of temperature which could be experimentally applied was found to be independent of the column length for lengths greater than 100 cm.

With the exception of acetone, all the solutes concentrated at the column foot. In single-solute solutions a maximum rate was observed at an intermediate concentration, which in the sugar series was at a lower molar concentration the higher the molar weight of the solute. With the two-solute solutions sucrose-glucose and sucrose-glycerol little or no relative separation was obtained, contrary to expectation based on the single-solute data.

Initial rates increased with rise in mean temperature and with increase in the applied temperature gradient.

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XXXII.—An Elementary Treatment of Thermal Diffusion in Gaseous and Liquid Systems. By Mowbray Ritchie, Chemistry Department, University of Edinburgh. (With Three Text-figures.)

(MS. received March 25, 1946. Read July 1, 1946)

It has been stated that appreciation of the principles of thermal diffusion in gases is difficult without detailed study of the transport equations for actual as opposed to ideal "kinetic theory" molecules. Such difficulty is obviously enhanced for the liquid state where even the simple diffusion law in relation to concentration is not fully developed (*cf.* Fürth, 1945). It was desired to have some simple if very approximate theory by which the main features noted in the previous paper might be examined; at the same time, any such theory must in principle be capable of reproducing the essentials of thermal diffusion in gaseous systems.

Thermal diffusion is thus considered for both gases and liquids in an "elementary strip". The expression for liquids is then combined with the convection effect of the diffusion column. The general approach to the problem was as follows.

Consider a volume of a binary gaseous or liquid mixture, represented by the elementary strip ABCD (fig. 1), where the faces AB and CD are maintained at fixed temperatures  $T - \Delta T$  and  $T + \Delta T$ . It is experimentally observed that the concentration of one species increases in

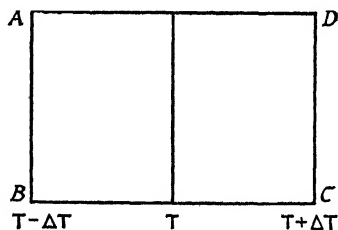


FIG. 1.

one temperature region and decreases in the other. By simple kinetic theory there is no explanation of this phenomenon. A molecule of one species will diffuse from the intermediate  $T$  region in the direction of, say, the  $T + \Delta T$  region at a rate determined by the conditions. At the same time the reverse process may be visualised as taking place to an equal extent; if the conditions along the path of transfer are the same, no net change in concentration will occur. This will apply also to the other type of molecule in the mixture, although the actual distance travelled in unit time will be different. No relative concentration gradient is therefore to be expected, because, essentially, the conditions encountered in the path between the two regions are regarded as the same no matter what the direction of transfer. It has become apparent that the general explanation of the observed phenomenon lies in the fact that a molecule from the  $T$  region on entering the  $T + \Delta T$  region does not at once lose the diffusional characteristics evident in the  $T$  region. The effect may thus be described as a persistence of molecular characteristics. If then a  $T$  molecule be regarded as "projected" into a  $T + \Delta T$  region, the reverse "projection" of a  $T + \Delta T$  molecule will not exactly restore the original situation, because the diffusional characteristics are not now the same throughout the path of travel. The same will apply, though to a different degree, to the other type of molecule, and a relative concentration change will then be found.

To determine the direction of concentration change as a function of pressures or concentrations, masses and diameters, a molecule is considered as diffusing from the central  $T$  region separately into the two regions of higher and lower temperatures; the difference between such rates of diffusion for the two kinds of molecule, on the condition that the total pressure must

remain the same throughout the system, will give a measure of the concentration change. In as much as the gas system must be capable of such treatment to give results in agreement with experiment, a gas mixture is first considered.

#### GAS SYSTEM

The relative rate of diffusion of a molecule A through a gas X at temperature T may be taken as  $D_X^A/[X]$ , where  $D_X^A$  is a diffusion constant containing molecular weight and diameter factors only. If the A molecules are of concentration  $[A]$ , the number of A molecules diffusing in a given direction in unit time may be taken as  $[A]D_X^A/[X]$ .

Where the A molecules diffuse through a binary mixture of  $X_1$  and  $X_2$  molecules, the time taken to diffuse a given distance in the mixture may be taken as the sum of the times for the constituents considered singly. Thus for one A molecule,

$$1/D_{X_1X_2}^A = [X_1]/D_{X_1}^A + [X_2]/D_{X_2}^A, \quad (1)$$

and hence

$$D_{X_1X_2}^A = [X_1]/D_{X_1}^A + [X_2]/D_{X_2}^A)^{-1}.$$

Let the binary gas mixture be represented by two different molecular species of molecular weights,  $M_1$ ,  $M_2$ , diameters  $\sigma_1$ ,  $\sigma_2$ , at pressures  $[X_1]$ ,  $[X_2]$  at temperature T. Since the total gas pressure must be constant, the corresponding concentrations at  $T + \Delta T$  and  $T - \Delta T$  will be

$$\frac{[X_1]T}{(T + \Delta T)}, \quad \frac{[X_2]T}{(T + \Delta T)} \quad \text{and} \quad \frac{[X_1]T}{(T - \Delta T)}, \quad \frac{[X_2]T}{(T - \Delta T)}.$$

The number of  $M_1$  molecules passing from T to  $T + \Delta T$  will be proportional to

$$(\Delta X_1)' = [X_1] \left[ \frac{[X_1]T}{(T + \Delta T)} / D_{X_1}^{X_1} + \frac{[X_2]T}{(T + \Delta T)} / D_{X_2}^{X_1} \right]^{-1} = [X_1] \frac{(T + \Delta T)}{T} D_{X_1X_2}^{X_1}.$$

This flow must be balanced by a movement of the whole mass of binary gas mixture in the reverse direction, where now the relative numbers of  $M_1$  and  $M_2$  molecules so returning will be determined by the relative concentrations of  $M_1$  and  $M_2$ , viz.

$$(\Delta X_1)'[X_1]/([X_1] + [X_2]) \quad \text{and} \quad (\Delta X_2)'[X_2]/([X_1] + [X_2]).$$

The number of  $M_1$  molecules passing from the T region to the  $T - \Delta T$  region is similarly treated; then, by applying the same considerations to the  $M_2$  molecules and summing the various terms, we find the difference in the numbers of  $M_1$  and  $M_2$  molecules passing from the colder to the hotter region, *i.e.* the separation, to be

$$\Delta S = \frac{4[X_1][X_2]}{([X_1] + [X_2])} \cdot \frac{\Delta T}{T} (D_{X_1X_2}^{X_1} - D_{X_1X_2}^{X_2}). \quad (2)$$

If the D terms, or their difference, be regarded as temperature independent over a short temperature range, we have on integration,

$$S \propto \frac{[X_1][X_2]}{([X_1] + [X_2])} (D_{X_1X_2}^{X_1} - D_{X_1X_2}^{X_2}) \log T. \quad (3)$$

The diffusion coefficient  $D_X^A$  may be put as proportional to  $(1/M_A + 1/M_X)^{\frac{1}{2}} \sigma_{AX}^2$  when  $M_A$  and  $M_X$  are the respective molecular weights and  $\sigma_{AX}$  is the sum of their radii. Thus by equation (1),

$$D_{X_1X_2}^{X_1} \propto \left[ \frac{\sigma_{11}^2[X_1]}{(1/M_1 + 1/M_1)^{\frac{1}{2}}} + \frac{\sigma_{12}^2[X_2]}{(1/M_1 + 1/M_2)^{\frac{1}{2}}} \right]^{-1}, \quad (3a)$$

with a corresponding term for  $D_{X_1X_2}^{X_2}$ .

The expression (3) is then in accordance with the essential features of gaseous thermal diffusion as follows:—

(1) The logarithmic dependence on temperature is in agreement with the expressions derived by Chapman and Cowling (1939) and by Fürth (1942).

(2) No separation is of course obtained when  $[X_1]$  or  $[X_2]$  is zero, and there must be one value of  $[X_1]$  and  $[X_2]$ , for which the separation is a maximum. This will not necessarily occur at  $[X_1] = [X_2]$ , because there are concentration factors in the diffusion terms also; the position of the maximum will be determined by the relative masses and diameters. For a given binary mixture the initial separation will, however, be independent of the total pressure for given relative values of  $[X_1]$  and  $[X_2]$ .

(3) If the diameters of the gas molecules are equal, the sign of the separation will be determined by the relative molecular weights. The expression is positive if  $M_1 < M_2$ , *i.e.* the lighter molecules will accumulate in the hot region. If the masses are equal, the molecules of smaller diameter will similarly accumulate in the hot region.

(4) A numerical illustration of the application of the above expression may be given for the case of two isotopes, when the masses are different but the diameters and any force law correction may be taken as the same for the two species. The numerical value of the thermal diffusion factor will depend on the effective diameter, but the ratio  $k_T$  of this factor to the coefficient  $D$  of ordinary diffusion will not contain a diameter term and may thus be compared with the value derived from more rigorous theory.

With  $[X_1] + [X_2] = 1$ , and with the ratio of the isotopic masses not far from unity, the coefficient  $D$  may be taken as  $D = (1/M_1 + 1/M_2)^{1/2} \sigma^2 ([X_1] + [X_2])$ . The ratio  $k_T$  then becomes

$$k_T = \frac{D_T}{D} = [X_1][X_2] \left[ \left\{ [X_2] + \frac{[X_1]}{\sqrt{2}} \left( 1 + \frac{M_1}{M_2} \right)^{1/2} \right\}^{-1} - \left\{ [X_1] + \frac{[X_2]}{\sqrt{2}} \left( 1 + \frac{M_2}{M_1} \right)^{1/2} \right\}^{-1} \right]. \quad (4)$$

Furry, Jones and Onsager (1939) quote Enskog's result for this particular case as

$$k_T = \frac{1.05}{1.8} \frac{(M_2 - M_1)}{(M_2 + M_1)} [X_1][X_2], \quad (5)$$

and Chapman's earlier expression as

$$k_T = \frac{1.7}{3} \frac{(M_2 - M_1)}{(M_2 + M_1)} \cdot \frac{[X_1][X_2]}{(9.15 - 8.25[X_1][X_2])}; \quad (6)$$

while experimental determination of  $k_T$  led Furry, Jones and Onsager to suggest that

$$k_T = 0.35 \frac{(M_2 - M_1)}{(M_2 + M_1)} [X_1][X_2] \quad (7)$$

be used as a provisional value for the design of apparatus. The following table shows values of  $k_T$  calculated for  $M_1 = 100$ ,  $M_2 = 110$  by each of the above expressions.

TABLE I

$X_1$	(4)	(5)	(6)	(7)
0.1	0.0024	0.0038	0.0029	0.0015
0.5	0.0060	0.0106	0.0095	0.0042
0.9	0.0023	0.0038	0.0029	0.0015

Formula (4) above may therefore be used to give reasonable values of the ratio for such conditions.

(5) Examination of equation (3) shows that for given diameters and masses the sign of the separation is always positive, or negative, or zero, independent of the relative concentration; for a particular binary mixture there is no indication of a change of sign as the relative concentrations are altered. Grew (1944), however, observed experimentally such a change in mixtures of neon and ammonia. Here the respective molecular weights and diameters are approximately the same, and the sign of the separation is then very sensitive to small changes in the diameter factors. The effect may be simply treated in relation to the general equation (3) on the basis that the magnitude of the effective collision diameter of the ammonia molecule is somewhat greater in an ammonia-ammonia collision than in an ammonia-neon collision, by virtue of the



"softness" of the ammonia as compared with the neon molecule. If we take certain standard values for the collision diameters, equation (3a) may be amended by the insertion of a factor  $1 + \lambda$  to give

$$D_{x_1 x_2}^{x_1} \propto \left[ \frac{[X_1](1 + \lambda)\sigma_{1,1}^2}{(1/M_1 + 1/M_1)^{\frac{1}{2}}} + \frac{[X_2]\sigma_{1,2}^2}{(1/M_1 + 1/M_2)^{\frac{1}{2}}} \right]^{-1}. \quad (3b)$$

Relative values of  $S$  (equation (3)) have been calculated for a constant temperature by taking

$$\text{NH}_3 = M_1 = 17, \quad \text{Ne} = M_2 = 20, \quad \sigma_{1,1} = 2.38, \quad \sigma_{2,2} = 2.30, \quad \sigma_{1,2} = \sigma_{2,1} = 2.34, \quad 1 + \lambda = 1.03;$$

it is remarkable that this simple amendment reproduces very satisfactorily not only the correct change in sign at a mole fraction of 0.25 as in Grew's experiments, but also the numerical relative separations in each region. This is shown in fig. 2, where the continuous curve was calculated on the above basis, with a proportionality constant of  $5 \times 10^{-6}$ , and Grew's experimental points are indicated by  $\bigcirc$ .

(6) It is obvious that thermal diffusion with actual molecules cannot strictly be dealt with on the simple theory resulting in equation (3), but must be largely dependent on the force law

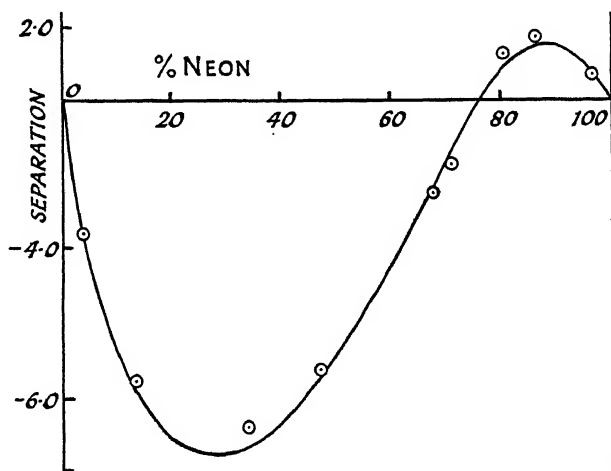


FIG. 2.

governing the collisional approach of the molecules. An indication of the effect of such a force law may be presented as follows.

The above simple theory of thermal diffusion is based essentially on the assumption that the rate of diffusion on the higher temperature side is greater than that on the lower temperature side because of the decrease in molecular concentration at the higher temperature. The rate of diffusion is in fact determined by the collision numbers of the diffusing molecule; if the number of collisions per unit time is the same in both temperature regions no thermal diffusion will occur. The number of collisions per second will be proportional to the effective collision area and to the velocity of the molecule, since the pressure remains constant (Glasstone, 1940), *i.e.*  $Z \propto V\sigma^2$ . If the repulsive force between the molecules be represented by  $F = -K/r^s$ , where  $r$  is the distance between the molecules and  $K$  is a constant, then, following Frenkel (1940),  $V\sigma^2$  may be derived in terms of  $V$  by the method of dimensions. Since  $K/r^s$  has the dimensions of force  $MLT^{-2}$ ,  $K$  has dimensions  $ML^{s+1}T^{-2}$ . Now  $mV^2$  is of dimensions  $ML^2T^{-2}$ , so that  $K/mV^2$  is of dimensions  $L^{s-1}$ . The radius  $\sigma$  of dimension  $L$  must then be proportional to  $(K/mV^2)^{\frac{1}{s-1}}$  or to  $V^{\frac{-2}{s-1}}$  for constant  $m$  and  $K$ . Hence the number of collisions  $V\sigma^2$  is proportional to  $V \cdot V^{\frac{-4}{s-1}} = V^{\frac{s-5}{s-1}}$ . The velocity  $V$  will increase with rise in temperature, but if  $s = 5$  (Maxwellian molecules), the number of collisions will be proportional to  $V^0$ , *i.e.* will be constant and independent of temperature; thermal diffusion will not be observed.

If  $s > 5$ , the number of collisions undergone by a low temperature molecule "projected"

into a high temperature region will tend to be less than the number of collisions undergone by a high temperature molecule in the high temperature region; the diffusing molecule will move further on the hot side than on the cold. This is the state of affairs corresponding to the elementary theory above; high values of  $s$  tend to approximate to kinetic theory conditions.

If  $s < 5$ , the number of collisions will be proportional to  $1/V^n$  where  $n$  is positive. Here higher temperatures and increased  $V$  mean a smaller number of collisions, and the diffusing molecule will then move further on the cold side.

#### LIQUID SYSTEM

In deriving an equation for thermal diffusion in the liquid state by the general principles above, it is necessary to obtain an expression for the rate of diffusion of a molecule through a liquid medium. Glasstone, Laidler and Eyring (1941, p. 525) by the theory of absolute reaction rates as applied to viscosity and diffusion, consider an expression which may be put in the form  $D = F e^{\epsilon_0/kT}$ ; here the factor  $F$  contains not only the masses  $M_1$  and  $M_2$  of solute and solvent molecules, but also molecular volume and temperature parameters, and  $\epsilon_0$  is the potential energy barrier for viscous flow.

When the diffusing molecule is of the same kind as the solvent molecule, as in self-diffusion, the diffusion coefficient above may be reduced (*loc. cit.*, p. 519) to the approximate value

$$D = \frac{kT}{a_1 \eta}, \text{ where } a_1 \text{ is the diameter of the diffusing molecule and } \eta \text{ is the viscosity of the medium.}$$

When a large molecule diffuses in a medium of relatively small molecules, it is regarded as probable (*loc. cit.*, p. 520) that the movement of solvent is the rate-determining process, and the

coefficient becomes  $D = \frac{kT}{a\pi a_1 \eta}$ , where  $a$  is a factor in the neighbourhood of unity. These two expressions are to be compared with the Stokes equation for "macroscopic" diffusing particles

$$D \propto \frac{1}{3\pi a_1 \eta}. \text{ On this basis it is to be expected that in the case of a sugar or glycerol molecule}$$

in aqueous solution the rate of diffusion would become relatively greater as the concentration increases, quite apart from the viscosity factor. For the present purpose we may put

$$D \propto \frac{T}{a_1 \eta} \text{ and include in the diameter any variation in rate with the relative sizes of diffusing and solvent molecules.}$$

An elementary strip is then considered as before (fig. 1). The solute molecules are represented as of mass  $M_1$ , diffusional diameter  $a_1$ , molecular volume  $V_1$  at concentration  $X_1$ , with corresponding symbols,  $M_2$ , etc. for the solvent. The number of molecules of type  $M_1$  passing from the  $T$  region through the  $T + \Delta T$  region is then proportional to  $(\Delta X_1)' = \frac{X_1}{a_1} \frac{T}{\eta_{T+\Delta T}}$ , corresponding to a molecule at temperature  $T$  passing through a region of viscosity  $\eta_{T+\Delta T}$ . This flow of  $M_1$  molecules must be balanced by a reverse flow of both solute and solvent molecules because of the constant pressure conditions. The volume of  $M_1$  molecules entering the  $T + \Delta T$  region is  $(\Delta X_1)' V_1$ , and since these eventually attain the  $T + \Delta T$  equilibrium the volume increase will be  $(\Delta X_1)' V_1 \rho_1 / \rho'_1$ , where  $\rho_1$  is the density of  $M_1$  molecules at  $T$ , and  $\rho'_1$  is the corresponding density at  $T + \Delta T$ . This volume of solution is then to be returned to the  $T$  region. Of this volume, the volume of  $M_1$  molecules present may be taken as

$$(\Delta X_1)' V_1 \frac{\rho_1}{\rho'_1} \frac{V_1 X_1}{(V_1 X_1 + V_2 X_2)},$$

and therefore the number of  $M_1$  molecules reaching the  $T$  region is

$$(\Delta X_1)' \frac{V_1}{V_1 \rho'_1} \frac{\rho_1}{(V_1 X_1 + V_2 X_2)} \frac{V_1 X_1}{\rho_1} = (\Delta X_1)' \frac{V_1 X_1}{(V_1 X_1 + V_2 X_2)}.$$

Similarly the number of  $M_2$  molecules returned in this process is

$$(\Delta X_1)' \frac{V_1}{V_2} \frac{V_2 X_2}{(V_1 X_1 + V_2 X_2)} \cdot \frac{\rho_1 \rho'_2}{\rho'_1 \rho_2}.$$

Expressions of similar type may then be obtained for an initial flow of  $M_2$  molecules, and the entire series of operations performed similarly for flow towards the  $T - \Delta T$  region. Examination of expansions of solutes and solvent over the relatively small temperature region involved shows that  $\rho_1 \rho_2' / \rho_1' \rho_2$  will not be far from unity, and we may put  $\rho_1 \rho_2' / \rho_1' \rho_2 = 1 + a$  with  $\rho_1' \rho_2 / \rho_1 \rho_2' = 1 - a$ ; further, if the expansions be taken as proportional to the temperature, so that  $\rho / \rho' = 1 + \alpha(\Delta T)$ ,  $a$  may be replaced by  $(\alpha_1 - \alpha_2)\Delta T$ , where  $\alpha_1$  and  $\alpha_2$  are the expansion coefficients per degree for solute and solvent respectively. By summation of the twelve different rates, the separation rate between  $M_1$  and  $M_2$  molecules in the direction  $(T - \Delta T) \rightarrow (T + \Delta T)$  is then found to be

$$\Delta S \propto A + B, \quad (8)$$

where

$$A = \frac{X_1 X_2}{(V_1 X_1 + V_2 X_2)} (V_1 + V_2) \left( \frac{1}{d_1} - \frac{1}{d_2} \right) (1/\eta_{T+\Delta T} - 1/\eta_{T-\Delta T})$$

and

$$B = (\alpha_1 - \alpha_2) \frac{X_1 X_2}{(V_1 X_1 + V_2 X_2)} (V_1/d_1 + V_2/d_2) (1/\eta_{T+\Delta T} + 1/\eta_{T-\Delta T}) \cdot \Delta T.$$

A proper appreciation of the factors of this expression can only be obtained by adequate integration, which in this case adds further complexities. Some idea of the relative importance of the A and B terms can, however, be obtained directly. The expansions of liquids over the temperature range concerned is generally small, and their differences are smaller still. We may, therefore, first consider the implications of the A term alone. The viscosity of a solution always decreases with rise in temperature; the sign of the separation will be determined by the difference of the diffusion diameter reciprocals. For  $d_1 > d_2$  the expression is negative; large molecules should therefore migrate to the low temperature region and, in the vertical cascade system, will tend to collect at the column foot. The separation will naturally be zero when  $X_1$  or  $X_2$  is zero, and a maximum will occur with changing concentration, the position of which, however, will be determined by viscosity as well as by the direct concentration values. If the difference between the diffusion diameters is small, the effect of the B term will become increasingly important.

If the diffusion diameter be taken as the cube root of the apparent molecular volume (see Table II below, p. 312), then diameters decrease in the order sucrose, glucose, glycerol, acetone, water. In all cases, then,  $(1/d_1 - 1/d_2)$  is negative and indicates concentration of solute at the column foot. In the case of acetone,  $(1/d_1 - 1/d_2)$  has the smallest numerical value, and, other things being equal, we would thus expect the B term to exert its greatest influence on the acetone-water system. Further, on the assumption that the expansions of solutes and solvent are respectively the same in solution as in the free state, the series of increasing expansions is water, sugar, glycerol, acetone. Here the value of the sugar expansion has been estimated from the expansion of concentrated solutions, since normally the expansion of a solution lies between that of solvent and solute. In all cases  $(\alpha_1 - \alpha_2)$  is positive; where this factor is largest and  $(1/d_1 - 1/d_2)$  is smallest, as in acetone-water, there will therefore be a decided tendency for  $\Delta S$  to become positive also, corresponding to solute concentration at the column top.

The effects of altered concentration and temperature are more complex. In the case of the former, increasing solute concentration in the sugar series means increased solution viscosity. The temperature change of viscosity for a constant temperature range is here greatest at high concentrations; in dilute solutions  $(1/\eta_{T+\Delta T} - 1/\eta_{T-\Delta T})$  in term A is small as compared with  $(1/\eta_{T+\Delta T} + 1/\eta_{T-\Delta T})$  in term B, and the effect of the B term will thus be greatest at low concentrations. Neglect of the B term will thus give ratios of experimental to calculated values which will increase as the concentration of solute increases. Other factors being equal, such a drift in ratios will be most pronounced for high  $V_1$  values in term B.

Before comparison between calculated and experimental values can be attempted, integration of equation (8) must be coupled with conversion from the elementary strip of fig. 1 to the cascade convection system. Both are complex, and in view of the many assumptions already made, integration of the A term alone of equation (8) was carried out.

With

$$\Delta S \propto T(1/\eta_{T+\Delta T} - 1/\eta_{T-\Delta T}) \quad \text{or} \quad dS \propto Td(1/\eta),$$

the variation of viscosity with temperature may be represented by  $\eta = Pe^{Q/T}$ . Then

$$S \propto (Q/P) \int e^{-Q/RT} T^{-1} dT.$$

This integration gives a convergent series which for aqueous solutions ( $Q = 2000$  cal. approx.) may be represented by the first term only: thus

$$S_{T_1}^T = k'(T_2/\eta_2 - T_1/\eta_1), \quad \text{where } k' \text{ does not include } Q \text{ or } P.$$

In the cascade convection system, the rate of relative movement of the two streams of liquid will be determined largely by the difference in density of the streams and by the viscosity at the mean temperature. The following expression is then obtained, increase in concentration at the column foot being given the positive sign:—

$$S = \frac{k(\rho_{T_1} - \rho_{T_2})}{\eta_T} \cdot \frac{X_1 X_2}{(V_1 X_1 + V_2 X_2)} (V_1 + V_2) \left( \frac{1}{d_2} - \frac{1}{d_1} \right) (T_2/\eta_2 - T_1/\eta_1). \quad (9)$$

Here  $T_2 - T_1$  is the applied temperature gradient,  $T$  is the mean temperature  $(T_2 + T_1)/2$ ,  $T'_2 = (T_2 + T)/2$  and  $T'_1 = (T + T_1)/2$ .

For comparison with this equation, the experimentally observed changes in concentration  $\Delta X_1$  require conversion to  $S_{\text{exp}}$  values. If  $X_1$  and  $X_2$  are initial concentrations in grm. mol. per litre, and  $X'_1$  and  $X'_2$  are the final concentrations after the period of thermal diffusion,

$$S_{\text{exp}} = (X'_1 - X'_2) - (X_1 - X_2) = (X'_1 - X_1) - (X'_2 - X_2) = \Delta X_1 - \Delta X_2.$$

But

$$V_1 X_1 + V_2 X_2 = V_1 X'_1 + V_2 X'_2 = 1000,$$

and hence

$$\Delta X_2 = -V_1(\Delta X_1)/V_2 \quad \text{and} \quad S_{\text{exp}} = (V_1 + V_2)(\Delta X_1)/V_2.$$

*Variation of Mean Temperature.*—When the temperature gradient is constant but the mean temperature is varied for a solution of constant concentration, relative rates should then be approximately expressed by

$$\Delta X_1 = k_1(T_2/\eta_2 - T_1/\eta_1)/\eta_T, \quad (10)$$

on the assumption that in the temperature range concerned the density of the solution may be considered as linearly related to the absolute temperature. The application of this equation is shown for 30 per cent. solutions of glucose, sucrose and glycerol by the continuous curves of fig. 3 of the previous paper (p. 301). For viscosity in millipoises, the values of the constants  $k_1$  were taken as 0.032 (glucose), 0.019 (sucrose) and 0.026 (glycerol). The agreement between theory and experiment is such that in each case the variation in mean temperature may be adequately represented by the above equation.

*Variation of Temperature Gradient.*—When the mean temperature is constant and the temperature gradient alone is varied, the assumption that density varies linearly with temperature gives

$$\Delta X_1 = k_2(T_2 - T_1)(T_2/\eta_2 - T_1/\eta_1)/\eta_T. \quad (11)$$

For the glucose solution in question (30 per cent. glucose)  $k_2$  was taken as  $5.5 \times 10^{-4}$ , with viscosity in millipoises. For the experimental conditions, satisfactory reproducibility of observed rates was more difficult than usual, this being connected with the manipulation of the heating and cooling streams, but, as shown in fig. 4 of the preceding paper (p. 302), the calculated continuous curves are again in fair agreement with experiment.

It may be here remarked that replacement of the factor  $(T_2/\eta_2 - T_1/\eta_1)$  by  $(1/\eta_2 - 1/\eta_1)$  causes little difference in calculated relative separation values. The general equation (8) may be put in the form

$$\Delta S \propto \left( \frac{bT}{\eta_{T+\Delta T}} - \frac{cT}{\eta_{T-\Delta T}} \right);$$

the fact that the variation of temperature may be represented by the A term alone of equation (8) does not require that the B factor is entirely negligible.

*The Sign of the Separation and the Effect of Concentration Change.*—For comparison with experiment, relative rates have been calculated by equation (9), omitting the proportionality constant, on the assumption that the diffusion diameter  $d$  is given by  $(M/\rho)^{\frac{1}{3}}$ . Since glucose and sucrose are solids, the apparent solute density in solution has been obtained from the expression  $V_1X_1 + V_2X_2 = 1000$  (where  $X_1$  and  $X_2$  are concentrations in grm. mol. per litre) on the assumption that the molecular volume of water remains at 18. The normal molecular weights were employed. The resulting values vary very slightly with the concentration of the solution used: for the present purposes 10 per cent. solutions were considered. Data are shown in Table II.

TABLE II

Substance	M	V	$d$	$\rho_{calc}$	$\rho_{obs}$	$(d_1 - d_2)$	$(1/d_2 - 1/d_1)$
Water	18	18	2.6	0.998	0.998	..	..
Acetone	58	65.6	4.0	0.885	0.79	1.4	0.135
Glycerol	92	70.2	4.1	1.311	1.26	1.5	0.141
Glucose	180	109	4.8	1.65	1.56 (solid)	2.2	0.177
Sucrose	342	206	4.9	1.66	1.59 (solid)	3.3	0.181

For comparison the ordinary values of the densities of the pure substances are given under  $\rho_{obs}$ ; for each solute the calculated density is appreciably greater, but was adopted as corresponding more closely to the conditions to which the general equation is considered to apply. The results of such calculations are given in Table III.

TABLE III.— $T_2 = 373$ ,  $T_1 = 293$ 

$X_1$	$X_2$	$X_1X_2$	$\eta_{80}$	$\left(\frac{T_2}{\eta_2} - \frac{T_1}{\eta_1}\right)$	$\rho_{40}$	$\rho_{80}$	$\rho_{40} - \rho_{80}$	$S_{calc}$	$S_{exp}$	$\frac{S_{exp}}{S_{calc}}$
<i>Glycerol</i>										
1.16	51.0	59.0	5.79	89.3	1.023	1.017	0.0060	0.0675	0.047	0.70
2.41	46.0	111	7.34	76.7	1.049	1.0425	0.0065	0.093	0.083	0.89
3.45	42.0	145	9.63	64.3	1.0695	1.062	0.0075	0.090	0.089	1.00
4.59	37.5	172	11.75	54.1	1.093	1.085	0.0080	0.0785	0.082	1.04
5.93	32.1	191	17.0	40.1	1.120	1.1105	0.0095	0.053	0.057	1.10
8.03	23.8	191	33.6	20.0	1.162	1.150	0.0120	0.0168	0.023	1.38
<i>Glucose</i>										
0.555	52.0	28.9	5.81	95.8	1.03052	1.00875	0.0218	0.234	0.033	0.141
1.11	48.6	54.0	7.36	71.6	1.06775	1.04544	0.0223	0.262	0.074	0.28
1.665	44.9	74.8	10.4	57.4	1.10201	1.07954	0.0225	0.210	0.099	0.47
2.22	41.5	92.1	13.8	41.6	1.14481	1.12211	0.0227	0.142	0.102	0.72
2.77	38.0	105.5	19.9	30.2	1.17792	1.15498	0.0229	0.082	0.087	1.05
<i>Sucrose</i>										
0.292	52.0	15.0	5.88	85.8	1.03050	1.00868	0.0218	0.229	0.034	0.15
0.614	48.1	29.6	7.88	67.2	1.07135	1.04909	0.0223	0.270	0.082	0.30
0.876	44.9	39.4	10.9	52.8	1.10465	1.08230	0.0224	0.262	0.091	0.34
1.170	41.4	48.4	15.7	41.1	1.14165	1.11910	0.0226	0.138	0.081	0.58
1.460	37.8	55.2	22.2	27.0	1.17800	1.15510	0.0231	0.075	0.065	0.88
<i>Acetone</i>										
2.74	45.6	125	12.0	29.2	0.9781	0.9676	0.0102	0.035	-0.043	-1.3
5.48	35.5	194	13.7	26.0	0.9527	0.9387	0.0140	0.058	-0.216	-3.7
8.23	24.9	205	12.2	29.4	0.9186	0.8990	0.0196	0.109	-0.412	-3.7
10.96	13.3	146	7.8	35.0	0.8672	0.8472	0.0200	0.148	-0.341	-2.3

In acetone, the mean temp. was taken as 30° C., with  $T_1 = 10^\circ$  C.,  $T_2 = 50^\circ$  C.; values of  $\rho$  are given for 18° and 38° C. All values of viscosities and densities were obtained by means of the International Critical Tables, with the exception of glucose solution densities, which were calculated on the basis that the increase in density for concentrations in grm. per 100 c.c. is the same for glucose as for sucrose.

As shown by the inconstancy of the ratios of the final columns, equation (9) does not rigidly account for the variation of separation rate with changing concentration, and it is therefore to be concluded that the B term of equation (8) cannot be here neglected. The sign of the separation in glycerol, glucose and sucrose is, however, correct, and the drift of the ratios is in agreement with the general expectations previously discussed. There is, indeed, some indication of an approach to the same ratio value at high concentrations. With these solutes also the calculated values show maxima which are not far removed from the experimental values. This is shown in fig. 3, where the discontinuous curves show the calculated values for glycerol and sucrose, the values of the proportionality constant  $k$  having been chosen in each case to make the maximum calculated and experimental rates equal. In these cases the calculated maxima occur at concentrations less than that observed experimentally; the converse is true for acetone.

As already indicated, the small value of  $(1/d_2 - 1/d_1)$  and the large coefficient of expansion of acetone tend to change the sign of the calculated separation. It is further probable that  $(1/d_2 - 1/d_1)$ , as already calculated for acetone-water, is subject to a considerable error, correction of which would favour the same tendency. The other solutes are all akin to water in the occurrence of hydroxyl groups, with consequent hydroxyl bonding in aqueous solution, and

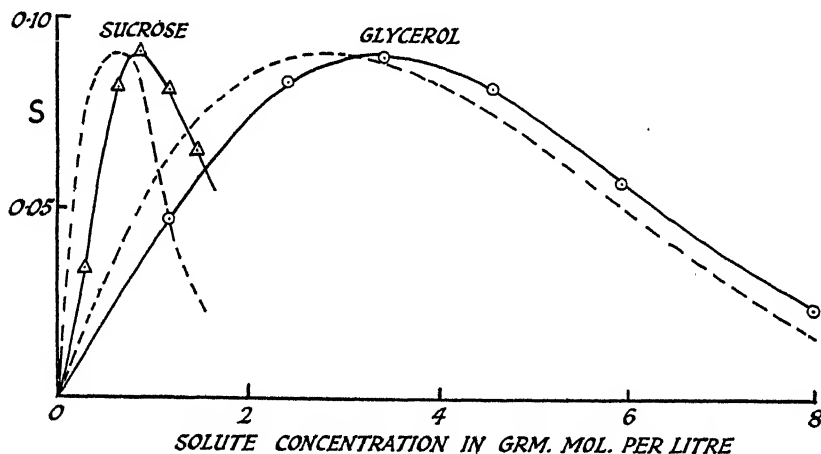


FIG. 3.

therefore their relative diffusion diameters as estimated from the molecular volume may not be far wrong. A relative diameter calculated in the same way for acetone, which must be taken as having little bonding with water, may be expected to be sensibly in error;  $1/d_2 - 1/d_1$  will be even smaller than the previously calculated value.

With changing concentration of acetone the viscosity factors do not alter widely; the relative constancy of the final ratios of the acetone table above is therefore not unexpected.

The experimental results of the preceding paper have shown that appreciable separation of sugar solutes in aqueous solution by thermal diffusion is not easily achieved. For solutes of  $d_1$  value much greater than that of water,  $1/d_1$  will be negligible, and examination of the A term of the general equation (9) shows that the rate  $\Delta X_1$  for a given  $X_1$  concentration will then be proportional to  $V_2$ , *i.e.* become constant, more particularly when viscosity and density values are similar for the different sugar solutions. In the same way the contribution to the separation  $\Delta X_1$  by the B term, already not predominant, will not be much altered by the altering  $V_1$  and  $d_1$  values; only a small change in relative rate is to be expected, this being again in agreement with experiment.

It is thus to be generally concluded that in thermal diffusion in a binary liquid mixture, the substance with the larger diffusion diameter will tend to collect in the cold region if the expansions of the two liquids are equal: if the diffusion diameters are equal, the liquid with the higher expansion will collect in the hot region.

Korsching and Wirtz (1940) carried out an extensive series of thermal diffusion experiments involving organic compounds, where the difference in diameters may be expected to be small. In

the systems for which density data are available, viz. water-methyl alcohol, water-ethyl alcohol, water-butyl alcohol, benzene-thiophene, benzene-cyclohexane, cyclohexane-carbon tetrachloride, cyclohexane-*n*-hexane, chlorobenzene-toluene, *n*-hexane-*n*-octane, *n*-hexane-carbon tetrachloride, all with one exception conform to the rule that the compound of higher expansion migrates to the top of the diffusion column. The exception is cyclohexane-carbon tetrachloride, where cyclohexane accumulated at the column top. Here, however, density data at 10° and 30° indicate only a very slightly higher expansion for the tetrachloride (1.0246 as against 1.0241); the expansions are so similar that the diameter factor is presumably predominant. Of considerable interest is the mixture benzene-cyclohexane; here the densities are appreciably different (0.889 and 0.788 respectively at 10°), but the expansions between 10° and 30° are again very similar (1.0245 and 1.0241). In this case Korsching and Wirtz recorded no separation.

The general conclusions here reached will naturally require modification if other factors become predominant, *e.g.* activity coefficients, osmotic pressure effects, change of effective diffusional diameter with changing concentration. It is further to be emphasised that the general theory and equations above have been built on the assumption of a diffusion equation, which can only be regarded as correct for molecules which are of large size as compared with the molecules of the medium in which they diffuse. For such "macroscopic" molecules, the  $\alpha$  value will be as determined from the apparent molecular volume. In a solution when the diffusing molecule is of the same dimensions as the solvent molecule, this is not necessarily true, because a certain free space must be envisaged as surrounding the more or less incompressible molecule. This free space, which is again connected with the density in that for incompressible molecules of approximately the same volume the reciprocal of the density is some measure of the free space, will be determined by the attraction between the molecules. The ordinary diffusion coefficient in solution is a complex function which involves not only the diameters and masses of diffusing and solvent molecules, but also the free space in the liquid; in as much as free space, density and expansion coefficients all depend on molecular attraction, it is to be concluded that internal pressures must be considered in the further study of thermal diffusion in solution.

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### SUMMARY

An elementary theory of thermal diffusion applicable to gaseous and liquid systems has been developed. This is based on the difference of diffusional characteristics of a molecule considered as diffusing through two different temperature regions, when the pressure is constant throughout.

For gaseous systems, the resultant expression is shown to be in general accordance with experimental variation of temperature, mass, and diameter factors, and is further developed to include isotopic separation, change of sign of separation with concentration, and general force law considerations.

A similar approach to thermal diffusion in solution, combined with the convection effect of a "cascade" system, gives an expression which is in general agreement with the results of experimental variation of mean temperature and temperature gradient for aqueous solutions of sucrose, glucose and glycerol. The simple expression does not account rigidly for the sign of separation or the effect of altered concentrations. These discrepancies are discussed in relation to the general formula; it is concluded that in addition to the diffusion diameters, the relative thermal expansions of solute and solvent are of importance in this connection.

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## XXXIII.—Applications of Elliptic Functions to Wind Tunnel Interference.

By L. M. Milne-Thomson

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It is well known that the problem of wind tunnel interference with an aerofoil is reducible to the two-dimensional problem of determining the additional upwash velocity  $v$  due to the presence of the tunnel boundaries. The local corrections to incidence and to drag coefficient are then  $\epsilon$  and  $\epsilon C_L$  where  $\epsilon = v/V$ ,  $V$  being the wind speed and  $C_L$  the lift coefficient (Glauert, 1930; Pistolesi, 1932).

Consider a wind tunnel whose cross-section is a given curve in the  $z$ -plane. Let the region interior to this curve be mapped on the region interior to a circle of unit radius in the  $Z$ -plane by the mapping function  $z=f(Z)$ . With vortices of strengths  $k$  at  $z_1$  and  $-k$  at  $z_2$  there will correspond vortices  $k$  at  $Z_1$  and  $-k$  at  $Z_2$ , and the complex potential  $w$  in the  $Z$ -plane is therefore (Milne-Thomson, 1938)

$$w = ik \log (Z - Z_1) - ik \log (Z - Z_2) \mp (ik \log (Z - 1/\bar{Z}_1) - ik \log (Z - 1/\bar{Z}_2)). \quad (1)$$

Here the upper sign is taken for a closed working section for which the stream function is constant on the circle, and the lower sign for an open working section for which the velocity potential is constant on the circle. The bar denotes the conjugate complex quantity.

If  $w_F$  denotes the complex potential when the tunnel boundary is absent, we can write  $w = w_F + w_I$ , so that  $w_I$  denotes the interference potential. Since

$$w_F = ik \log (z - z_1) - ik \log (z - z_2),$$

we have the interference velocity given by

$$\left. \begin{aligned} -u + iv &= \frac{dw_I}{dz} = \frac{dw}{dZ} \frac{dZ}{dz} - \frac{dw_F}{dz} \\ &= \frac{ik}{f'(Z)} \left( \left( \frac{1}{Z - Z_1} - \frac{1}{Z - Z_2} \right) \mp \left( \frac{1}{Z - 1/\bar{Z}_1} - \frac{1}{Z - 1/\bar{Z}_2} \right) \right) \\ &\quad - ik \left( \frac{1}{z - z_1} - \frac{1}{z - z_2} \right). \end{aligned} \right\} \quad (2)$$

This gives the interference velocity at the point  $z$  due to the vortex pair  $k$  at  $z_1$ ,  $-k$  at  $z_2$ . The interference due to a continuous distribution of vorticity along the span can be obtained by integration.

For simplicity we shall consider only those cases in which the span  $B$  of the aerofoil lies on the real axis in the  $z$ -plane with its centre at the origin, and in which the points of the span map into points of the real axis in the  $Z$ -plane, the two origins corresponding. We can then write

$$z_1 = \frac{1}{2}x, \quad z_2 = -\frac{1}{2}x, \quad Z_1 = \frac{1}{2}X, \quad Z_2 = -\frac{1}{2}X \quad (3)$$

all real quantities. For elliptic loading we put  $4\pi k = -Gd\sqrt{(1-x^2/B^2)}$  and integrate across the span from  $x=0$  to  $x=B$ ,  $G$  being the circulation round the centre section of the aerofoil. The last term of (2) will contribute the downwash due to elliptic loading, namely  $iG/2B$ , so that (2) now gives, since  $u=0$ ,

$$\frac{4\pi v}{G} - \frac{2\pi}{B} = \frac{1}{f'(Z)} \int_0^B \left( \left( \frac{1}{Z - X/2} - \frac{1}{Z + X/2} \right) \mp \left( \frac{1}{Z - 2/X} - \frac{1}{Z + 2/X} \right) \right) \frac{xdx}{B^2\sqrt{(1-x^2/B^2)}}. \quad (4)$$

Note that the variables are  $X$  and  $x$  where  $\frac{1}{2}x = f(\frac{1}{2}X)$ .

As a particular application of the general formula (4) consider the elliptical cross-section of major axis  $2a$  and minor axis  $2b$ .

The interior of the ellipse is mapped on the interior of the unit circle by

$$z = c \sin \phi t, \quad Z = m^{\frac{1}{2}} \operatorname{sn} t, \quad c^2 = a^2 - b^2, \quad \phi = \pi/2K, \quad a = c \cosh \frac{1}{2}\phi K', \quad b = c \sinh \frac{1}{2}\phi K', \quad (5)$$

where  $m$  is the squared modulus and  $K, K'$  are the real and imaginary quarter periods of the elliptic functions. To see this, observe that the elliptic functions  $\operatorname{sn}(\frac{1}{2}iK' + t)$ ,  $\operatorname{sn}(-\frac{1}{2}iK' + t)$  are such that a simple pole of one is a simple zero of the other, and therefore their product is an elliptic function devoid of poles and is consequently a constant, in fact,

$$\operatorname{sn}(\frac{1}{2}iK' + t) \operatorname{sn}(-\frac{1}{2}iK' + t) = m^{-\frac{1}{2}}. \quad (6)$$

Thus if  $t$  is real, the two functions are conjugate imaginaries, and therefore

$$|\operatorname{sn}(\frac{1}{2}iK' + t)| = m^{-\frac{1}{2}}.$$

Thus as  $t$  describes the straight line from  $\frac{1}{2}iK' + K$  to  $\frac{1}{2}iK' - K$  in the  $t$ -plane, we see from (5) that  $Z$  describes a semicircle of unit radius in the  $Z$ -plane, and that  $z$  describes a semi-ellipse in the  $z$ -plane, whence the correspondence is easily established (Neville, 1944).

Thus with the span of the aerofoil along and centred on the major axis we may write:

$$\left. \begin{aligned} \frac{1}{2}x &= c \sin \phi t = \frac{1}{2}B \sin \phi r, & \frac{1}{2}X &= m^{\frac{1}{2}} \operatorname{sn} t, \\ z &= c \sin \phi T, & Z &= m^{\frac{1}{2}} \operatorname{sn} T. \end{aligned} \right\} \quad (7)$$

Then (4) gives

$$\frac{2Bv}{G} - 1 = \frac{B \operatorname{cn} T \operatorname{dn} T}{2\pi c \cos \phi T} \int_0^K (f_1(t) \mp f_2(t)) \sin \phi r dr, \quad (8)$$

where

$$f_1(t) = \frac{1}{\operatorname{sn} T - \operatorname{sn} t} - \frac{1}{\operatorname{sn} T + \operatorname{sn} t}, \quad f_2(t) = f_1(t + iK'), \quad (9)$$

the last relation being obtained by the use of (6).

When the span of the aerofoil coincides with the distance between the foci, (8) can be integrated in compact form. In this case  $B = 2c$  and  $r = t$ . If we integrate  $(f_1(t) \mp f_2(t)) \cos \phi t$  round the rectangle whose corners are  $\pm K \pm iK'$ , indented to exclude the poles  $\pm T \pm iK'$ , and observe that  $f_1(t \pm iK') = f_2(t)$ ,  $f_2(t \pm iK') = f_1(t)$ , we find that the right-hand member of (8) becomes

$$\frac{\pm 1 - \cosh \phi K'}{\sinh \phi K'} = -\frac{b}{a} \quad \text{or} \quad -\frac{a}{b}.$$

This shows that  $v$  is constant across the span. Also  $\pi BG = 2SVC_L$ , where  $S$  is the plan area of the aerofoil, and so if  $C = \pi ab$ , the area of the tunnel section, we get

$$e = \frac{SC_L}{4C} \frac{b}{a+b} \quad \text{or} \quad -\frac{SC_L}{4C} \frac{a}{a+b},$$

according as the working section is closed or open, results obtained by Glauert (Glauert, 1932) by a different method.

Rosenhead (Rosenhead, 1933) has discussed the elliptic and rectangular section with the aid of theta functions. With the present method the rectangular section is treated by using the mapping

$$Z = sc \frac{1}{2} \phi z \operatorname{dn} \frac{1}{2} \phi z, \\ \frac{K}{a} = \frac{K'}{b} = \frac{\phi}{2},$$

where  $a$  is the breadth and  $b$  is the height.

## SUMMARY

A general formula is obtained for the interference velocity when an aerofoil with elliptically distributed circulation is in a closed or open wind tunnel of any cross-section. The mapping of the section on the interior of a circle is given in terms of the Jacobian elliptic functions appropriate to the ellipse and rectangle. The result is worked out for an aerofoil which spans the focal distance in a tunnel whose section is an ellipse.

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XXXIV.—Foundations of Relativity: Parts I and II. By A. G. Walker, D.Sc.,  
 Department of Mathematics, University of Sheffield. *Communicated by*  
 Sir EDMUND WHITTAKER, F.R.S. (With One Text-figure.)

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PART I

I. INTRODUCTION

RELATIVITY is the study of matter in motion, and the basis of a theory of relativity can be either physical, mathematical, or logical. It is physical if some of the elementary objects and relations are concepts derived from the external world and if certain of their properties are assumed as physically obvious. If, however, the elementary objects, etc. are defined as mathematical symbols and relations, and if the subsequent theorems are mathematical deductions from these definitions, then the theory may be described as mathematical. Lastly, the basis of a theory is logical if certain terms are undefined—and clearly stated as such—and if the theory is then developed strictly deductively from an explicit set of axioms and definitions. Analogous examples taken from geometry are the Euclidean, algebraic, and projective theories. The first, as developed by Euclid, has a physical basis, while the second is mathematical, a point being defined as an ordered set of numbers (co-ordinates) and a line as the class of points satisfying a linear equation. The third is logical, the undefined elements being point and line (an undefined class of points) and the axioms being those of incidence, extension, etc. Usually a physical theory comes first, to be followed by a mathematical and then by a logical theory, this last being so constructed that it includes previous theories when its undefined elements are replaced on the one hand by the conceptual physical objects and on the other hand by the symbolic mathematical objects. The construction of such a logical theory is not merely a matter of academic interest, for it can be regarded as an analysis of the previous theories. It tests, for example, the consistency and independence of their basic assumptions and definitions. It also indicates how a theory can be modified, with as little change as possible, so as to include some feature previously excluded. This can be particularly useful in the case of a physical theory which has been constructed to correspond as closely as possible to the external world, for such a theory may need continual modification to keep in step with observational data. For this reason the axioms of a logical theory should be not only consistent and independent but also simple, *i.e.* indivisible.

When we examine existing theories of relativity from this point of view, we see that the Newtonian theory is physical and that General Relativity is mathematical. Kinematical Relativity, although more recent than General Relativity, is again physical, for it starts with the physical concepts of temporal order, particle, light signal, and collinearity of particles and with certain assumed relations between them, particularly between light signals and temporal order. This physical basis is, however, much smaller than in previous theories, and the task of constructing a corresponding logical theory † is considerably simplified by this fact and by the novel but powerful deductive arguments initiated by E. A. Milne and continued by G. J. Whitrow. One contribution which this theory makes in line with modern ideas is the demonstration of the fact that it is sufficient to consider only observable, or “knowable”, quantities, *i.e.* those given by definable experiments which may be either practical or ideal when applied to the external world. This idea can be used with advantage in a logical theory, as will be seen later.

† A paper by G. C. McVittie (1942), entitled “Axiomatic Treatment of Kinematical Relativity”, is not, as the name implies, the statement of a logical theory in our sense. An examination of the “axioms” shows it to be physico-mathematical. These axioms were subsequently criticised by E. A. Milne and by G. J. Whitrow.

In the present series of papers a theory of cosmology is developed deductively from a logical basis consisting of four undefined elements and relations (*instant*, *particle-class of instants*, *temporal order*, and *signal correspondence*) and a number of axioms. As in Kinematical Relativity, the present theory is expressed entirely in terms of observables. An additional feature, however, is the supposition that *there is only one "observer"*, and that *all axioms, definitions, and "experiments" must be expressed in terms of primitive observations made by this observer*. The axioms are classified as *temporal* (1), *signal* (4), and *fundamental* (4), this last number being relevant only to Part II since it will be increased in a later part. Part I is devoted to the study of temporal order, and the consequences of the temporal and signal axioms as applied to particles in general and to properly defined collinear sets of particles in particular. By the end of Part I, with the addition of an axiom which is discarded later as redundant, we are in possession of a body of theorems which corresponds exactly to the physical basis of Kinematical Relativity. It appears, therefore, that this basis, when analysed, requires six axioms to make it logically sound.

The temporal and signal axioms are also consistent with General Relativity except, in certain cases, axiom S.4. The real trouble arises from axiom S.1, which is satisfied without question in space-times with zero or negative spatial curvature, but tends to exclude the multiple reception of a light signal which is a feature of space-time with positive curvature. This difficulty can be overcome by allowing the axiom to refer only to the "first arrival", but then axiom S.4 would be false as it stands and would need drastic modification to make it satisfactory in all cases. There is no doubt that a logical system which is consistent with *all* space-times of General Relativity is far more complicated than that given here, and for the present, therefore, we prefer to examine the more economical system.

In Part II we postulate and examine certain *fundamental particles*, which correspond to the fundamental particles in Kinematical Relativity and to those particles in General Relativity which are at rest relative to the matter near them. We are now beginning to construct a framework, or *sub-stratum* to use Milne's term, for a model universe, and certain features which we wish this model to possess are contained in the fundamental axioms. Since it is at first necessary to examine collinear sets of fundamental particles, Part II is restricted to these and contains only those fundamental axioms which are relevant to such sets. A later paper will give the axioms in full and will discuss the three-dimensional sub-stratum as a whole. Some of the results obtained in Part II correspond to those obtained by Milne and Whitrow for what they call "linear equivalences", but in our derivation we avoid making several of their assumptions. We do not assume, for example, that all fundamental particles are equivalent in pairs, or that the functions which arise are differentiable.

One point which may be of interest is that we meet, for what appears to be the first time, a criticism levelled against all previous theories. Previously it has been assumed without question that temporal order can be adequately described by the continuum of real numbers, *i.e.* that the ordered set of instants in the history of a particle is *ordinally-linear*. This is, of course, open to criticism in view of the fact that there are known to exist ordered sets of much more complicated types. In the present theory we do not make this assumption but start with a quite general type of order. It is then proved that temporal order is *ordinally-linear* as a consequence of the fundamental axioms. The non-assumption of ordinal-linearity makes it necessary to introduce a functional calculus which can be applied to any ordered set; this is done in § 8 of Part I, and the notation is used extensively in Part II. Some knowledge of ordered sets in general is assumed in these papers (see, for example, Hobson, 1921, and Sierpinski, 1928).

I wish to acknowledge here the debt which I owe to Professor E. A. Milne, both directly and through his published work of the last twelve years, a debt far greater than is indicated by specific references to his work.

## 2. UNDEFINED BASIS

The fundamental undefined element is an *instant*, and certain undefined mutually exclusive classes of instants are *particles*. Particles are denoted by  $\bar{A}$ ,  $\bar{P}$ ,  $\bar{Q}$ ,  $\bar{T}$ , etc., and instants of a particle are denoted by suffixes attached to the particle symbol.

At first one particle,  $\bar{A}$ , is preferred, this being the *particle-observer*. Between any two instants of  $\bar{A}$  there is an undefined relation, denoted by the symbol  $<$ , so that for distinct instants  $\bar{A}_x, \bar{A}_y$ , either  $\bar{A}_x < \bar{A}_y$  or  $\bar{A}_y < \bar{A}_x$ . When  $\bar{A}_x < \bar{A}_y$  we say that  $\bar{A}_x$  is *before*  $\bar{A}_y$ ; when  $\bar{A}_y < \bar{A}_x$ , then we write  $\bar{A}_y > \bar{A}_x$  and say that  $\bar{A}_y$  is *after*  $\bar{A}_x$ . The identity relation is  $\bar{A}_x = \bar{A}_y$ , and the three relations  $=, <, >$  are mutually exclusive.

Another undefined relation is a correspondence between the instants of any two particles. This will be called a *signal correspondence*, and for particles  $\bar{P}, \bar{Q}$  will be written  $\bar{P} \wedge \bar{Q}$ , equivalent to  $\bar{Q} \vee \bar{P}$ . It is not a symmetric relation, for in general  $\bar{P} \wedge \bar{Q}$  and  $\bar{P} \vee \bar{Q}$  are not identical. The signal axioms (§ 4) relate signal correspondences to temporal order and give rise to the desired properties of "light signals".

One of the rules followed in this work is to construct all axioms so that they can be expressed in terms of  $\bar{A}$ 's instants. We find later that every other particle is similar in all respects to  $\bar{A}$ , so that  $\bar{A}$  finally loses its preferential position. It will be seen that all axioms and definitions are *primitive*, i.e. involve only simple statements of identity or order in  $\bar{A}$ 's instants.

It may be argued that we are not in agreement with experience in taking our undefined element of time to be an instant, and that this element should be a *duration*, to be pictured as an interval. This is certainly true, and we hope later to replace the instants, temporal relations and the temporal axiom of the present paper by still more fundamental ideas in closer agreement with experience. These will give rise to instants as defined elements, and, except for signal correspondences which will then refer to durations, the remainder of the present paper will be valid.

*Added in proof.* A satisfactory theory on these lines is given in Walker, 1947, *instants* being defined in terms of a partially ordered set of *durations*. The set of instants arising in this way is found to be closed, and the definition of ideal instants in § 3 below is therefore unnecessary.

### 3. TEMPORAL ORDER

**Axiom T. 1.**—If  $\bar{A}_x < \bar{A}_y$  and  $\bar{A}_y < \bar{A}_z$ , then  $\bar{A}_x < \bar{A}_z$ .

We say in this case that  $\bar{A}_y$  is *between*  $\bar{A}_x, \bar{A}_z$ , and also  $\bar{A}_x, \bar{A}_z$ .

By this axiom the instants of  $\bar{A}$  form an ordered set. It is well known that there are many ordered sets other than those similar† to sets of real numbers in natural order. It will, however, appear later, as a consequence of the F-axioms, that the set  $\bar{A}$  is similar to such a number set, so that it will be possible to construct a numerical representation of the set. Such a representation we call a *clock*. The theorem which makes this possible will be proved in § 13 of Part II, and will now be given as an axiom with the understanding that it will be discarded later as redundant.

**Axiom T. X.**—The set  $\bar{A}$  possesses a denumerable sub-set such that between any two instants of  $\bar{A}$  there is a member of the sub-set.

This axiom implies that the set  $\bar{A}$  is dense-in-itself, and is the well-known necessary and sufficient condition for  $\bar{A}$  to be similar to an everywhere-dense set of numbers in natural order.

Even with axiom T. X we see that the set  $\bar{A}$  is not necessarily *closed*, i.e. may not be such that every bounded progression taken from  $\bar{A}$  possesses a limit in  $\bar{A}$ . We shall not, however, adopt an axiom of closure, for if the set is not already closed, then it can be augmented in a definable way so that it becomes closed.

Let  $\Sigma$  be a bounded ascending progression of instants taken from  $\bar{A}$ , and place in a class  $\mathcal{A}$  each instant of  $\bar{A}$  which is before some member of  $\Sigma$ , and in a class  $\mathcal{B}$  each instant of  $\bar{A}$  which is after all members of  $\Sigma$ . Then  $\mathcal{A}$  has no last instant, and  $\mathcal{B}$  has a first instant when and only when  $\Sigma$  has a limit in  $\bar{A}$ . When  $\Sigma$  has no such limit, we define an *ideal instant* as the pair of classes  $(\mathcal{A}, \mathcal{B})$ . Other ideal instants are defined in the same way by ascending and descending progressions which do not possess limits in  $\bar{A}$ , with the understanding that all related progressions define the same instant. If  $\bar{A}$  has no first instant or no last instant, then other ideal instants are defined by  $(O, \bar{A})$  and  $(\bar{A}, O)$ , where  $O$  is the null class.

† The word "similar" applied to two ordered sets has the definite meaning that there is a one-one correspondence between their elements which preserves the order.

A set  $A'$  is now defined as consisting of all the ordinary instants of  $\bar{A}$  together with all the ideal instants arising out of  $\bar{A}$ . This new set is ordered as follows:—

- (i) Order in  $\bar{A}$  is preserved.
- (ii) The ideal instant  $(\mathcal{A}, \mathcal{B})$  is before or after  $\bar{A}_x$  according as  $\bar{A}_x$  belongs to  $\mathcal{B}$  or  $\mathcal{A}$ .
- (iii) The ideal instant  $(\mathcal{A}, \mathcal{B})$  is before the ideal instant  $(\mathcal{A}', \mathcal{B}')$  if  $\mathcal{B}$  and  $\mathcal{A}'$  have an instant in common.

It follows at once that axiom T.1 holds for all instants of  $A'$ .

We have now augmented  $\bar{A}$  to form a well-defined ordered set  $A'$  which is closed, every progression in  $A'$  having a limit in the set. Also  $A'$  has a first instant  $(A'_1)$  and a last instant  $(A'_2)$ . The set which remains when these extreme instants are excluded will be denoted by  $A$ . If axiom T.X is assumed, then it follows that  $A$  is similar to an open interval of the arithmetic continuum.

This augmentation is a sound logical process. It is also significant physically when our theory is compared with a physical theory, a particular event, such as a collision between particles, being perfectly well described in time by a succession of events leading up to the main event.

#### 4. SIGNAL AXIOMS

**Axiom S.1.**—*The signal correspondence between the ordinary instants of any two particles is one-one.*

From this axiom it follows that order can be induced in the set of instants comprising any particle  $\bar{P}$  by means of the correspondence  $\bar{A} \wedge \bar{P}$ . Axiom T.1 is satisfied by these ordered instants, and  $\bar{P}$  can be augmented by the addition of ideal instants, sets  $P', P$  being defined as for  $A', A$ . Since  $\bar{P}$  is similar to  $\bar{A}$ , it is easily seen to follow that  $P', P$  are similar to  $A', A$  respectively. The implied correspondence between ideal instants is unique because corresponding ideal instants are limits of corresponding progressions, and it follows that axiom S.1 now holds for ideal as well as ordinary instants.

If instants  $P'_x, Q'_x$  correspond in  $P' \wedge Q'$ , we write  $P'_x \wedge Q'_x$ .

**Axiom S.2.**—*If  $\bar{P}, \bar{Q}, \bar{R}$  are any three particles,  $\bar{P}_x$  is any instant of  $\bar{P}$ , and  $\bar{P}_y$  is given by the signal chain  $\bar{P}_x \wedge \bar{Q}_x \wedge \bar{R}_x \vee \bar{P}_y$ , then  $\bar{P}_y \geq \bar{P}_x$ .*

**Axiom S.3.**— *$\bar{P}, \bar{Q}$  are any two particles and  $\bar{P}_x, \bar{P}_y$  any two instants of  $\bar{P}$  such that  $\bar{P}_x < \bar{P}_y$ . If  $\bar{P}_x \wedge \bar{Q}_x$  and  $\bar{P}_y \wedge \bar{Q}_y$ , then  $\bar{Q}_x < \bar{Q}_y$ .*

There is an axiom S.4, but this will be left until § 6. It should be noted that although for convenience we do not now refer back to  $\bar{A}$ , yet it is always possible to do so. Thus axiom S.2 could be written

$$\bar{A}_x \wedge \bar{P}_x \wedge \bar{Q}_x \wedge \bar{R}_x \vee \bar{P}_y \vee \bar{A}_y \supset \bar{A}_y \geq \bar{A}_x.$$

Strictly, there are other axioms covering the cases when two or more of the particles in the above axioms are identical. It is convenient, however, to cover these cases by allowing particles to be identical, with the convention that if  $\bar{Q}, \bar{R}$  are identical and  $\bar{Q}_x = \bar{R}_x$ , then  $\bar{Q}_x \wedge \bar{R}_x$  and  $\bar{Q}_x \vee \bar{R}_x$ . It follows from axiom S.2 that if  $\bar{P}_x \wedge \bar{Q}_x$ , then  $\bar{P}_x \wedge \bar{R}_x$ , and that if  $\bar{P}_x \vee \bar{Q}_x$ , then  $\bar{P}_x \vee \bar{R}_x$ , from which we see, as we would expect, that  $\bar{Q}$  and  $\bar{R}$  are always interchangeable.

From the definition of ideal instants and the extension of the signal correspondence to include them, it can easily be verified that axioms S.2 and S.3 hold for the augmented particle sets.

It will be remembered that the instants of any  $P'$  are ordered in accordance with the correspondence  $A' \wedge P'$ , and the extension of axiom S.3 to  $P', Q'$  now shows that every signal correspondence preserves order. We could now say that  $\bar{A}$  has lost its preferential position. In the language of relativity,  $\bar{A}$  is still the only "real" particle-observer, but to every other particle has been attached a "subordinate observer", and the structure is unaltered if we take one of these observers to be "real" instead of  $\bar{A}$ .

## 5. SIGNAL THEOREMS

The extreme instants of the augmented particle sets are special, and we are more often concerned with interior instants, *i.e.* those of the sets  $A, P, Q$ , etc. That these never correspond to extreme instants is clear from the following theorem, which is an immediate consequence of the extended signal axioms.

**Theorem 5.1.**—*If  $P'_F, P'_L, Q'_F, Q'_L$  are the first and last instants of  $P', Q'$  respectively, then  $P'_F \wedge Q'_F, Q'_F \wedge P'_L, P'_L \wedge Q'_L, Q'_L \wedge P'_F$ .*

From a continued application of axioms S.2 and S.3 we get†

**Theorem 5.2.**— *$P, Q, R, \dots, T$  are any particles, and  $P_x$  any instant of  $P$ . If  $P_x \wedge Q_x \wedge R_x \dots \wedge T_x$  and  $P_x \wedge T_y$ , then  $T_y \leq T_x$ .*

**Corollary.**—*If  $P_x \wedge Q_x \wedge P_y$ , then  $P_x \leq P_y$  (cf. theorem 5.5).*

**Definition.**—Particles  $P, Q$  coincide at instants  $P_x, Q_x$  if  $P_x \wedge Q_x$  and  $P_x \vee Q_x$ . This is written  $P_x \times Q_x$ .

From theorem 5.1, all particles coincide at their first instants and again at their last instants.

To prove that coincidence is transitive, suppose that  $P_x \times Q_x$  and  $P_x \times R_x$ , and that  $Q_x \wedge R_y$ . Then from theorem 5.2,

$$P_x \wedge Q_x \wedge R_y \quad \text{and} \quad P_x \wedge R_x \supset R_x \leq R_y,$$

and

$$Q_x \wedge P_x \wedge R_x \quad \text{and} \quad Q_x \wedge R_y \supset R_y \leq R_x.$$

Hence  $R_x = R_y$  and  $Q_x \wedge R_x$ . Similarly  $R_x \wedge Q_x$ , and we have:

**Theorem 5.3.**—*If  $P_x \times Q_x$  and  $P_x \times R_x$ , then  $Q_x \times R_x$ .*

We can also prove the following:—

**Theorem 5.4.**—*If  $P_x \times Q_x$  and  $R_x \wedge P_x \wedge R_y$ , then  $R_x \wedge Q_x \wedge R_y$ . For suppose that  $R_x \wedge Q_x \wedge R_y$ . Then  $R_x \wedge P_x \wedge Q_x \vee R_x$  and  $R_x \wedge Q_x \wedge P_x \vee R_x$ , and from the extended axiom S.2,  $R_x \leq R_x$  and  $R_x \leq R_x$ . Hence  $R_x = R_x$ , and similarly  $R_y = R_y$ .*

From the definition of coincidence and the corollary to theorem 5.2, we have immediately,

**Theorem 5.5.**—*If  $P$  does not coincide with  $Q$  at  $P_x$  and if  $P_x \wedge Q_x \wedge P_y$ , then  $P_x < P_y$ .*

We shall say that particles  $P, Q$  are *distinct* in an interval (of  $P$  or  $Q$ ) if there is an instant in the interval at which  $P, Q$  do not coincide. It will be shown later that if  $P$  does not coincide with  $Q$  at  $P_x$ , then this instant lies in an interval of  $P$  at each instant of which  $P$  does not coincide with  $Q$ .

It is convenient to define an order relation between certain pairs of instants not necessarily belonging to the same particle.

**Definition.**— $P_x < Q_y$  if  $P_x \wedge Q_x \leq Q_y$  provided, in the case of the equality,  $P$  is not coincident with  $Q$  at  $P_x$ . An equivalent statement is  $P_x \leq P_y \wedge Q_y$ , with the same proviso. If  $P_x < Q_y$ , then  $Q_y > P_x$ .

The following are easily deduced from previous theorems:—

**Theorem 5.6.**—(i) *If  $P_x < Q_y$  and  $Q_y < R_z$ , then  $P_x < R_z$ .*

(ii) *If  $P_x < Q_y$  and  $Q_y \times R_z$ , then  $P_x < R_z$ .*

(iii) *If  $P_x \times Q_y$  and  $Q_y < R_z$ , then  $P_x < R_z$ .*

These theorems also hold when two of the three particles are identical.

## 6. OPTICAL LINES

**Definition.**—A number of instants are in *optical line* if they belong one to each of a number of particles and if every instant is related by a signal correspondence to every other instant. Thus if  $P_x \wedge Q_x, Q_x \wedge R_x$  and  $P_x \wedge R_x$ , then  $P_x, Q_x, R_x$  are in optical line.

This corresponds physically to the primitive observation of particles "in line of sight".

† Space-time diagrams, such as those described systematically in § 9, can be conveniently used to suggest formal proofs of this and later theorems.



From theorem 5.4 we see that if  $P_x$  is in an optical line and if  $P_x \times Q_x$ , then  $Q_x$  is in the same optical line. From the definition and properties of general order given in § 5, it follows that *the instants of an optical line are ordered except for coincidences.*

**Axiom S.4.**—*If  $P$  does not coincide with  $Q$  at  $P_x$ , and if  $P_x Q_x R_x$  and  $P_x Q_x S_x$  are optical lines, then one at least of  $P_x R_x S_x$  and  $Q_x R_x S_x$  is an optical line.*

By considering all the various possible orders involved in this axiom, it can be deduced that both  $P_x R_x S_x$  and  $Q_x R_x S_x$  are optical lines, and that  $P_x, Q_x, R_x, S_x$  are in the same optical line in some order. Extending to any number of instants we have:

**Theorem 6.1.**—*All instants collinear optically with two non-coincident instants are in one optical line. This line is given in the same way by any two non-coincident instants of the set.*

## 7. COLLINEAR PARTICLES

Having defined optical lines, we can now define a linear set of particles, the idea being that such particles are at all times "traversed" by optical lines in both "directions". Also we wish to admit the possibility that although the particles are ordered along each optical line, this order can change from one such line to another, i.e. "in time".

**Definition.**—A set of particles, which includes  $T$ , is *linear* if through each instant of  $T$  there are two distinct optical lines each of which contains an instant from every particle of the set. It is clear from previous theorems that  $T$  can be any particle of the set.

From theorem 6.1 we have immediately:

**Theorem 7.1.**—*If particles  $P, Q$  are distinct in every interval of  $P$  (and  $Q$ ), then all particles collinear with  $P, Q$  form a linear set which includes  $P$  and  $Q$ . The set is determined in the same way by any two distinct particles belonging to it.*

It is clear that order in relation to particles of a linear set is one of "betweenness" at an instant. The essential relation, resulting from order along optical lines, follows from the fact that:

**Theorem 7.2.**—*If  $P, T, Q$  are collinear, and  $T_x$  is between instants of  $P, Q$  on one optical line, then  $T_x$  is between instants of  $P, Q$  on the other optical line through  $T_x$ . In this case we say that  $T$  is between  $P, Q$  at the instant  $T_x$ .*

Extending to any number of particles, we have:

**Theorem 7.3.**—*For each instant  $T_x$  of a particle  $T$  belonging to a linear set, every particle of the set can be placed in one of three classes  $\mathcal{C}, \mathcal{R}, \mathcal{L}$ , those in  $\mathcal{C}$  being coincident with  $T$  at  $T_x$  and the others being such that  $T$  at  $T_x$  is between any member of  $\mathcal{R}$  and any member of  $\mathcal{L}$  but not between any two members of  $\mathcal{R}$  or of  $\mathcal{L}$ .*

The classes  $\mathcal{R}$  and  $\mathcal{L}$  are the two sides of  $T$  at the instant  $T_x$ . We shall now state a theorem which will be proved in § 8.

**Theorem 7.4.**—*If two particles are on the same side (opposite sides) of  $T$  at one instant and on opposite sides (same side) of  $T$  at a later instant, then one of them coincides with  $T$  at some intermediate instant.*

From this theorem we see that in a linear set which includes  $T$ , two particles which are on the same side or on opposite sides of  $T$  at  $T_x$  remain so while  $T_x$  advances, until one of the particles coincides with  $T$ . Each side of  $T$  thus preserves its identity throughout an interval of  $T$  which contains no instant at which all the particles of the set coincide. In each such interval the two sides of  $T$  can conveniently be described as *right* and *left* respectively, the choice being arbitrary at one instant.

Through each  $T_x$  there are two optical lines traversing the set. We shall describe as the *right optical line* that one which contains instants such as  $P_x$ , where  $T_x \wedge P_x$  if  $P$  is to the right of  $T$  at  $T_x$  and  $T_x \vee P_x$  if  $P$  is to the left of  $T$  at  $T_x$ . The other is the *left optical line*.

Having defined the two sides of  $T$ , we can now define the right and left sides of any other particle of the linear set. One way of doing this is to say that  $Q$  is to the right of  $P$  at  $P_x$  (and  $Q_x$ ) if  $P_x, Q_x$  are on the same right optical line and  $P_x \wedge Q_x$ . Similarly for particles to the left of  $P$ .

From theorem 7.4 it follows that if no two particles of a linear set coincide, then the set is *permanently ordered*, the two sides of each particle remaining unchanged at all instants.

## 8. RECORD FUNCTIONS

Suppose that particles  $T$ ,  $P$  are linked by a chain of signal correspondences, and that instants  $T_x$ ,  $P_x$  correspond by this chain. Then it is clear from the signal axioms that  $P_x$  advances continuously (in the general sense applicable to ordered sets) as  $T_x$  advances over the set  $T$ . Chains of particular interest are those which start and finish at the same particle, say  $T$ . If such a chain starts at  $T_x$  and finishes at  $T_y$ , then we can conveniently use functional notation and write  $T_y = f(T_x)$ . This  $f$  is a symbol denoting a certain kind of correspondence between the instants of  $T$ , and will be called a *record function*. As mentioned above, all record functions are continuous and monotonic increasing. Examples of chains defining record functions are  $T \wedge P \wedge T$ ,  $T \vee P \vee T$ ,  $T \wedge P \wedge Q \vee T$ ,  $T \wedge P \vee Q \wedge T$ .

If  $f$ ,  $g$  are record functions and  $T_y = f(T_x)$ ,  $T_z = g(T_y)$ , then we write  $T_z = gf(T_x)$ , and call  $gf$  the *functional product*, or simply product, of  $f$  and  $g$ . Similarly for any number of functions. Also, if  $T_y = f(T_x)$ , then we write  $T_x = f^{-1}(T_y)$  and call  $f^{-1}$  the *inverse* of  $f$ . It follows at once from our definitions that all products and inverses of record functions are themselves record functions.

Functional powers  $f^n$ ,  $f^{-n}$  of  $f$  are defined in obvious ways and satisfy

$$f^p f^q = f^{p+q}, \quad (f^p)^q = f^{pq}$$

for all positive and negative integers  $p$ ,  $q$ ,  $f^0$  being the *identity function*. This identity function will be written  $I$ , so that  $I(T_x) = T_x$ .

Referring to  $T$  as in the above definitions, the simplest record function associated with another particle  $P$  is given by  $T \wedge P \wedge T$ ; this function will be denoted by  $P$ .

For any instant  $T_0$  at which  $T$  does not coincide with  $P$ , an ascending progression  $T_n$  and a descending progression  $T_{-n}$  are defined by  $T_p = P^p(T_0)$  for all integers  $p$ . If  $T_\omega$ ,  $T_{-\omega}$  are their respective limits (possibly  $T'_\omega$  and  $T'_\omega$ ), we can prove

**Theorem 8.1.**— *$T$  coincides with  $P$  at  $T_{-\omega}$  and again at  $T_\omega$  but at no intermediate instant.*

For if  $T_0 < T_x < T_1$ , then since  $P$  is an increasing function we have  $P(T_x) > P(T_0) = T_1$ . This shows that if  $T_x \wedge P_x$ , then  $P_x \wedge T_x$  is impossible, and  $T$  cannot coincide with  $P$  in the interval  $T_0 T_1$ , or at  $T_1$ . Similarly for each interval  $T_p T_{p+1}$ , so that finally  $T$  cannot coincide with  $P$  at any instant between  $T_{-\omega}$  and  $T_\omega$ . Now if  $T_n \wedge P_n$ , then  $P_n \wedge T_{n+1}$ , and in the limit we have  $T_\omega \times P_\omega$ , where  $P_\omega$  is the limit of  $P_n$ . Similarly  $T_{-\omega} \times P_{-\omega}$ , where  $P_{-\omega}$  is the limit of  $P_{-n}$ . The theorem is thus proved. It shows that *any instant at which  $T$  does not coincide with  $P$  lies within an interval of  $T$  at every instant of which  $T$  does not coincide with  $P$ .*

We are now in a position to prove theorem 7.4. Consider collinear particles  $T$ ,  $P$ ,  $Q$ , and referring to  $T$ , let  $f$  denote the function such that  $f(T_x)$  is the earlier of  $P(T_x)$ ,  $Q(T_x)$ , and so for each  $T_x$ . Then it is clear that  $f$  is continuous and monotonic increasing, and that if  $f(T_x) = T_x$ , then  $T$  coincides with either  $P$  or  $Q$  or both at  $T_x$ .

**Lemma.**—*If  $P$ ,  $Q$  are on the same side of  $T$  at  $T_0$ , then they are on the same side of  $T$  at every instant between and including  $T_0, f(T_0)$ .*

To prove this lemma, we can assume without loss of generality that  $P$  is between  $T$ ,  $Q$ , or coincides with  $Q$ , at  $P_0$  where  $T_0 \wedge P_0$ , and define other instants by  $P_0 \wedge Q_0$  ( $T_0 \wedge Q_0$ ),  $Q_1 \wedge P_0 \wedge T_1$  ( $Q_1 \wedge T_1$ ), and  $Q_0 \wedge T_2$ . Then from theorem 5.2,

$$P_0 \wedge Q_0 \wedge T_2 \quad \text{and} \quad P_0 \wedge T_1 \supset T_1 < T_2,$$

so that  $f(T_0) = T_1$ , the earlier of  $T_1$ ,  $T_2$ . Let  $T_x$  be any instant between  $T_0$ ,  $T_1$  or at  $T_1$ , and suppose that  $T$  is between  $P$ ,  $Q$  at  $T_x$ . Writing  $T_x \wedge P_x$ ,  $T_x \vee Q_x$  ( $Q_x \wedge P_x$ ), then from axiom S. 3,

$$\begin{aligned} T_0 \wedge P_0, \quad T_x \wedge P_x \quad \text{and} \quad T_0 < T_x \supset P_0 < P_x \\ Q_1 \wedge P_0, \quad Q_x \wedge P_x \quad \text{and} \quad P_0 < P_x \supset Q_1 < Q_x, \end{aligned}$$

and

$$Q_x \wedge T_x, \quad Q_1 \wedge T_1 \quad \text{and} \quad T_x < T_1 \supset Q_x < Q_1,$$

so that our supposition leads to a contradiction. The lemma is thus proved.

Returning now to the theorem, suppose that  $P$ ,  $Q$  are on the same side of  $T$  at  $T_0$ , and write  $T_n = f^n(T_0)$ . Applying the lemma to successive intervals  $T_n T_{n+1}$ , it follows that  $P$ ,  $Q$

are on the same side of  $T$  at every instant between  $T_0$  and  $T_\omega$ , where  $T_\omega$  is the limit of  $T_n$ . Since  $f$  is continuous,  $f(T_\omega) = T_\omega$ , so that  $T$  coincides with  $P$  or  $Q$  at  $T_\omega$ . If, therefore,  $P$ ,  $Q$  are on opposite sides of  $T$  at some instant after  $T_0$ , then this instant is also after  $T_\omega$ , and the theorem is proved. There is a similar argument for the alternative form of the theorem.

In the case of a linear set of particles which includes  $T$ , a *modified record function*, associated with  $P$  and denoted by  $P^*$ , is defined by  $P^*(T_x) = P(T_x)$ ,  $T_x$ , or  $P^{-1}(T_x)$  according as  $P$  is to the right of, coincides with, or is to the left of  $T$  at  $T_x$ . In many ways this function is more important than direct record functions, for  $P^*$  not only describes, in a sense, the "apartness" of  $T$  and  $P$  but it also indicates their relative position, since  $P^*(T_x) \geq T_x$  according as  $P$  is to the right or left of  $T$ .

**Theorem 8.2.**—*In a linear set of which  $T$  is a fixed and  $P$  a variable member, let  $T_0$  be any instant of  $T$  and let  $P_0$  be the instant of  $P$  which belongs to the right optical line through  $T_0$ . Then the order of the instants  $P_0$  along this optical line is similar to the order of the instants  $P^*(T_0)$  in  $T$ .*

To prove this, it is clearly true for particles  $P$ ,  $Q$  on opposite sides of  $T$  at  $T_0$ . For particles to the right of  $T$  at  $T_0$ , suppose that  $P$  is between  $T$  and  $Q$  on the right optical line through  $T_0$ . Then the relation  $\dagger T_1 < T_2$  in the proof of the above lemma is now equivalent to  $P^*(T_0) < Q^*(T_0)$ , as required. Similarly for particles to the left of  $T$ . Finally, it is clear that if  $P$ ,  $Q$  coincide on the right optical line through  $T_0$ , then  $P^*(T_0) = Q^*(T_0)$ . This completes the proof.

We deduce immediately:

**Theorem 8.3.**—*If  $T$ ,  $P$ ,  $Q$  are collinear and  $P^*(T_0) = Q^*(T_0)$ , then  $P$  coincides with  $Q$  at the instant of  $P$  which is on the right optical line through  $T_0$ .*

The idea of denseness can be applied to a linear set of particles, either in relation to itself or in relation to temporal order by means of theorem 8.2. The latter is the more significant, and we state the following definition:—

**Definition.**—A linear set of particles which includes  $T$  is *dense at  $T_0$*  if for any instant  $T_x > T_0$  there is a particle  $P$  of the set such that  $T_0 < P(T_0) < T_x$ .

## 9. CONCLUSION OF PART I

We have so far considered the general consequences of certain concepts and of the five temporal and signal axioms. No geometrical (space) axioms have been introduced, and spatial relationships have been expressed in terms of temporal order and signal correspondences. In particular we have defined and discussed linear sets of particles, and there has emerged the idea of spatial order in such a set, an order which can vary "in time". One important result is that the two "sides" of a particle  $T$  are well defined at each instant of  $T$  and persist in time until all the particles of the linear set coincide; also, if another particle crosses from one side of  $T$  to the other, then it must coincide with  $T$  at some instant. From this and other properties we derive the picture, familiar in classical kinematics, of particles moving in the same spatial line. The only differences are that our time and space are not necessarily ordinally-linear (*i.e.* similar to the arithmetic continuum), axiom T.X being necessary for this, and that no distance measure has yet been defined for the "interval" between two particles. These differences will be removed later.

From our results it follows that we can construct a convenient space-time diagram for a linear set of particles. Referring to rectangular axes in a plane, then the instants of a particle are represented by the points of a continuous smooth curve whose tangent always makes an angle of less than  $45^\circ$  with the  $y$ -axis, temporal advance being given by  $y$  increasing. A coincidence of two particles is denoted by an intersection of the corresponding curves. Right and left optical lines are represented by straight lines parallel to  $y = x$  and  $y = -x$  respectively. Strictly, this diagrammatic representation implies axiom T.X, but with care it can be used without assuming this axiom. Examples of the use of this diagram occur in Part II.

If axiom T.X is assumed, then, as stated in the Introduction, we have reached precisely the starting-point for previous relativity theories, particularly Kinematical Relativity. The

$\dagger$  The equality  $T_1 = T_2$  is now impossible because  $P_0$ ,  $Q_0$ ,  $T_1$  cannot be in optical line.

physical properties of temporal order, light signals, collinearity, etc. which were previously assumed as physically obvious have now been deduced as theorems, and it follows that these assumptions are logically equivalent to six axioms. This number does not, of course, include the explicit principles and hypotheses, such as the Cosmological Principle, upon which these others theories are based. In our theory, the axioms which correspond to such principles, etc. are the *fundamental axioms*, some of which are given and studied in Part II. It will there be shown that axiom T.X is a consequence of the fundamental axioms.

## PART II

### 10. SPECIAL RELATIONSHIPS

Before stating the fundamental axioms, we shall define and discuss certain special relationships between particles which are needed for these axioms. The first is that of symmetry.

**Definition.**—Particles  $P, Q$  are *symmetric about* a collinear particle  $T$  if they are on opposite sides of (or coincident with)  $T$  and satisfy  $P(T_x) = Q(T_x)$  for all  $T_x$ . An equivalent statement is  $P^*Q^*(T_x) = T_x$ .

**Theorem 10.1.**—If particles  $T, P, Q, R$  are collinear and if  $P, Q$  are symmetric about  $R$ , then  $Q^*(T_x) = R^*P^{*-1}R^*(T_x)$  for all  $T_x$ .

**Lemma.**—Particles  $T, P, Q$  are collinear,  $P_x$  is any instant and  $P_y = Q^*(P_x)$ . If  $P_x, T_x$  lie on a right optical line and if  $P_y, T_y$  also lie on a right optical line, then  $T_y = P^{*-1}Q^*(T_x)$ . This function can be regarded as the projection on to the set  $T$  of the function  $Q^*(P_x)$ .

Let  $P_y, T_z$  lie on a left optical line. Then we find in all cases,  $T_z = P^*(T_y)$  and  $T_z = Q^*(T_x)$ , whence  $T_y = P^{*-1}(T_z) = P^{*-1}Q^*(T_x)$ .

Turning to the theorem, then  $P, Q$  are symmetric about  $R$  if  $P^*Q^*(R_x) = R_x$  for all  $R_x$ . If, therefore,  $T_x, T_y, T_z$  are defined to lie on the right optical lines through  $R_x, Q^*(R_x)$ , and  $P^*Q^*(R_x)$  respectively, then we are given that  $T_z = T_x$ . Now by the lemma we have

$$T_y = R^{*-1}Q^*(T_x), \quad T_z = R^{*-1}P^*(T_y),$$

whence  $T_z = R^{*-1}P^*R^{*-1}Q^*(T_x) = T_x$  for symmetry, and the theorem follows.

If  $P, Q$  are symmetric about a collinear particle  $T$ , then we can say that each of  $P, Q$  is the *reflection* of the other in  $T$ .

**Definition.**—A linear set of particles is *completely symmetric* if it contains the reflection of every member in every other member.

The following theorem is seen at once by induction, using the last theorem and the definition of modified record functions.

**Theorem 10.2.**—If  $T, P$  are members of a completely symmetric linear set of particles, then for each integer  $p$  there is a member whose modified record function is  $P^{*p}(T_x)$ , the corresponding unmodified record function being  $P^n(T_x)$  where  $n = |p|$ .

The next special relationship is that of commutation:

**Definition.**—If  $T, P, Q$  are collinear particles, then  $P, Q$  are *commutative with respect to*  $T$  if  $P^*Q^*(T_x) = Q^*P^*(T_x)$ .

This relationship is symmetric in  $P$  and  $Q$ , and we shall prove that it is symmetric in all three particles, *i.e.* that:

**Theorem 10.3.**—If  $P, Q$  are commutative with respect to a collinear particle  $R$ , then  $P, R$  are commutative with respect to  $Q$ .

To prove this, let  $T$  be a particle collinear with  $P, Q, R$  (it can be identical with one of these). Then by the lemma of theorem 10.1, the projection of the function  $P^*Q^*(R_x)$  on to the set  $T$  is  $R^{*-1}P^*(T_y)$  where  $T_y = R^{*-1}Q^*(T_x)$ , *i.e.* is  $R^{*-1}P^*R^{*-1}Q^*(T_x)$ , and the projection of  $Q^*P^*(R_x)$  is  $R^{*-1}Q^*R^{*-1}P^*(T_x)$ . Equating these and pre-multiplying by  $R^*$ , then  $P, Q$  are commutative with respect to  $R$  if

$$P^*R^{*-1}Q^*(T_x) = Q^*R^{*-1}P^*(T_x). \quad (1)$$

Pre- and post-multiplying by  $P^{*-1}$  and inverting both sides, we get the same equation but with

Q and R interchanged. The theorem is thus proved. It also follows that Q, R are commutative with respect to P.

**Definition.**—A linear set of particles is a *commutative set* if every sub-set of three particles satisfies the commutative relation.

**Theorem 10.4.**—A linear set of particles, one of which is *T*, is commutative if *T*, *P*, *Q* are commutative for every pair *P*, *Q* of the set.

For referring to *T*, we have for any three particles *P*, *Q*, *R*,  $P*Q* = Q*P*$ ,  $P*R* = R*P*$ , and  $Q*R* = R*Q*$ . From these relations we can deduce (1), and the theorem is proved.

Lastly there is the *fundamental relation*,† which is defined for any three particles but which reduces to commutation in the case of collinear particles. This relation for *P*, *Q*, *R* will be written  $P/QR$  and is symmetric in *Q* and *R*.

**Definition.**—For any particles *P*, *Q*, *R* and any instant  $P_x$ , let other instants of *P*, *Q*, *R* be defined by the signal chains

$$P_x \wedge Q_x \wedge R_x \wedge Q_y \vee P_y, \quad P_x \wedge R_y \wedge Q_z \wedge R_z \vee P_z.$$

Then *P*, *Q*, *R* satisfy the relation  $P/QR$  if  $P_y = P_z$  for every  $P_x$ .

**Theorem 10.5.**—If *P*, *Q*, *R* are collinear particles satisfying the relations  $P/QR$ ,  $Q/PR$  and  $R/PQ$ , then *P*, *Q*, *R* are commutative. Conversely, if *P*, *Q*, *R* are collinear and commutative, then the three fundamental relations are satisfied.

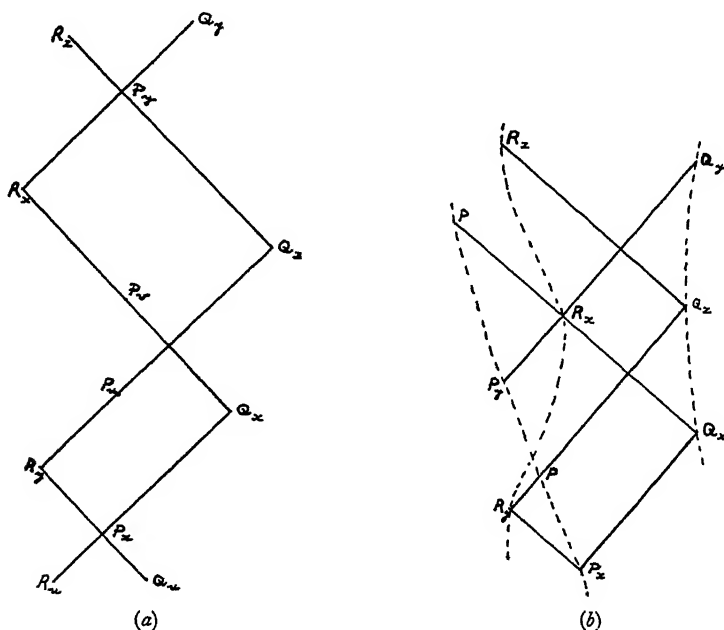


FIG. 1

*Note.*—Particle lines are drawn (broken) in (b) but not in (a) since it is not definite whether they do or do not intersect in this case. The positions of  $P_u$  and  $P_v$  are also not definite, all that is known being that *P* is between *Q* and *R* at these instants.

To prove this we must deduce  $Q*R*(P_t) = R*Q*(P_t)$  for every  $P_t$ . Suppose firstly that  $P_x$  is an instant at which *P* is between *Q*, *R*. Then except for an interchange of *Q* and *R*, fig. 1 (a) is the only possible diagram consistent with collinearity and the relation  $P/QR$ . In fig. 1 (b), for example, is an attempt to construct an alternative diagram assuming that *R* is between *P*, *Q* at  $R_x$ , but it shows that  $R_x \vee P_y$  is impossible; other alternatives are similarly

† This relation appears at first sight to be artificial, but it will be seen later to be equivalent to the statement that *T*'s estimates of the "distances"  $QR$  and  $RQ$  are the same. For collinear particles, this relation is contained in Milne's assumption of equivalence.

ruled out. Constructing now  $P_u$ ,  $P_v$ ,  $Q_u$  and  $R_u$  as in fig. 1 (a), we deduce at once from this diagram

- (i)  $P_v = Q^*R^*(P_u) = R^*Q^*(P_u)$ ,
- (ii)  $Q_u = P^*R^*(Q_v) = R^*P^*(Q_v)$ ,
- (iii)  $R_u = P^*Q^*(R_v) = Q^*P^*(R_v)$ .

If  $P$  is between  $Q$ ,  $R$  at  $P_t$ , then the diagram applies with  $P_t$  in place of  $P_u$  (and possibly  $Q$ ,  $R$  interchanged), and (i) gives the desired relation. If  $P$  is to the right of both  $Q$  and  $R$  at  $P_t$ , then the diagram applies with  $P$  and  $Q$  interchanged (and then possibly also  $Q$  and  $R$ ), and with  $P_t$  in place of  $Q_v$ ; the desired relation is now given by (ii). If  $P$  is to the left of both  $Q$  and  $R$  at  $P_t$ , then the diagram applies with  $P$  and  $R$  interchanged (and then possibly also  $Q$  and  $R$ ), and with  $P_t$  in place of  $R_u$ ; the desired relation is now given by (iii). Finally, the cases with coincidences follow from the above by continuity, and the theorem is proved. The converse of the theorem is at once seen to be true from the diagrams illustrating commutation.

## 11. THE FUNDAMENTAL AXIOMS

We are now in a position to state the fundamental axioms as they concern linear sets of particles. Unlike previous axioms, these do not refer to all particles but only to a special set  $\mathcal{F}$  of *fundamental particles*. The set  $\mathcal{F}$  contains linear sub-sets, and we shall only be concerned with these in the present paper. We shall therefore consider only those axioms or their special cases which refer to linear sets. The axioms for the whole set  $\mathcal{F}$  will be given in a later paper.

**Axiom F. 1.**—*The fundamental relation is satisfied by every sub-set of three particles in  $\mathcal{F}$ .*

**Axiom F. 2.**—*If  $P$ ,  $Q$  are any two distinct particles in  $\mathcal{F}$ , then the set  $\mathcal{F}_1$  of particles in  $\mathcal{F}$  which are collinear with  $P$ ,  $Q$  is dense at every instant of some member of  $\mathcal{F}_1$ .*

**Axiom F. 3.**—*Every linear sub-set of  $\mathcal{F}$  is completely symmetric.*

**Axiom F. 4.**—*There are at least two distinct particles in  $\mathcal{F}$ .*

In a later paper, when we add other axioms which give three spatial dimensions, axiom F. 3' will be replaced by an axiom F. 3 of which it is a special case and which gives what we shall call complete symmetry in  $\mathcal{F}$ .

From axiom F. 4 we see that there is at least one linear sub-set of  $\mathcal{F}$ . Such a sub-set we shall write as  $\mathcal{F}_1$ , and we shall now be concerned only with  $\mathcal{F}_1$ , a typical linear sub-set of  $\mathcal{F}$ . From axiom F. 1 and theorem 10.5, we have that  $\mathcal{F}_1$  is a commutative set. From axiom F. 2,  $\mathcal{F}_1$  contains at least one particle, which we shall write as  $T$ , such that  $\mathcal{F}_1$  is dense at every instant of  $T$ . Later we shall prove that this property holds for every particle of  $\mathcal{F}_1$ .

Referring to  $T$ , let  $P^*$ ,  $Q^*$ , etc. denote the modified signal functions of particles of  $\mathcal{F}_1$ . Then from theorem 10.4, since  $\mathcal{F}_1$  is a commutative set, every one of these functions commutes with every other. It follows that every functional product of any number of these functions commutes with every other functional product.

From axiom F. 3',  $\mathcal{F}_1$  is a completely symmetric set, so that if  $P$ ,  $R$  are any two members, then there is another member whose modified record function is  $R^*P^{*-1}R^*$ , this being the reflection of  $P$  in  $R$  by theorem 10.1. From the commutative property, this function is equivalent to  $P^{*-1}R^{*2}$ .

## 12. PERMANENT ORDER IN $\mathcal{F}_1$

**Theorem 12.1.**—*No particle of  $\mathcal{F}_1$  distinct from  $T$  (the member of  $\mathcal{F}_1$  mentioned in axiom F. 2) coincides with  $T$  at any instant. (The first and last instants of the set  $T'$  are excluded by our notation.)*

Suppose that  $P$  is a member of  $\mathcal{F}_1$  which is distinct from  $T$  but coincides with  $T$  at one or

more instants. Then there is an interval, say  $T_0T_1$  ( $T_1 > T_0$ ), such that  $T, P$  coincide at either or both of  $T_0, T_1$  but at no instant between. Suppose that  $T, P$  coincide at  $T_0$ . Then from axioms F.2 and F.3' there is a member of  $\mathcal{S}_1$ , say  $Q$ , which is on the right of  $T$  at  $T_0$  and is such that  $T_0 < Q(T_0) < T_1$ . Now from the commutative property of  $\mathcal{S}_1$  we have

$$P^*Q^*(T_0) = Q^*P^*(T_0) = Q^*(T_0),$$

since  $P^*(T_0) = T_0$  is a consequence of coincidence. Hence  $T, P$  coincide at the instant  $Q^*(T_0)$ , which is  $Q(T_0)$  and lies between  $T_0$  and  $T_1$ . We therefore have a contradiction. There is a similar argument supposing that  $T, P$  coincide at  $T_1$ , and the theorem is therefore proved.

**Corollary 1.**—*If  $P$  is a member of  $\mathcal{S}_1$ , then for all instants of  $T$ , either  $P(T_w) = P^*(T_w)$  or  $P(T_w) = P^{*-1}(T_w)$ , and the record functions  $P, Q, \dots$  of particles of  $\mathcal{S}_1$  are commutative in pairs.*

This follows from the definition of modified record functions and from the properties of commutative functions.

**Corollary 2.**— *$T$  has no first or last instant.*

For from elementary signal properties,  $T$  would coincide with every other particle at such an instant (cf. theorem 5.1).

We can now prove the following important theorem:—

**Theorem 12.2.**—*No two particles of  $\mathcal{S}_1$  coincide except at their first and last instants.*

**Lemma.**—*Particles  $P, Q$  of  $\mathcal{S}_1$  are identical if  $P^{*2} = Q^{*2}$ .*

For suppose that  $P^*(T_w) \geq Q^*(T_w)$  for some  $T_w$ . Then from the commutative property and the fact that  $P^*, Q^*$  are strictly increasing functions, we have

$$P^{*2}(T_w) \geq P^*Q^*(T_w) = Q^*P^*(T_w) \geq Q^2(T_w),$$

which contradicts the given equality. Hence  $P^*(T_w) = Q^*(T_w)$  for all  $T_w$  and the lemma is proved by theorem 8.3.

Suppose now that  $P, Q$  are distinct particles of  $\mathcal{S}_1$  which coincide, so that by theorem 8.3 there is a  $T_0$  for which  $P^*(T_0) = Q^*(T_0)$ . By axiom F.3' let  $R$  be the reflection of  $T$  in  $P$  and let  $S$  be the reflection of  $R$  in  $Q$ . Then from theorem 10.1 and the commutation property we have  $R^* = P^{*2}$  and

$$S^* = Q^*R^{*-1}Q^* = Q^*P^{*-2}Q^* = P^{*-2}Q^{*2}. \quad (2)$$

From  $P^*(T_0) = Q^*(T_0)$  it follows that  $Q^*P^{*-2}Q^*(T_0) = T_0$ , so that  $S$  coincides with  $T$  at  $T_0$ . Hence from theorem 12.1,  $S$  must be identical with  $T$ , whence from (2),  $P^{*2} = Q^{*2}$ . This, however, is impossible by our lemma, since  $P, Q$  are given distinct. Our supposition is therefore false, and the theorem is proved.

We have now proved that  $\mathcal{S}_1$  is a permanently ordered linear set.

### 13. THE ORDINAL-LINEARITY OF TIME

We are now in a position to deduce from the fundamental axioms the following theorem:—

**Theorem 13.1.**—*The set  $T$  of instants is ordinally-linear, i.e. axiom T.X is now true.*

Let  $\dagger T_n$  be a descending progression of instants with a limit, say  $T_w$ , in  $T$ . Then by axiom F.2 we can choose particles  $P$  in  $\mathcal{S}_1$  such that

$$P(T_w) < T_n, \quad P(T_w) < P(T_{n+1}) \quad (n = 1, 2, \dots). \quad (3)$$

Let  $\Sigma$  be the set of instants

$$P^p(T_w) \quad (n = 1, 2, \dots, p = 0, \pm 1, \pm 2, \dots),$$

repetitions being excluded. Then  $\Sigma$  is denumerable, and we shall prove that this sub-set of  $T$  satisfies the conditions of axiom T.X, i.e. that between any two instants of  $T$  is a member of  $\Sigma$ .

$\dagger$  The existence of such a progression follows from axiom F.2.

Let  $T_x, T_y$  be any two instants ( $T_x < T_y$ ). Then from axiom F.2 there is a particle  $P$  of  $\mathcal{S}_1$  such that

$$T_x < P(T_x) < T_y. \quad (4)$$

Since  $P$  does not coincide with  $T$  at  $T_\omega$  and since  $T_\omega$  is the limit of the descending progression  $T_n$ , it follows that there is an  $n$  such that  $T_n < P(T_\omega)$ , whence from (3),  $P(T_\omega) < P(T_\omega)$ . We deduce that  $P(T_i) < P(T_i)$  for all instants of  $T$  because by theorem 12.2,  $P, P$  do not coincide. In particular, therefore, and from (4),

$$P(T_x) < P(T_x) < T_y. \quad (5)$$

Since  $P$  does not coincide with  $T$ , the ascending and descending progressions  $P^r(T_\omega)$ ,  $P^{-r}(T_\omega)$  ( $r = 1, 2, \dots$ ) have no limits in  $T$ , so that  $T$  is covered by the intervals

$$(P^n(T_\omega), P^{n+1}(T_\omega)),$$

where  $n$  is fixed and  $p = 0, \pm 1, \pm 2, \dots$ . There is therefore a  $p$  such that

$$P^p(T_\omega) < T_x < P^{p+1}(T_\omega). \quad (6)$$

From this and (5), since  $P$  is an increasing function, we have

$$P^{p+1}(T_\omega) = PP^p(T_\omega) < P(T_x) < T_y. \quad (7)$$

Finally, from (6) and (7),

$$T_x < P^{p+1}(T_\omega) < T_y,$$

which shows as required that a member of  $\Sigma$  lies between  $T_x$  and  $T_y$ .

From the above theorem and the second corollary to theorem 12.1 we have at once, remembering that the set  $T'$  is closed,

**Theorem 13.2.**—*The set  $T$  is ordinally similar to an open interval of the arithmetic continuum.*

**Corollary 1.**—*The set  $T'$  is ordinally similar to a closed interval of the continuum.*

**Corollary 2.**—*Every particle-set  $P'$  of instants is ordinally similar to a closed interval of the continuum.* This follows from the fact that a signal correspondence is one-one and preserves order.

We now see from the foregoing theorems that there is a one-one correlation between the instants of a particle  $P$  and the numbers of an open interval of the arithmetic continuum such that order is preserved, the relations "before" and "after" corresponding to "less than" and "greater than" respectively. Such a correlation we call a *clock* attached to  $P$ , and the word "instant" can be used when referring to the corresponding number, or "clock reading". From the definition it follows that a clock reads continuous time. A clock attached to a particle is not unique, i.e. can be *regraduated*. A regraduation corresponds to a transformation of the form  $t' = \psi(t)$ , where  $t, t'$  are old and new clock readings and  $\psi$  is continuous and monotonic increasing.

#### 14. $\tau$ -CLOCKS AND SPATIAL DISTANCE

For a given clock attached to  $T$ , each record and modified record function is equivalent to a c.m.i. (continuous monotonic increasing) function of a numerical variable  $t$  in some interval  $< a, b >$ . From the fundamental axioms and theorems it follows that the functions  $P^*, Q^*, \dots$  given by particles of  $\mathcal{S}_1$  can now be regarded as functions of  $t$  and have the following properties:—

- (i) Each function is complete and node-free  $\dagger$  in  $< a, b >$ .
- (ii) Each function commutes with each other function.
- (iii) The set of functions contains a sequence which converges uniformly to the identity function.

$\dagger$  The present terminology, and the definition and study of related sets of functions referred to later, are given in Walker, 1946. A *node* of  $f(x)$  is a root of  $f(x) = x$ , and a function  $f$  is *complete and node-free* in an interval  $< a, b >$  if it is c.m.i. in this interval and has nodes at  $a$  and  $b$  but at no point between.



These are exactly the properties which functions must possess to ensure that they form a *related set*, as was shown in a recent study of commutative functions (Walker, 1946, § 13).

A definitive property of a related set is that there is a function  $\psi$ , c.m.i. and with extreme values 0 and  $\infty$  in the interval  $\langle a, b \rangle$ , such that each function of the set can be expressed in canonical form  $\psi^{-1}\alpha\psi$  for some constant  $\alpha > 0$ . Applying this to the set of modified record functions and then regraduating  $T$ 's clock from  $t$  to  $\tau$  by means of the transformation  $\tau = \log \psi(t)$ , we finally get the following theorem:—

**Theorem 14.1.**—*There is a clock attached to  $T$  and reading time  $\tau$  such that the modified record function for every particle  $P$  of  $\mathcal{F}_1$  is of the form  $P^*(\tau) \equiv \tau + 2x$ , where  $x$  is a constant which varies for different particles.*

**Definitions.**—A clock satisfying the conditions of the above theorem will be called a  $\tau$ -clock, and for a given  $\tau$ -clock, the constant  $x$  in the theorem will be called the *distance TP*.

We observe that the distance TP is positive, zero or negative according as  $P$  is to the right of, coincident with, or to the left of  $T$ . Also, from theorem 10.1, we see that  $P, Q$  are symmetric about  $R$  if the distances from  $T$  satisfy

$$TP + TQ = 2TR. \quad (8)$$

**Theorem 14.2.**—*For a given  $\tau$ -clock attached to  $T$ , the set  $X$  of distances from  $T$  to the particles of  $\mathcal{F}_1$  is everywhere-dense in the continuum.*

To prove this we first see that axiom F.2 implies that the set  $X$  contains a series  $x_n (x_n \neq 0)$  such that  $x_n \rightarrow 0$ . Now from (8) and the complete symmetry of  $\mathcal{F}_1$  we see that if  $x$  belongs to  $X$ , then  $px$  belongs to  $X$  for all integers  $p$ . The theorem now follows from the fact that the set  $px_n$  is contained in  $X$  and is everywhere-dense in the continuum.

Since each member of  $X$  is a limit point, we have:

**Corollary.**—*The set  $\mathcal{F}_1$  is dense at every instant of every particle of the set.*

We now have a clear picture of  $\mathcal{F}_1$  as a one-dimensional space in which each point corresponds to a particle and is determined by its distance from a fixed point. Although the set  $X$  of distances is everywhere-dense, it is not necessarily closed; it could, for example, be denumerable. If, however,  $X$  is not closed, then it is possible to define ideal particles, just as we defined ideal instants in § 3, so that they have the properties possessed by particles of  $\mathcal{F}_1$  and so that when  $\mathcal{F}_1$  is augmented by their addition, then the augmented set  $X$  is closed.

Without assuming that  $X$  has been augmented in this way, we shall prove the following theorem concerning the uniqueness of  $\tau$ -clocks:—

**Theorem 14.3.**—*A  $\tau$ -clock is unique except for regraduations of the form  $\tau' = A\tau + B$  ( $A > 0$ ), i.e. arbitrary changes of unit and zero. All distances from  $T$  are affected proportionately when the unit is changed.*

Consider a regraduation  $\tau' = \phi(\tau)$  where  $\phi$  is c.m.i. Then a record function  $\tau + 2x$  transforms into the new record function  $\phi\{2x + \phi^{-1}(\tau)\}$ , and this has the required form  $\tau' + 2x'$  if

$$\phi(2x + y) = \phi(y) + 2x', \quad (9)$$

where  $x'$  depends upon  $x$  but not upon  $y$ . The regraduation therefore gives a new  $\tau$ -clock if this equation is satisfied for all  $y$  and for all  $x$  of  $X$ . Writing  $\theta(\tau) = \phi(\tau) - \phi(0)$ , then we find  $2x' = \theta(2x)$ , and (9) becomes

$$\theta(2x + y) = \theta(2x) + \theta(y).$$

This must be satisfied for all  $y$  and all  $x$  of  $X$ , and therefore all  $x$  since  $X$  is everywhere-dense and  $\theta$  must be continuous. Hence  $\theta(k) = k\theta(1)$  for every rational number  $k$ , and hence for all  $k$  by continuity, so that  $\phi(\tau)$  is of the form  $A\tau + B$  where  $A, B$  are constants.

Conversely, it is at once seen that every regraduation of this kind gives a  $\tau$ -clock, a record function  $\tau + 2x$  transforming into the function  $\tau' + 2Ax$ . The original distances are thus multiplied by  $A$  to give the new distances, this then being the effect of a change of unit.

15. CONSTRUCTION OF A  $\tau$ -CLOCK

Having proved a number of existence theorems concerning  $\tau$ -clocks, we now come to the problem of constructing a  $\tau$ -clock directly from primitive observables, without first postulating an arbitrary clock and without having to solve complicated functional equations. There are two arbitrary elements in the construction of a  $\tau$ -clock, corresponding to the arbitrary constants in the affine transformation admitted by such a clock. We choose therefore any instant  $T_0$  of  $T$  and any other particle  $P$  of  $\mathcal{F}_1$  and define a clock in terms of these; if  $T_0$  or  $P$  are changed, then the clock will undergo an affine regraduation.

Let  $Q$  be any particle of  $\mathcal{F}_1$ . Then from the properties of  $\mathcal{F}_1$  it follows that if  $p, q$  are integers,  $Q^q(T_x) >, =$  or  $< P^p(T_x)$ , and the relation is the same for all  $T_x$ . Also, both inequalities occur for different choices of  $p, q$ . Hence a unique number  $\lambda > 0$  is defined as a section of the rationals by

$$q\lambda \gtrless p \quad \text{according as} \quad Q^q(T_x) \gtrless P^p(T_x).$$

This number  $\lambda$  is an observable associated with  $T, P$ , and  $Q$ , and in particular,  $\lambda = 0$  when  $Q = T$ ,  $\lambda = 1$  when  $Q = P$ , and  $\lambda = p$  when  $Q = P^p$ .

A clock is now attached to  $T$  which is defined to read  $\lambda q$  at the instant  $Q^q(T_0)$ , and so for all integers  $q$  and all particles  $Q$  of  $\mathcal{F}_1$ . It can be verified that these clock-readings have the required numerical order in correspondence with temporal order, that they form an everywhere-dense set in the continuum, and that the instants they record form an everywhere-dense set in  $T$ . The extension to all other instants of  $T$  is therefore determined by continuity, and a complete clock has been constructed for  $T$ .

That this is a  $\tau$ -clock can be verified directly or by means of the existence theorems of § 14. For if a  $\tau$ -clock is chosen to read 0, 1 at instants  $T_0, P(T_0)$  respectively, then  $P$ 's record function is  $P(\tau) \equiv \tau + 1$  and from the above definition of  $\lambda$ ,  $Q$ 's record function is  $Q(\tau) \equiv \tau + \lambda$ . The reading of  $Q^q(T_0)$  is therefore  $\lambda q$ , and the  $\tau$ -clock agrees with our constructed clock at all these instants and therefore at all instants of  $T$  by continuity. The clock we constructed is therefore this  $\tau$ -clock.

 16. SIMILAR  $\tau$ -CLOCKS AND SIMULTANEITY

Having discussed clocks attached to one particle, we now come to the problem of attaching clocks to all particles of  $\mathcal{F}_1$  so that they are in some sense "similar" and "synchronised" and give rise to a unique definition of simultaneity. We shall first define a clock for each particle in terms of a given  $\tau$ -clock attached to  $T$  and show that all these clocks are  $\tau$ -clocks with the desired properties. Later, in § 17, we shall examine the problem from a different point of view and deduce that the clocks previously defined are the only clocks having certain general properties.

Suppose that a  $\tau$ -clock has been attached to  $T$ , and let  $P$  be a particle of  $\mathcal{F}_1$  whose distance from  $T$  is  $x$ . For any  $\tau$ , let  $T_\tau$  be the instant whose clock reading is  $\tau$ , and let  $P_{\tau+x}$  be the instant of  $P$  on the right optical line through  $T_\tau$  (i.e.  $T_\tau \wedge P_{\tau+x}$  or  $T_\tau \vee P_{\tau+x}$  according as  $P$  is to the right or left of  $T$ ). Then we attach a clock to  $P$  which is defined to read  $\tau + x$  at the instant  $P_{\tau+x}$ , and so for every  $\tau$ . A clock is similarly attached to every other particle of  $\mathcal{F}_1$ .

To prove that  $P$ 's clock is a  $\tau$ -clock, let  $Q$  be any other particle of  $\mathcal{F}_1$  and let the distance  $TQ$  be  $y$ . Then the modified record functions of  $P$  and  $Q$  with respect to  $T$  are  $\tau + 2x, \tau + 2y$  respectively, and from the lemma of theorem 10.1 we deduce at once that the modified record function of  $Q$  with respect to  $P$  is  $\tau + 2(y - x)$  in terms of  $P$ 's time  $\tau$ . This is of the required form, so that  $P$ 's clock is a  $\tau$ -clock.

From the function  $\tau + 2(y - x)$  derived above we deduce that if  $PQ$  is the distance of  $Q$  from  $P$  defined in relation to  $P$ 's clock, then  $PQ = TQ - TP$ . In particular,  $PT = -TP$ , i.e.  $T$  is the same numerical distance from  $P$  as  $P$  is from  $T$ . This fact leads us to describe the clocks attached to  $T$  and  $P$  as *similar*.

If now we regraduate  $T$ 's clock by means of  $\tau' = A\tau + B$ , then the new distance  $TP$  is  $x' = Ax$ , and we have

$$\tau' + x' = A\tau + B + Ax = A(\tau + x) + B.$$

This shows that when T's clock is regraduated affinely, then P's clock undergoes exactly the same regraduation, and *all* distances are multiplied by the same constant.

Simultaneity between T and P is defined as the one-one correspondence in which instants with the same clock-reading correspond, P's clock being defined in terms of T's as described above. It is easily deduced from our definitions that these correspondences are transitive, *i.e.* that instants of P and Q are simultaneous if they are both simultaneous with the same instant of T. We also see that the simultaneity correspondence between any two particles of  $\mathcal{T}_1$  is independent of the particular  $\tau$ -clock attached to T, since all clocks undergo the same affine transformation when T's clock is regraduated.

### 17. GENERAL SIMULTANEITY

A general definition of simultaneity, which closely resembles Milne's definition of equivalence, is as follows:—

**Definition.**—A *simultaneity* between particles P, Q (not necessarily fundamental) is a one-one correspondence between their instants, which we write as  $P_x \sim Q_x$ , such that

- (i)  $P_x < P_y, P_x \sim Q_x$  and  $P_y \sim Q_y \supset Q_x < Q_y$ ;
- (ii)  $P_x \sim Q_x, P_x \wedge Q_y$  and  $Q_x \wedge P_y \supset P_y \sim Q_y$ .

We see from (i) that the correspondence is ordinal and continuous. Also, if  $P_x \sim Q_x$ , then  $Q_x \sim P_x$ .

If, for a given simultaneity,  $P_x \sim Q_x \wedge P_y$ , then we write  $P_y = \theta(P_x)$ , and call the function  $\theta$  defined in this way a *signal function*, a name which arises out of Whitrow's study of equivalences. We observe that the same function is given by  $P_x \wedge Q_y \sim P_y$ . A signal function is clearly continuous and monotonic increasing.

Writing  $P_y = \theta(P_x)$  and  $P_z = \theta(P_y) = \theta^2(P_x)$ , then if  $P_y \sim Q_y$ , we have  $P_x \wedge Q_y \wedge P_z$ , *i.e.*  $P_y = Q(P_x)$  in the notation of record functions. Hence, for all  $P_x$ ,

$$\theta^2(P_x) = Q(P_x). \quad (10)$$

Conversely every c.m.i. function  $\theta$  which satisfies (10) gives rise to a simultaneity between P and Q, the correspondence being  $P_x \sim Q_x$  where  $Q_x$  is given by  $Q_x \wedge \theta(P_x)$ .

This last result shows how a simultaneity can be constructed and how arbitrary it is. A solution of the functional equation (10) for  $\theta$  may not exist when the ordered set P is non-linear, but always exists when P is linear (Milne and Whitrow, 1938), as is now the case from the second corollary of theorem 13.2.

We shall say that a set of particles *admits simultaneity* if simultaneities can be found for all pairs such that they are transitive, *i.e.*  $P_x \sim Q_x$  and  $P_x \sim R_x$  imply  $Q_x \sim R_x$ . We shall not discuss such sets at length, but it can be verified with the aid of diagrams that if T, P, Q are collinear and admit simultaneity, then

- (i) if  $T_x \sim P_x \sim Q_x$ , the three particles have the same order on every optical line through  $T_x, P_x$ , or  $Q_x$ ;
- (ii)  $\theta\phi = \phi\theta$  where  $\theta, \phi$  are the signal functions of P, Q respectively with respect to T.

Since  $\theta^2 = P$  and  $\phi^2 = Q$ , it follows from (ii) that  $PQ = QP$  and then from (i) that  $P^*Q^* = Q^*P^*$ . Hence:

**Theorem 17.1.**—*A collinear set of particles which admits simultaneity is commutative.*

It is easily seen that commutation is less restrictive than simultaneity, for  $P^*Q^* = Q^*P^*$  does not imply  $PQ = QP$ . Nor even does  $PQ = QP$  imply the existence of  $\theta, \phi$  such that

$$\theta^2 = P, \quad \phi^2 = Q, \quad \theta\phi = \phi\theta. \quad (11)$$

We also notice that for a set of particles, simultaneity is not a primitive observable. The primitive ("observable") functions are the record functions, and the question of simultaneity depends upon whether solutions of equations similar to (11) for  $\theta$  and  $\phi$  do or do not exist. This cannot be tested by a sequence of primitive experiments.

Turning now to the set  $\mathcal{F}_1$ , suppose that a  $\tau$ -clock has been attached to T. Then for a particle P of  $\mathcal{F}_1$ , we have  $P^*(\tau) = \tau + 2x$  whence  $P(\tau) = \tau + 2|x|$ , so that  $TQ = QP$  for all pairs.

For simultaneity, we want to find if possible a  $\theta$  for each P such that  $\theta^2 = P$  and all the  $\theta$ 's commute in pairs, each  $\theta$  being c.m.i. Since  $\theta\phi = \phi\theta$  implies  $\theta\phi^2 = \phi^2\theta$ , it follows that each  $\theta$  must commute with all the P's, i.e. with all functions of the form  $\tau + k$  ( $k > 0$ ). Hence each  $\theta$  must satisfy

$$\theta(\tau + k) = \theta(\tau) + k$$

for all  $\tau$  and all  $k > 0$ , and this leads at once to the form  $\theta(\tau) = \tau + \lambda$ , where  $\lambda$  is any constant. Substituting now in  $\theta^2 = P = \tau + 2|x|$ , we find that the corresponding  $\theta(\tau)$  must be  $\tau + |x|$ . Thus the signal function for each particle is unique and defines exactly the simultaneity given in § 16. These simultaneities are clearly transitive, and we have:

**Theorem 17.2.**—*The set  $\mathcal{F}_1$  admits simultaneity in a unique way, this being equivalent to the definition given in § 16.*

### 18. PARTICLES COLLINEAR WITH FUNDAMENTAL PARTICLES

We have not so far considered, in relation to time keeping, particles which do not belong to an  $\mathcal{F}_1$ . The general case will be discussed in a later paper, but we can now prove the following:—

**Theorem 18.1.**—*If a particle A is collinear with those of  $\mathcal{F}_1$ , then A coincides with a particle of  $\mathcal{F}_1$  at every instant of A. (It is assumed here that the set  $\mathcal{F}_1$  has been augmented as described in § 14.)*

Let T be a particle of  $\mathcal{F}_1$ . Then for any instant  $A_x$  of A there is an instant  $T_x$  of T which lies on the right optical line through  $A_x$ . From the spatial properties of  $\mathcal{F}_1$  described in § 14 there is for any instant  $T_y$  a particle P of  $\mathcal{F}_1$  such that  $P^*(T_x) = T_y$ , and taking  $T_y$  to be  $A^*(T_x)$ , it follows that there is a P such that  $P^*(T_x) = A^*(T_x)$ . Hence from theorem 8.3, A coincides with P at  $A_x$ .

When similar  $\tau$ -clocks have been attached to the particles of  $\mathcal{F}_1$  then a unique clock can be attached to any particle A which is collinear with those of  $\mathcal{F}_1$ . For any instant  $A_x$  of A, let P be the member of  $\mathcal{F}_1$  which coincides with A at  $A_x$  and let  $A_x \times P_x$ . Then A's clock is defined so that the reading at  $A_x$  is the clock-reading of P at  $P_x$ . It is easily verified that the clock so defined has the desired order and continuity properties.

The position of a particle A such as the above can be specified at any instant  $A_x$  by defining its distance at this instant from a fixed particle T of  $\mathcal{F}_1$ . This is simply the distance from T to the particle of  $\mathcal{F}_1$  which coincides with A at  $A_x$ . Thus finally there is associated with each instant of A a clock-reading,  $\tau$ , and a spatial co-ordinate,  $x$ . These numbers are also given directly in terms of T's clock-readings by  $\tau = \frac{1}{2}(\tau_1 + \tau_0)$ ,  $|x| = \frac{1}{2}(\tau_1 - \tau_0)$ , where  $\tau_0, \tau_1$  are the readings of instants  $T_0, T_1$  such that  $T_0 \wedge A_x \wedge T_1$ . This agrees with Milne's conventions for measuring epoch and distance (with  $c = 1$ ).

The "motion" of A (relative to  $\mathcal{F}_1$ ) is described by the functional relation of  $x$  to  $\tau$ , and it follows from our axioms that such a function is single-valued and continuous.

This completes our discussion of linear sub-sets of fundamental particles.

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XXXV.—Graphite Crystals and Crystallites. I. Binding Energies in Small Crystal Layers. By Mary Bradburn (Royal Holloway College, London University), C. A. Coulson (Wheatstone Physics Laboratory, King's College, London), and G. S. Rushbrooke (Chemistry Department, Leeds University). (With Two Text-figures.)

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# 1. INTRODUCTION\*

THE process of building up a crystal or metallic lattice from the individual atoms is a very complex one: at present it is not properly understood theoretically. In part this is due to difficulties in dealing with large, but not infinitely large, systems. Thus, on the one hand the infinite lattice has been fully discussed, and on the other hand so also has the small aggregate such as the diatomic molecule. But between these two extremes there lies an important field for which practically no theoretical results are known. Indeed the only work of this kind seems to be that of Taylor, Eyring and Sherman (1933), who considered systems containing up to 8 atoms in  $s$ -states; although this work is interesting it is of little value for discussing the properties of the finite crystallites which exist with many more atoms than 8. In this connection it seems generally agreed now that in the process of crystallisation the nuclei around which the larger crystals grow never contain less than 25 atoms. And, moreover, the Heitler-London method used by these writers is not readily extended to larger aggregates.

These larger aggregates, which we shall refer to as crystallites, occur both in metallic and non-metallic systems. They have a particular importance in the case of carbon, where, in the pyrolysis of natural coals, cellulose, lignin and certain large aromatic molecules (Blayden, Gibson and Riley, 1943), chars are obtained containing layers of carbon atoms with between 38 and 150 atoms in a plane. In view of the importance of these crystallites in metallurgical and other practice, we have studied their properties and we present in this paper an account of our work. The methods may be applied, with a little more difficulty, to other systems. But there is an advantage in dealing first with the graphite crystallites: for X-ray evidence makes it very clear that these small aggregates resemble the large graphite crystal in having a planar structure, so that in each layer plane there is the familiar hexagonal pattern of carbon atoms, with a C—C bond length about 1.42 Å., whereas the individual planes are separated by a much greater distance of some 3.4 Å. This large distance shows that the forces between the planes are largely non-specific Van der Waals forces. Along any one plane there are stronger forces of metallic or molecular valence type. This conclusion is reinforced by the fact that in many cases there is no correlation between the individual planes, which occupy random relative positions so far as translations and rotations are concerned (Warren's turbostratic system, 1941). For this reason we can, provisionally, neglect all inter-layer effects and consider only the single layers separately. This does not mean that these other effects are really non-existent, but rather that we must proceed stage by stage, so that a proper understanding of the forces between the atoms in one plane must precede any discussion of the forces between neighbouring planes.

This decision greatly simplifies our problem. For a crystallite layer with 50 to 100 carbon atoms arranged hexagonally is like a huge condensed aromatic molecule. In fact it lies between a molecule like coronene (24 carbon atoms) and the infinite graphite layer plane. Now it has been shown that the methods of wave mechanics may be applied successfully to a study of coronene (Coulson, 1944) and to the infinite graphite layer (to be published). We may

\* The work contained in this paper was largely done when the authors were members of the staff of the Mathematics Department of University College, Dundee.

therefore confidently expect the same methods to work satisfactorily with these crystallites of intermediate size.

There are many problems associated with these layer systems. In this paper we shall discuss the binding energy in terms of the size and shape of the crystallite; this will lead us to an estimate of the C—C bond length and its dependence upon shape. In this way we shall get some indication of the manner in which the crystallites build up to form a larger layer in the process of graphitisation. Finally, we shall consider the question of chemical stability of these crystallites with regard to hydrogen addition. The conclusions of this latter discussion, however, are less reliable than the earlier ones, since they involve the absolute, instead of the relative, magnitudes of certain constants whose values are not known with certainty.

## 2. DETAILS OF MODEL ASSUMED

We must now describe more fully the model that we use to represent the crystallite layer. We shall suppose that there are  $N$  carbon atoms arranged hexagonally in a plane according to the scheme shown in fig. 1. We suppose also (i) that all the C—C distances are equal, and

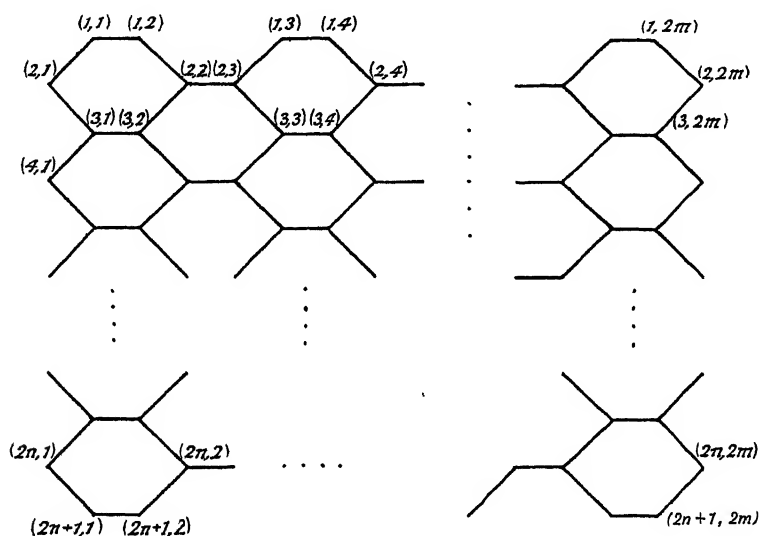


FIG. 1

(ii) that all the carbon atoms are in the trigonal state first introduced by Pauling. (For a description of this state, see Coulson, 1941). Now in the first place it is almost certain that the C—C links are not all equal in length throughout any one layer, and in a later paper one of us will discuss this effect with particular reference to approximately circular molecules of the form  $C_xH_y$ , where  $x=6n^2$ ,  $y=6n$ , and  $n=1, 2, \dots$ . It is sufficient for the present to state that although there is greater disparity between the links near the edge of the crystallite than in the centre, the effect is too small to make any serious difference to our conclusions in the present paper; a detailed allowance for this variation would make our calculations quite impracticable. In the second place, assumption (ii) requires us to suppose that the boundary, or surface, carbon atoms have their third trigonal valence satisfied. For that reason we shall imagine that these edge atoms have each one attached hydrogen atom. This makes them resemble the condensed hydrocarbon molecules which have often been discussed before with considerable success (Lennard-Jones and Coulson, 1939). This assumption, without which it would at present be impossible to make our calculations, is quite reasonable, though it is certainly not completely true to the actual situation. For, although Riley (1939) has shown that at fairly low temperatures (*e.g.*  $< 600^\circ \text{C.}$ ) there is enough hydrogen present to saturate all the edge valencies, we have nevertheless neglected the possibility, which certainly occurs to some extent, of acetylenic "tails" such as  $\text{C} \equiv \text{CH}$ , and we have also disregarded completely the effect of the remaining inorganic matter, chiefly oxygen, which is known to play an important

part in the chemistry of coal. However, it is very probable that this latter effect is produced largely on the matrix within which the layer crystallites are embedded; in any case, until more experimental evidence is available with regard to the location and function of these inorganic substances, it is not possible to include them in our discussion.

To summarise—we discuss the energies of the electrons in large condensed systems such as those of fig. 1, supposing that all the C—C bonds are equal in length and all the carbon atoms are in the trigonal, or aromatic state with three localised single bonds, and one mobile, or  $\pi$ -electron free to move over the carbon framework. The non-mobile electrons, responsible for the basic single bonds, are sufficiently localised for us to treat them as ordinary molecular valence bond electrons, making due allowance, when necessary, for their compression beyond the normal C—C bond length. To the energy of these basic bonds we have to add that of the mobile  $\pi$ -electrons whose wave functions have a nodal surface coincident with the layer plane. Our main problem is to calculate this latter energy.

A word is necessary concerning the shapes of these pseudo-molecules. These have been chosen so that the secular equation is soluble; and this condition severely restricts the possible arrangements of the  $N$  carbon nuclei. We have chosen the rectangular shape shown in fig. 1 because it is then possible to use some results of D. E. Rutherford (1947) without further analysis. By varying  $m$  and  $n$  we are able to give a wide variety to the crystallite shape, and in particular we are able to answer the important question: does a layer of this type build up more easily in roughly square or filamentous shape? It will be recognised that  $m$  and  $n$  denote the numbers of hexagons along the two sides of the molecule. Thus  $m = 1$  denotes the polyacene series with particular cases naphthalene ( $m = 1, n = 2$ ), anthracene ( $m = 1, n = 3$ ); and  $n = 1$  denotes the polyphenyl series with particular cases diphenyl (2, 1) and *p*-terphenyl (3, 1). We may also note that perylene is (2, 2) and chalkacene is (3, 2).

If  $N, B, H$  denote the total number of carbon atoms, C—C bonds and complete hexagons respectively, it is not difficult to show that

$$\begin{aligned} N &= 2m(2n + 1), \\ B &= 6mn + m - n, \\ H &= 2mn - m - n + 1. \end{aligned} \quad (1)$$

### 3. CALCULATIONS

Our method of calculation follows closely the method of molecular orbitals (Lennard-Jones and Coulson, 1939), in which we suppose each of the  $N$  mobile electrons to have a molecular orbit embracing all the nuclei, and represented to our degree of approximation, by a linear sum of separate atomic orbits. Thus if  $\psi_{rs}$  denotes the atomic  $2p_z$  orbit of the carbon atom labelled ( $r, s$ ) in fig. 1, the  $z$ -axis being directed normal to the layer plane, we write for the molecular orbital,

$$\Psi = \sum_{r=1}^{2m} \sum_{s=1}^{2n+1} c_{rs} \psi_{rs}, \quad (2)$$

where the coefficients  $c_{rs}$ , which have a definite set of values for each allowed energy level, are given by solution of the secular equations. There is one secular equation for each of the  $N$  carbon atoms. As usual, electrons are allotted to these orbits, two at a time, with opposed spins, so that the Pauli Exclusion Principle is satisfied, in order of increasing energy. To this degree of approximation the energies are additive, so that the total energy is just the sum of the component energies. There are, of course,  $N$  allowed energies in this "mobile shell"; but, by the "starring process" theorem of Coulson and Rushbrooke (1940) they occur symmetrically about the middle. And thus the lower half (bonding) of the mobile shell is, in each case, completely filled, the upper half (anti-bonding) being completely empty except in excited states with which we shall not be concerned.

The secular equations which give the energy  $E$  and the coefficients  $c_{rs}$  in (2) are

$$c_{rs}(E_0 - E) + \sum_i' \sum_j' c_{ij} \beta_{ij, rs} = 0, \quad 1 \leq i \leq 2m, 1 \leq j \leq 2n + 1, \quad (3)$$

where, if  $\mathbf{H}$  is the effective Hamiltonian for each separate electron,

$$\begin{aligned} \int \psi_{rs} \mathbf{H} \psi_{rs} d\tau &= E_0 && \text{for all } (r, s), \\ \int \psi_{rs} \mathbf{H} \psi_{ij} d\tau &= \beta_{ij, rs} = \beta && \text{if } (r, s) \text{ and } (i, j) \text{ are neighbours,} \\ &= 0 && \text{otherwise.} \end{aligned} \quad (4)$$

There is one equation (3) for each allowed value of  $r$  and  $s$ . The dash ' in the summation denotes that we omit the terms  $i=r, j=s$ .  $E_0$  is the energy of one of these  $\pi$ -electrons when confined to one nucleus, and  $\beta$  is the so-called resonance energy for a single electron jump to a neighbouring nucleus, assumed to have the same value for every pair of neighbours. In this approximation we neglect the overlap integral, putting

$$\begin{aligned} \int \psi_{rs} \psi_{ij} d\tau &= 1 && \text{if } (r, s) \text{ is the same as } (i, j), \\ &= 0 && \text{otherwise.} \end{aligned} \quad (5)$$

In a later paragraph we shall re-examine the legitimacy of this neglect of the overlap between nearest neighbours.

With these approximations the summation in (3) is merely over the neighbours of  $(r, s)$ . If  $(r, s)$  is an internal atom there are three neighbours; if it is a boundary atom there are only two. These types are illustrated by the typical examples below, in which, again, the numbering of the atoms follows that of fig. 1.

*Internal atom (2, 3)*

$$c_{23}(E_0 - E) + c_{13}\beta + c_{22}\beta + c_{33}\beta = 0.$$

*Boundary atom (1, 3)*

$$c_{13}(E_0 - E) + c_{14}\beta + c_{23}\beta = 0. \quad (6)$$

If we put

$$\begin{aligned} \epsilon &= \text{binding energy} = E_0 - E, \\ x &= \epsilon/\beta, \end{aligned} \quad (7)$$

these two equations may be written

$$\begin{aligned} c_{23}x + c_{13} + c_{22} + c_{33} &= 0, \\ c_{13}x + c_{14} + c_{23} &= 0. \end{aligned} \quad (8)$$

There are  $N$  such equations. If we eliminate the  $N$  coefficients  $c_{rs}$  we obtain a determinantal equation (secular equation) of order  $N$ . It is a rather formidable determinant, which we shall not write down explicitly, but Rutherford (1947) has shown that it may be evaluated. He shows that if we introduce quantities  $z_k$  and  $\phi_k$  defined by

$$\begin{aligned} z_k &= 2 \cos \frac{k\pi}{2(n+1)}, \\ x^2 &= 1 + z_k^2 + 2z_k \cos \phi_k, \end{aligned} \quad (9)$$

then the value of the determinant is

$$\Delta = (x^2 - 1)^m \prod_{k=1}^n \frac{z_k^{2m+1} \sin(2m+1)\phi_k + z_k^{2m} \sin 2m\phi_k}{z_k \sin \phi_k}. \quad (10)$$

The roots of the secular equation  $\Delta = 0$  are

$$x^2 = 1 \quad (m \text{ times}), \quad (11)$$

together with the roots (other than  $\phi = 0$  or  $\pi$ ) of the  $n$  equations

$$z_k \sin(2m+1)\phi_k + \sin 2m\phi_k = 0, \quad k = 1, 2, \dots, n. \quad (12)$$

The total number of roots of all these equations must be  $N$ . Now from (1)  $N = 4mn + 2m$ . But the equation (11) contributes  $2m$  roots, and each of the equations (12) contributes  $2m$  roots of  $\phi_k$ , that is,  $4m$  roots of  $x$ . This latter fact is most easily seen by expanding

$$\frac{z \sin(2m+1)\phi + \sin 2m\phi}{\sin \phi}$$



as a power series in  $\cos \phi$  of degree  $2m$ . By combining all the roots thus obtained we complete the necessary total  $N$ . It is well known that all the roots of real Hermitean determinants of this kind are real.

Our procedure has been to choose values of  $m$  and  $n$  (*i.e.* to choose a certain length and breadth for the molecule in fig. 1) and then to solve the equations (12) numerically. In this way we determine the individual energies in terms of the resonance integral  $\beta$ . By summing over the occupied orbits we obtain the total mobile binding energy  $\Sigma \epsilon$ . Let us denote this by  $\mathcal{E}$ . This quantity is important for two reasons. First, it allows us to calculate  $\mathcal{E}/N$ , which is the total mobile binding energy per carbon atom. Secondly, it allows us to calculate  $\mathcal{E}/B$ , which is the total mobile binding energy per carbon-carbon bond.

$\mathcal{E}/N$  is not itself a measure of the resonance energy, but this quantity may easily be calculated as follows. In any single Kekulé structure each carbon atom is attached to one end of a double bond, so that the number of double bonds is  $\frac{1}{2}N$ . If these double bonds were "fixed", the remaining bonds being pure single bonds, their binding energy would be  $\frac{1}{2}N \times 2\beta = N\beta$ . The difference between this and the calculated binding energy, viz.  $\mathcal{E} - N\beta$ , is sometimes called the resonance energy, but is more aptly called the delocalisation energy, since it measures, at least approximately, the gain in energy due to the delocalising, or metallic character, of these electrons. Now a table of  $\mathcal{E}/N\beta$  is given below (Table I) for various values of  $m$  and  $n$ . The resonance energy per atom is found in terms of  $\beta$  by subtracting unity from each entry in the table.

TABLE I.—VALUES OF  $\mathcal{E}/N\beta$ . (MEAN MOBILE BINDING ENERGY PER C ATOM)

$n \backslash m$	1	2	3	4	5	6	7	8	15	19	$\infty$
1	1.3333	1.3684	1.3796	1.3851	1.3886	1.3906	1.3921	1.3935	1.3976	1.3987	1.4028
2	1.3653	1.4122	1.4314	1.4424		1.4547	1.4584				1.4826
3	1.3762	1.4279	1.4507		1.4731	1.4789	1.4835				1.5125
4	1.3817			1.4751		1.4917					
5	1.3851		1.4677								
6	1.3872	1.4442									
7	1.3889										
8	1.3899		1.4774				1.5167		1.5342		
$\infty$	1.3983	1.4612	1.4960	1.5136	1.5240						1.5761

This table also shows certain values for the cases when  $m$  or  $n$  is infinite. These are obtained somewhat differently, and are discussed in paper II.

It may be objected that no allowance has been made for the compression energy of the basic single C—C bonds. As the bond lengths do not change by much from molecule to molecule, this compression energy will be about constant. Thus although Mulliken, Rieke and Brown (1941 and later papers) have shown that this energy is a significant fraction of the total binding energy, we may safely regard it as included in the empirically determined resonance energy  $\beta$ . Thus, if we may anticipate a little, we may use the bond lengths shortly to be deduced, and collected in Table III, to estimate the variations in compressional energy among the molecules of Table I. If we omit the smallest molecule of this system given by  $m=1$ ,  $n=1$ , the mean bond length varies over a total range of 0.02 Å. If we consider only those molecules for which  $m$  and  $n$  exceed 3, the range of bond length is 0.008 Å. Now Mulliken, Rieke and Brown (1941) have shown that in the benzene molecule this compressional energy amounts to 35 KCals per mole for the six bonds. This suggests, on a proportional basis, that if the mean bond length varies over a range of 0.02 Å., variations of  $\beta$  do not exceed  $\frac{1}{2}$  KCal, and if the range is less than 0.008 Å., the variations are less than  $\frac{1}{5}$  KCal. These changes in  $\beta$  are smaller than is implied in the approximation of our model in section 2, and justify us, in this present connection, in neglecting the variations of compressional energy from one molecule to another.

Table II shows the corresponding values of  $\mathcal{E}/2B\beta$ . Now, according to the theory developed by Coulson (1939),  $\mathcal{E}/B$  is immediately related to the average order of the C—C bonds. In fact, the mean mobile bond order  $p$  is given by  $p = \mathcal{E}/2B\beta$ . This is the quantity shown in

Table II. If we know the bond order we can use the curve of order against length to deduce the corresponding bond length. In Table III these mean lengths are shown measured in A. units. There is still some uncertainty with regard to the lengths of the fundamental C—C, C=C and C≡C bonds, particularly the C=C bond. But if we take these as 1.542, 1.331 and 1.202 A. respectively, we obtain the bond lengths given in Table III. In this table we give the lengths to 0.001 A. The uncertainty in C=C means that no reliance whatever may be placed upon the absolute values shown in the table. But there is every reason to believe that the relative values are quite correctly spaced. It may indeed easily happen that the absolute values are not far wrong, for the bond length in the full graphite layer ( $m=n=\infty$ , calculations in a later paper) is predicted to be 1.416 A., and observed values are 1.415 A. (Taylor, 1941) or 1.421 A. (Nelson and Riley, 1945). But even if this almost exact coincidence of values is fortuitous, we may safely claim that the differences between the mean bond lengths for different combinations of  $m$  and  $n$ , as given in Table III, are significant. It will be recognised that the figures given in these tables are entirely independent of the precise numerical value of  $\beta$ .

TABLE II.—VALUES OF  $\mathcal{E}/2B\beta$ . (MEAN MOBILE ORDER OF CARBON BONDS)

$n \backslash m$	1	2	3	4	5	6	7	8	15	19	$\infty$
1	0.6667	0.6220	0.6036	0.5936	0.5875	0.5832	0.5800	0.5778	0.5701	0.5682	0.5611
2	0.6301	0.5884	0.5726	0.5644		0.5562	0.5538				0.5389
3	0.6194	0.5789	0.5642		0.5524	0.5493	0.5472				0.5338
4	0.6141			0.5532		0.5463					
5	0.6111		0.5584								
6	0.6090	0.5701									
7	0.6076										
8	0.6065		0.5553				0.5401		0.5336		
$\infty$	0.5993	0.5620	0.5512	0.5449	0.5408						0.5254

TABLE III.—MEAN CARBON-CARBON BOND LENGTHS (A.)

$n \backslash m$	1	2	3	4	5	6	7	8	15	19	$\infty$
1	1.389	1.397	1.400	1.402	1.403	1.404	1.405	1.405	1.407	1.408	1.413
2	1.396	1.403	1.406	1.408		1.409	1.410				1.415
3	1.399	1.404	1.408		1.411	1.411	1.412				1.415
4	1.399			1.410		1.412					
5	1.399		1.409								
6	1.399	1.406									
7	1.400										
8	1.400		1.410				1.413		1.415		
$\infty$	1.402	1.409	1.411	1.412	1.413						1.416

## 4. DEDUCTIONS FROM RESULTS OF TABLES I-III

We may make several deductions from the values shown in Tables I-III.

(a) Table I shows that an increase in size of crystallite always leads to a greater binding energy per atom. This is found along every column and every row of the table. We shall shortly compare this with the results of Taylor, Eyring and Sherman, who found that 5 sodium atoms were less stable than 4. Our table shows that after the first few atoms are put together  $\mathcal{E}/N\beta$  changes only very slowly from structure to structure, gradually increasing with the size of the molecule. This increase, to be sure, has only been established for a change from one of our regular symmetrical structures to the next one in either direction: and at first, with small molecules, we may expect an irregular variation of  $\mathcal{E}/N\beta$  when we add 1 or 2 further atoms. Such erratic behaviour, however, would not be anticipated when the structures are larger, and we may suppose that after a certain minimum size  $\mathcal{E}/N\beta$  becomes a smooth function of  $N$ .

Combining this conclusion with the work of Taylor, Eyring and Sherman, it follows that it is not until a crystallite has built up to at least 20 or 30 atoms that it possesses the power of attracting further atoms with increasing facility. This is in keeping with a good deal of evidence summarised by Taylor, Eyring and Sherman, and fits in with the minimum number of about 25 atoms for crystallisation nuclei.

These calculations, we must emphasise, refer to the resonance energy of the  $\pi$ -electrons, but they would probably be valid for other types of binding. However, we must exercise a little caution in extending them directly. For in our case only one half of the allowed levels of the molecular  $\pi$ -electrons are filled: in terms of the familiar theory of metals, the energy band from the  $\pi$ -electrons is only half filled, since it will accommodate two electrons per atom, and our systems provide only one. If we had completely filled the band, our approximation would have given a total energy equal to  $NE_0$ , and resonance would have played no part in the total binding. Thus what we have shown is that with an incompletely filled, or half-filled, band, the larger the layer the more bonding it becomes.

(b) If we consider the two homologous series represented by  $m=1$  (polyacenes) and  $n=1$  (polyphenyls), we see from Tables I and II that when we compare systems with an equal number of hexagons, as given by the formula in (1), then the resonance energy per atom is slightly greater for the polyacene series than for the polyphenyl series, but that the energy per bond is distinctly less. However, when we consider, in section 5, the effect of introducing the overlap integral, previously neglected, we find that both the energy per bond and the energy per atom are greater for the polyphenyl series. It seems most likely that this latter calculation is the more accurate. This greater stability of the polyphenyl systems may be attributed to the larger number of unexcited Kekulé structures which are possible.

(c) Table III shows that although there is a distinct increase in the length of the carbon-carbon bond as the size of the crystallite increases, this increase is not large. By the time that as many as 10 hexagons have been built together, the mean bond length differs from that of the infinite layer by not more than 0.01 Å. This is in general agreement with the experimental results of Taylor (1941). J. H. de Boer (1940) has deduced a similar increase as the crystallite increases in size, but his argument, which differs from ours, predicts considerably larger bond length changes than now seem to occur. De Boer's argument appears not to make an adequate distinction between  $\sigma$ - and  $\pi$ -type electrons on the boundary atoms.

(d) Table II shows that increasing the size of the crystallite in either direction reduces the order of the bond, so that, as Table III shows, the carbon-carbon distance is actually increased. This is, at first, a rather surprising condition, for we should have expected greater resonance to be associated with shorter bonds. It is due to the fact that the resonance energy per atom depends on  $\mathcal{E}/N$ , whereas the mean bond order depends on  $\mathcal{E}/B$ . As (1) shows,  $N$  and  $B$  vary differently in terms of  $m$  and  $n$ .

(e) Table I allows us to discuss the effect of resonance energy on the shape; and in particular to answer the question: if the crystallite layer contains  $N$  carbon atoms, will the resonance, or mobile, character of the  $\pi$ -electrons favour a long, thin, filamentous structure or a more nearly square one? It is most important to recognise that there are other factors as well as the resonance energy which help to determine the shape. Thus the availability of other atoms (H, N, O) to saturate the edge trigonal valencies, the relative strengths and numbers of these bonds in relation to the basic single C—C bonds, are factors that need to be considered. We shall make some rather tentative suggestions regarding their effects later in this paper. But it is desirable to distinguish these various factors and to consider separately their influence on the shape.

Now if  $N$  is constant, the permitted values of  $m$  and  $n$  are such as to lie on approximately parallel curves running from bottom left to top right of Table I. It is evident that the resonance energy per atom is greatest somewhere near the middle of such lines. We conclude that resonance, considered as an effect by itself, prefers an approximately square, or circular structure rather than a long strip either of polyphenyl or polyacene type.

We may make this discussion of the effect of resonance on shape more explicit by comparing together three different structures for all of which  $N=66$ . If we refer to the structure in terms of  $(m, n)$ , these are (11, 1), (1, 16) and (3, 5). The first of these is a polyphenyl chain of 11 hexagons, the second is a condensed polyacene with 16 hexagons side by side, and the third is

very nearly square. The resonance energies per atom are now strictly comparable. They are shown below.

	Polyphenyl (11, 1)	Polyacene (1, 16)	Square (3, 5)
Resonance energy per atom	$0.392\beta$	$0.398\beta$	$0.473\beta$

This makes it quite clear that resonance itself gives a strong tendency to favour a square rather than an oblong configuration. If  $\beta$  is taken to be about 20 KCals, the energy difference above is of the order of 1.5 KCals per atom. This is significantly large.

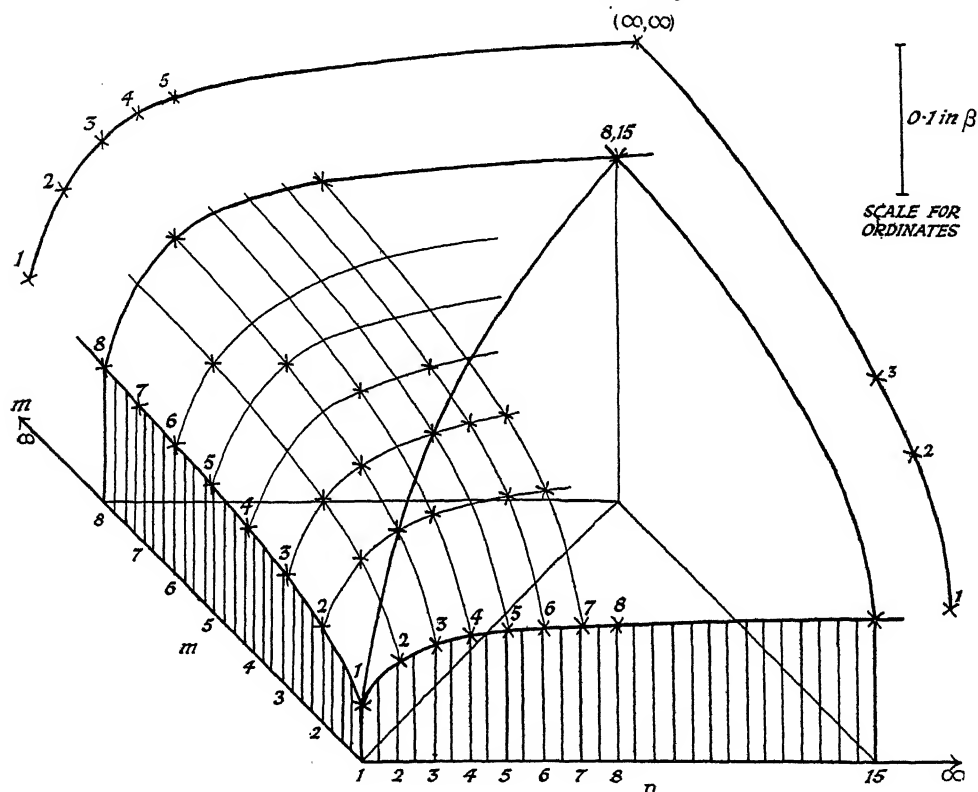


FIG. 2.—Graphical representation of energy per atom in terms of  $m$  and  $n$ . Values of  $\mathcal{E}/N\beta$  are shown by the height above the basal plane. This plane corresponds to a value  $\mathcal{E}/N\beta = 1.3$ . Note the scale of energy in the top right-hand corner.

This dependence of  $\mathcal{E}/N\beta$  upon the values of  $m$  and  $n$  may be exhibited graphically by regarding  $\mathcal{E}/N\beta$  as a function of  $m$  and  $n$ , and plotting a surface in which the base plane contains the  $m$  and  $n$  axes, and in which the ordinate is the value of  $\mathcal{E}/N\beta$ . This ordinate is, of course, strictly determined only for integral  $m$  and  $n$ , but as the perspective diagram of fig. 2 shows, the calculated points lie on a smooth surface. In fig. 2 the height of the basal plane corresponds to  $\mathcal{E}/N\beta = 1.3$ . The thin lines are lines of constant  $m$  and  $n$  along the surface, and the line of greatest slope (*i.e.* the line joining points of greatest resonance energy per atom for a given total number of atoms  $N$ ) is shown along the diagonal. The crosses mark the calculated points.

(f) The variations in  $\mathcal{E}/N\beta$  shown in Table I imply that the heats of combustion of various shapes and sizes of crystallite are not by any means identical. If allowance is not made for this, significant errors may be made. For example, if we consider the (3, 5) structure, for which  $\mathcal{E}/N\beta = 1.4731$  and compare it with the infinite layer for which  $\mathcal{E}/N\beta = 1.5761$ , or the (8, 15) structure where  $\mathcal{E}/N\beta = 1.5342$ , and if we take  $\beta = 20$  KCals approximately, the corresponding differences in heat of combustion or sublimation amount to about 2.1 and 1.2 KCals per mole respectively. It is interesting to note here that the crystal layers found by Riley

(1939) actually have sizes intermediate between those of the (3, 5) and (8, 15) structures. Such differences in energy are within the accuracy of modern precision calorimetry. If the heat of sublimation of graphite is taken to be 124.1 KCals per mole (Herzberg, 1937) they amount to between 1 and 2 per cent.

Now it happens that there are some extremely careful experiments of Dewey and Harper (1938) which provide some indication of the validity of our work. These writers prepared anthracite cokes at temperatures between 900 and 1300° C., and they found a steadily diminishing hydrogen content. Riley (1939) would have expressed this by saying that the cokes prepared at the higher temperatures had a larger condensed aromatic structure than those at the lower temperature. According to our present account this should imply a larger resonance energy, and hence a smaller heat of combustion. Dewey and Harper measured the heat of combustion as a function of the percentage of hydrogen and found a smooth variation. In particular, the difference in heat of combustion between cokes prepared at 900° and 1300° was no less than 2.4 KCals per mole. Assuming that the hydrogen had the same heat of combustion as ordinary gaseous hydrogen, they calculate that this difference would have been expected to be 3.0 KCals per mole. The remaining 0.6 KCals per mole, which is considerably greater than the experimental error, must lie in a difference in energy between the two cokes. This will itself arise partly from the increased size, so that we may expect differences of this order of magnitude in the resonance energy of the two sizes of layer. In the case of graphite layers, a complete numerical correlation cannot be expected till we have a fuller understanding of the proportion and location of the small amounts of inorganic matter present in the cokes, and of the nature of the intercrystalline boundaries. But it is satisfactory that the order of magnitude of energy difference that we predict theoretically agrees so nicely with that actually obtained.

Another rough comparison of order of magnitude may be made with some equally accurate experiments of Jessup (1938), who measured the heat of combustion of diamond and found variations of 0.12 KCals per mole between diamonds of size 2.5  $\mu$  and 39.5  $\mu$ . Only one quarter of this could be attributed to free valencies at the edge of the crystallite; as there are no foreign atoms present, the rest must presumably be largely due to differences in resonance energy on account of the difference in size. And, in fact, the direction of the change is in accord with this interpretation. Strict comparison is not possible because diamond is a three-dimensional crystal and we have dealt only with two-dimensional ones. Also, the sizes of Jessup's crystallites were rather larger than those for which we have made our calculations. But the importance of the comparison is that it shows again that our order of magnitude for change of resonance energy with size is indeed correct. Further work on this and other crystallites may enable a more precise test of the absolute magnitude to be made.

(g) The considerations above lead to the further question: how large must a layer of carbon atoms be before it may be regarded as graphite and not as a large molecule approximating to graphite? The answer to this question, which is given from Tables I to III, seems to be that we require something like 50 carbon atoms (the (3, 5) structure has  $N = 66$ ), and these must form a condensed system with at least two, and preferably three, hexagons in each direction. Such a structure would have an area of not less than 100 square Angstroms.

## 5. INCLUSION OF THE OVERLAP INTEGRAL

In this section we reconsider one of the approximations that was previously made in section 3. We assumed in (5) that the overlap integral  $S$  between the atomic orbitals of two adjacent carbon atoms was negligible. Actually its value is about 0.25. In view of this it behoves us to examine the validity of arbitrarily neglecting it. The matter has been discussed for simpler molecules by Wheland (1942), who showed that no appreciable error was involved in this simplification. We shall adopt a somewhat similar procedure here.

We are to include the overlap between neighbouring orbitals, giving all such integrals the same value  $S$ . If we include  $S$ , the fundamental secular equations (6) become

$$\begin{aligned} c_{23}(E_0 - E) + (c_{13} + c_{22} + c_{33})(\beta - ES) &= 0, \\ c_{13}(E_0 - E) + (c_{14} + c_{23})(\beta - ES) &= 0. \end{aligned} \quad (13)$$

Let us put  $E_0 - E = \epsilon$ , as in (7), and also

$$\gamma = (E_0 - E)/(\beta - ES). \quad (14)$$

Then the fundamental equations (13) become exactly the same as (8) except that the variable is now  $\gamma$  and not  $x$ . Our previous analysis may be applied just as before, but the roots of the secular determinant give us  $(E_0 - E)/(\beta - ES)$  and not  $(E_0 - E)/\beta$ . We may, however, transform (14) by putting

$$\beta - E_0 S = \gamma. \quad (15)$$

Then

$$E_0 - E = \epsilon = \frac{\gamma\gamma}{1 - S\gamma}. \quad (16)$$

According to (16), the binding energy of each orbital is found in terms of the new integral  $\gamma$  which replaces the former  $\beta$ . It is a simple matter to convert all the previous energy levels into the revised form, and then to determine the total mobile energy  $\mathcal{E} = \sum \epsilon$ . From this we could proceed, just as in sections 3, 4, to draw up tables of  $\mathcal{E}/N\gamma$  and  $\mathcal{E}/2B\gamma$ . But we shall only reproduce a table of  $\mathcal{E}/N\gamma$ , as our object is merely to confirm that the inclusion of  $S$  makes no more difference to the order of energies than with the simpler molecules discussed by Wheland.

It may perhaps be mentioned that the inclusion of  $S$  introduces a considerable asymmetry into the energy distribution—as is shown for infinite strips in paper II—although it will be seen from (16) that previously bonding orbitals remain bonding, and anti-bonding ones remain anti-bonding. We hope to discuss this asymmetry in another place. In our notation  $\gamma$  is a negative constant, so that we obtain bonding orbitals by taking all the negative values of  $\gamma$ ; *i.e.* all the negative roots of the former secular determinant. It will also be recognised that apart from a normalising constant, the coefficients  $c_{rs}$  in the final molecular orbitals are unaffected by the inclusion of  $S$ .

Our calculations have been carried through, where definite numerical values are required, under the hypothesis that  $S = 1/4$  and  $\gamma = -40$  KCals. Wheland (1942) suggests  $\gamma = -38$  KCals, and Mulliken, Rieke and Brown take a somewhat larger value. A really strict theory would need to take account of the individual variations of  $S$  and  $\gamma$  for the links of different length. But such a procedure is not justified for our present purposes, and it would certainly not change the nature of our conclusions. These are:

(a) An increase of either  $m$  or  $n$  increases the binding energy per atom, so that the perspective diagram of  $\mathcal{E}/N\gamma$  closely resembles that shown in fig. 2 for  $\mathcal{E}/N\beta$ . Table IV shows the actual values of  $\mathcal{E}/N\gamma$ .

TABLE IV.—VALUES OF  $\mathcal{E}/N\gamma$ . BINDING ENERGY PER ATOM WITH ACCOUNT TAKEN OF THE OVERLAP  $S$

$n \backslash m$	1	2	3	4	5	6	7	8	15	19	$\infty$
1	0.9778	0.9864	0.9861	0.9856	0.9849	0.9843	0.9839	0.9836	0.9825	0.9822	0.9812
2	0.9876	1.0027	1.0083	1.0113		1.0159	1.0171				1.0254
3	0.9912	1.0093	1.0177		1.0269	1.0289	1.0312				1.0434
4	0.9930			1.0299		1.0367					
5	0.9941		1.0267								
6	0.9948	1.0165									
7	0.9953										
8	0.9955		1.0321				1.0507		1.0589		
$\infty$	0.9984	1.0242	1.0430								

(b) There is only one exception to the rule (a), notably the polyacene series with  $m = 1$ . Here, for  $n$  greater than 2,  $\mathcal{E}/N\gamma$  actually decreases very slightly with increase of  $n$ . This predicted decrease, which is less than  $\frac{1}{2}$  per cent., makes no difference to our final conclusion (c), though it does leave undecided the relative stabilities per atom (not per bond) of the  $\pi$  electrons in the polyacene and polyphenyl series.

(c) The greatest stability for a given total number  $N$  of atoms occurs in the approximately square configuration. We give, below, the resonance energies per atom for the three structures mentioned in section 4, and for which  $N = 66$ .

	Polyphenyl (11, 1)	Polyacene (1, 16)	Square (3, 5)
Resonance energy per atom	0.196γ	0.183γ	0.227γ

The preference for a square shape is again shown clearly; in fact the (3, 5) structure is the most stable of the three by about 14 per cent. In the earlier approximation this figure was 17 per cent. Numerically these two extra stabilities amount to 1.24 and 1.50 KCals respectively.

All this shows that the effect of introducing the overlap  $S$  is not to make any really significant change in our earlier conclusions.

## 6. STABILITY WITH RESPECT TO HYDROGEN

We have already discussed, in section 4 (e), the effect of resonance energy on the shape of a crystallite layer, showing that this makes for a square rather than an oblong crystallite. But, as we then emphasised, resonance energy is not the only factor influencing this shape. The total energy of a crystallite comprises three terms:

- The energy of the underlying carbon structure of single carbon-carbon bonds.
- The energy of the mobile  $\pi$ -electrons (effectively, the resonance energy which we have already considered).
- The boundary energy (*cf.* section 4 (e)).

We have still to include the contributions from (a) and (c).

The energy contribution (c) arises from the necessity of our saturating the free valencies of those carbon atoms, on the boundaries of the crystallite, which have only two carbon neighbours. The energy contribution of a free valency of this kind has never been calculated, and it is not our present purpose to do so. We shall rather assume that the whole carbon crystallite is edged with hydrogen atoms. We thus confine our attention to pure hydrocarbons. This enables us most simply to illustrate the effect of the energy contribution (c). We shall further, for definiteness, assume that the hydrogen atoms round the boundaries of the crystallite are drawn from a sufficient supply of diatomic hydrogen. Then, since there are  $6nm + m - n$  C—C  $\sigma$ -bonds, and  $4m + 2n$  C—H  $\sigma$ -bonds, the combined effect of the energies (a) and (c) is given by

$$(6nm + m - n)\mathcal{E}_{\text{C-O}} + (4m + 2n)(\mathcal{E}_{\text{C-H}} - \frac{1}{2}\mathcal{E}_{\text{H-H}}),$$

where  $\mathcal{E}_{\text{C-O}}$  denotes the energy of a C—C single ( $\sigma$ )-bond, etc. Now, for a given number of carbon atoms, it is easily shown that this part of the energy is *smallest* for a square crystallite, provided that  $2\mathcal{E}_{\text{C-H}} > \mathcal{E}_{\text{C-O}} + \mathcal{E}_{\text{H-H}}$ , which is almost certainly true and will, indeed, be the case for almost all conceivable edge atoms. In other words, the energy contributions (a) and (c) act in opposition to the energy contribution (b) in that while resonance energy makes for square, disc-like crystallites, the energy contributions (a) and (c) together favour long, oblong structures. In fact we shall find that, in our particular field of pure hydrocarbons, the effect of the energies (a) and (c) is as important as the resonance energy (b).

The three structures, viz. (11, 1), (1, 16) and (3, 5), whose resonance energies we have already compared in section 4 (e), provide a good illustration. Following Pauling (1939), we shall take the energies  $\mathcal{E}_{\text{C-O}}$ ,  $\mathcal{E}_{\text{C-H}}$  and  $\mathcal{E}_{\text{H-H}}$  as 58.6, 87.3 and 103.4 KCals respectively. As is usual in these problems, any other self-consistent set of alternatives to these would give equivalent results. Then, with  $\beta = 19.5$  KCals (Wheland, 1934):

(i) Polyphenyl (11, 1)	C—C bonds, $76 \times 58.6$	= 4453.6
	C—H bonds, $46 \times 87.3$	= 4015.8
	$\pi$ -electrons, $66 \times 19.5 \times 1.392$	= 1791.5
		<hr/>
		10260.9

(ii) Polyacene (1, 16)	C—C bonds, $81 \times 58.6$	= 4746.6
	C—H bonds, $36 \times 87.3$	= 3142.8
	$\pi$ -electrons, $66 \times 19.5 \times 1.398$	= 1799.2
		<hr/> 9688.6
	$5H_2 = 12 \times 103.4$	= 517.0
		<hr/> 10205.6
(iii) Square (3, 5)	C—C bonds, $88 \times 58.6$	= 5156.8
	C—H bonds, $22 \times 87.3$	= 1920.6
	$\pi$ -electrons, $66 \times 19.5 \times 1.473$	= 1895.8
		<hr/> 8973.2
	$12H_2 = 12 \times 103.4$	= 1240.8
		<hr/> 10214.0

This shows that the "square" is now less stable than the polyphenyl. This conclusion is reinforced if we take account of the overlap integral (section 5). Taking  $\gamma = 38$  KCals, the three energies for comparison become 10967, 10871 and 10894 KCals respectively. This is satisfactory in that the polyphenyl energy is now sufficiently greater than that of the other two for us to be confident that the relative order will not be affected by the effects of steric hindrance between the boundary hydrogens in the former case, and in agreement with the experimental experience that polyphenyls are more readily prepared than polyacenes. We must stress, however, that the calculated energies are relative energies only; we have not attempted to give a numerical value to the constant energy contribution  $66 E_0$ . Further, although the figures shown relate to a particular choice of the resonance integrals  $\beta$  and  $\gamma$ , small alterations in the values of these two quantities would not affect the conclusions that we have just made.

There remains, however, the wider question of whether, on the basis of energy considerations alone, we should expect small units to join together to form larger ones. We have made a number of calculations in this field, but the results are too inconclusive to be given in any great detail. Two examples must suffice.

(i) The energies of two polyphenyls  $m=4$ ,  $n=1$  amount (without overlap) to 7600 KCals, while that of the single polyphenyl  $m=8$ ,  $n=1$ , together with a hydrogen molecule, is 7596 KCals; suggesting that polymerisation will not take place. Including the overlap integral, the two energies for comparison become 8118 and 8110 KCals respectively.

(ii) The energies of two polyacenes  $m=1$ ,  $n=4$  amount (without overlap) to 5529 KCals, while that of the polyacene  $m=1$ ,  $n=8$ , together with the ethylene molecule  $C_2H_4$  which we can form from the unused carbon and hydrogen atoms, amounts to 5522 KCals; again suggesting, though not conclusively, that polymerisation will not take place. Including the overlap integral (and taking  $\gamma = 38$  KCals), the energies for comparison become 5905 and 5869 KCals, so that our former tentative conclusion receives stronger support.

The results of many such calculations, involving more elaborate structures than those we have here considered, lead us to the following observations:—

- The energies of the various sets of hydrocarbons which it is possible to form from a given number of hydrogen and carbon atoms are surprisingly close together, and to the accuracy of our present knowledge of  $\beta$  and  $\gamma$  it is barely possible to distinguish between them.
- In no case examined do the energy considerations suggest that simple structures will combine to form large laminar crystallites.
- The inclusion of the overlap integral strengthens, rather than weakens, the basis of observation (b) above.

Therefore, tentatively, we conclude that on the basis of energy considerations alone no such building-up process will occur at low temperatures.

But we must stress that our conclusion, if valid, applies strictly only at the absolute zero



of temperature. We have not taken account of energy contributions from excited electronic states, nor of the energies and entropies associated with the vibrations, etc. of the hydrocarbon structures. Particularly in the case of the zero-point energy this may be important in deciding the relative stabilities of small and large crystallites. On the other hand, experimental support may be lent to our conclusion from the observation (Blayden, Gibson and Riley, 1943) that it is only at relatively high temperatures that hydrogen is given off from coals, and larger graphitic crystallites begin to form.

## 7. CONCLUSION

The methods and techniques illustrated in this paper may be applied to other structures, in particular to three-dimensional crystallites as well as two-dimensional ones. The writers are considering the problem of a system of atoms in a cubic lattice, the atoms being supposed to be in  $s$ -states. In the case of insulators, where the whole of the conduction band is filled, there are considerable divergences from symmetry in the density distribution of the levels.

It is possible to treat other shapes of graphite layer in the same way. For example, we may consider the energies of layer planes having the shape of a lozenge, or parallelogram, with angle  $60^\circ$ . We have not, however, thought it worth while to report such calculations here, for they would only confirm the general conclusions already reached with the rectangular arrays.

Taylor, Eyring and Sherman have stressed the fact that many surface films of atoms, containing one or two layers of adsorbed atoms, have markedly different properties from those associated with a three-dimensional structure. It should be possible to consider such systems along the lines of this present paper.

In conclusion, our thanks are due to the British Iron and Steel Research Association for providing a calculating machine with which parts of these calculations were performed.

## SUMMARY

Calculations are made of the resonance energy, bond order and bond length in a series of graphitic layers of varying size. Carbon-carbon bond lengths appear to vary very little in size with increasing number of carbon atoms, in agreement with experiment. But variations in resonance energy are significant, and indicate clearly that resonance, by itself, favours an approximately square, rather than oblong, shape. But in the case of such layers in equilibrium in the presence of molecular hydrogen, the most stable layer containing a given number of carbon atoms is of the long, thin polyphenyl type. Some tentative calculations suggest that polymerisation of smaller groups to larger ones should be endothermic, in agreement with the experimental fact that the formation of larger graphitic crystallites during carbonisation occurs, with emission of hydrogen, only at high temperatures.

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(Issued separately September 22, 1948)

XXXVI.—Graphite Crystals and Crystallites. II. Energies of Mobile Electrons in Crystallites Infinite in One Direction. By C. A. Coulson (Wheatstone Physics Laboratory, King's College, London) and G. S. Rushbrooke (Chemistry Department, Leeds University). (With Three Text-figures.)

(MS. received September 18, 1946. Read February 3, 1947)

I. INTRODUCTION

IF, in the notation of paper I in this series (Bradburn, Coulson, and Rushbrooke, preceding paper; see especially fig. 1), the numbers  $m$  and  $n$  are both very large, then the corresponding crystallite becomes equivalent to a two-dimensional plane graphite layer of effectively infinite extent. We shall not, however, discuss this particular case at present, but confine our attention in this paper to those crystallites in which either  $m$  or  $n$  is large, but not both. Since by "large" we mean "effectively infinite", we shall refer to such a crystallite as an *infinite strip*. The width of the strip will then be denoted by the parameter,  $m$  or  $n$ , which is still finite. Fig. 1 shows sections of some typical strips, and, in conjunction with fig. 1 of the preceding paper, will sufficiently explain the nomenclature.

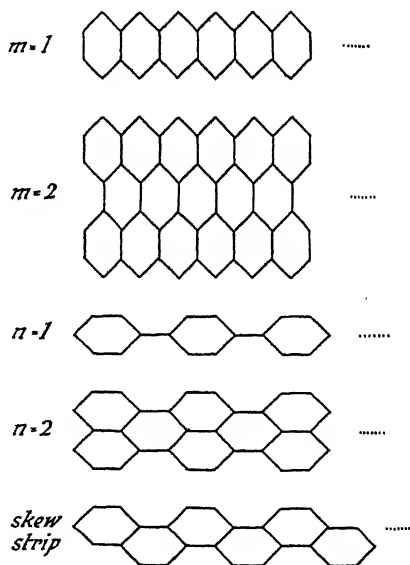


FIG. 1.—Some typical graphitic strips for which the energy bands have been calculated.

It is not suggested that graphitic strips of this kind have any chemical interest or importance. It is, perhaps, rather unlikely that they would ever arise in practice. But they do have a certain theoretical interest, in that they provide a one-dimensional analogue of a metal (*cf.* Rijnaw, 1934). As either  $m$  or  $n$  becomes very large, the possible energy levels for the mobile electrons become closer and closer together, eventually forming continuous "bands" of energy levels very much as in the well-known theory of ordinary three-dimensional metals. And we shall find that some infinite strips would behave as conductors and others as insulators. We end with a discussion of the extent to which such behaviour can be predicted from simple chemical bond diagrams.

We have confined our attention entirely to determining the energies of the mobile electrons, and we shall not deal either with bond lengths or total binding energies. These are of rather secondary importance for such hypothetical structures.

## 2. ENERGIES OF MOBILE ELECTRONS FOR INFINITE STRIPS: WITHOUT OVERLAP

As in paper I we shall first ignore the influence of the overlap integral  $S$ , and later consider how the results so obtained have to be modified when this is included. In I the effect of this overlap integral was found to be relatively unimportant so far as binding energies were concerned: most of the effect of  $S$  could, in fact, be absorbed into the empirically determined resonance integral  $\beta$ . Although this still remains true of the binding energies of the infinite strips discussed here, the influence of the overlap integral on the shapes of the energy bands is pronounced, as we shall see in § 3.

(a)  $n$  finite,  $m \rightarrow \infty$

When  $n$  is finite and  $m$  tends to infinity, the energy per carbon atom of the mobile electrons can be calculated at once from equation I (10) (equation (10) of paper I). For each of the  $n$  values of  $k$ , the  $2m$  values of  $\phi_k$  which make  $\Delta = 0$  are easily seen to be uniformly distributed over the range  $0 \leq \phi_k < \pi$ . Consequently, from equation I (9), the limiting value of  $\mathcal{E}/N\beta$  as  $m \rightarrow \infty$  is given by

$$\frac{\mathcal{E}}{N\beta} \rightarrow \frac{1}{2n+1} + \frac{2}{2n+1} \sum_{k=1}^n \frac{1}{\pi} \int_0^\pi \sqrt{(1+z_k^2+2z_k \cos \phi)} d\phi, \quad (1)$$

where

$$z_k = 2 \cos \frac{k\pi}{2(n+1)}.$$

The first term on the right-hand side of (1) derives from the  $m$  roots  $x=1$  of the equation  $\Delta=0$  (remembering that  $N=2m(2n+1)$  and each occupied level is doubly filled). Each integral in (1) can be expressed straightforwardly as a complete elliptic integral, and its numerical value found from the tables. In this way limiting values of  $\mathcal{E}/N\beta$  for the infinite strips  $n=1, 2, \dots, 5$ ,  $m \rightarrow \infty$  have been computed, and are given in Table I.

TABLE I

$m \rightarrow \infty$ $n =$	1	2	3	4	5
$\mathcal{E}/N\beta$	1.3983	1.4612	1.4960	1.5136	1.5240
$\mathcal{E}/N\gamma$	0.9984	1.0242	1.0430	..	..
$n \rightarrow \infty$ $m =$	1	2	3	Skew Strip	
$\mathcal{E}/N\beta$	1.4028	1.4826	1.5125	1.4373	
$\mathcal{E}/N\gamma$	0.9812	1.0254	1.0434	1.0144	

Moreover, from (1) we can also derive the actual distribution of the individual energy levels. For let  $N(\epsilon)d\epsilon$  denote the number of levels in the range  $\epsilon$  to  $\epsilon+d\epsilon$ . Then  $N(\epsilon)$  is the sum of

- (i)  $m$  pairs of roots  $\epsilon/\beta = \pm 1$ ,
- (ii)  $n$  energy bands of density  $N_k(\epsilon)$ ,  $k=1, 2, \dots, n$ ,

each band  $k$  corresponding to the value of  $k$  in I (9).  $N_k(\epsilon)$  is easily calculated if we remember that there are  $2m$  levels uniformly distributed between  $\phi_k=0$  and  $\phi_k=\pi$ , and that  $x dx = -z_k \sin \phi_k d\phi_k$ . In fact,

$$\begin{aligned} N_k(\epsilon) &= \pm \frac{x}{z_k \sin \phi_k} \frac{2m}{\pi} \\ &= \pm \frac{4m}{\pi\beta} \frac{\epsilon/\beta}{\sqrt{4z_k^2 - (1+z_k^2 - (\epsilon/\beta)^2)}}. \end{aligned} \quad (2)$$

When we use the  $\pm$  sign, here and later, it must be understood that the choice must always be such as to make  $N$  non-negative.

Some of these energy bands are shown in fig. 2a. But before they can be plotted they must be "normalised"; details of the normalisation adopted are given below (§ 3).

(b)  $m$  finite,  $n \rightarrow \infty$

Except in the particular case  $m=1$ , there is no simple formula for the mean energy, or the density of the energy levels, for a strip of the type  $n \rightarrow \infty$ .

When  $m=1$ , straightforward calculations give

$$\frac{\epsilon}{\beta} = \pm \frac{1}{2} \left\{ 1 \pm \sqrt{9 + 8 \cos \frac{k\pi}{n+1}} \right\}, \quad k=1, 2, \dots, n,$$

so that the mean energy is given on evaluating a complete elliptic integral, as in case (a). So also, in this particular case, we can find the distribution of the roots analytically, obtaining

$$N(\epsilon) = \pm \frac{n}{\pi\beta} \{M(y_1) + M(y_2)\},$$

where

$$M(y) = y \sqrt{1 - \left( \frac{4y^2 - 9}{8} \right)^2},$$

and

$$y_1 = \frac{\epsilon}{\beta} + \frac{1}{2}, \quad y_2 = \frac{\epsilon}{\beta} - \frac{1}{2}.$$

There is, however, no concentration of isolated roots in this case at  $\frac{\epsilon}{\beta} = \pm 1$ .

For  $m > 1$  no such analysis is possible, and we must proceed numerically. The calculations are laborious, and we have dealt only with the two further cases  $m=2$  and  $m=3$ . The continuous variable is now  $z$  or  $\theta$ , where  $z = 2 \cos \theta$  and  $\theta = k\pi/2(n+1)$ . The method adopted was to find numerically the energies corresponding to evenly spaced values of  $\theta$  over the range  $0 < \theta < \pi/2$ . We actually took  $\theta = 0, \pi/16, \pi/8, \dots, \pi/2$ , and for each value of  $\theta$  we located the values of  $\phi$  for which  $\Delta = 0$ . The corresponding energies were then found, without trouble, from I(9). These energies lie in  $2m$  bands, and we have located nine points on each band. The total energy of the mobile electrons, and hence the mean energy per atom, can now be found by numerical integration over each band, using Simpson's rule or an appropriate eight-strip formula. The results so obtained are given in Table I. We believe that they are correct to all the figures given.

Similarly the distribution, or density, of the energy levels can be found by numerical differentiation of each energy band. This was done graphically, and consequently the numerical results are not particularly accurate; but the energy bands shown in fig. 2b will be sufficiently reliable for comparison purposes.

(c) A "Skew" Strip

We have considered also one other such infinite strip, a "skew" strip of which the cross-section is shown in fig. 1. The general theory of the energy levels of such a strip is not included in the results of I, and is given in outline in the Appendix to the present paper. The mean energy of a mobile electron and the distribution of the energy levels are given by analytical formulæ similar to those above. Numerically the results are shown in Table 1 and fig. 2b. The chief interest of this strip lies in a comparison of the mean energy of its mobile electrons with those of the strip  $m=1$ ,  $n \rightarrow \infty$ . While the number of C—C bonds (and C—H bonds at the edges) is the same for the two strips, the resonance energy of the mobile electrons is greater for the skew array. Thus the skew array, *i.e.* the more "condensed" array, is the more stable. As a matter of fact, the increased stability of the skew type holds for smaller as well as larger systems: the simplest example is that phenanthrene, with three rings in a skew configuration, is about 14 KCals more stable than anthracene, with three rings in a straight row (Wheland, 1944, p. 69).

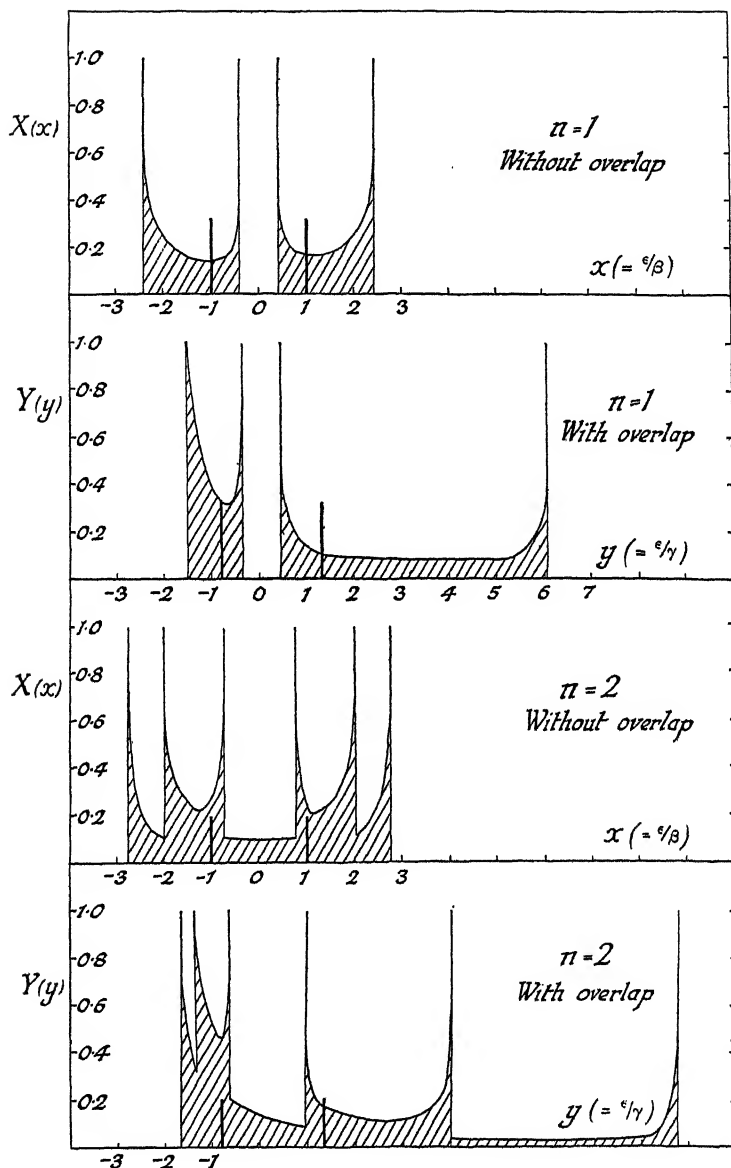


FIG. 2a.—Density distribution of energy levels for the infinite polyphenyl-type strips. For a description of the strips see fig. 1. The thick black ordinates denote concentrated levels.  $X(x)$  and  $Y(y)$  denote the densities without and with inclusion of overlap; in each case the normalisation is such that the total area under the curve, including the concentrated levels, is 2.

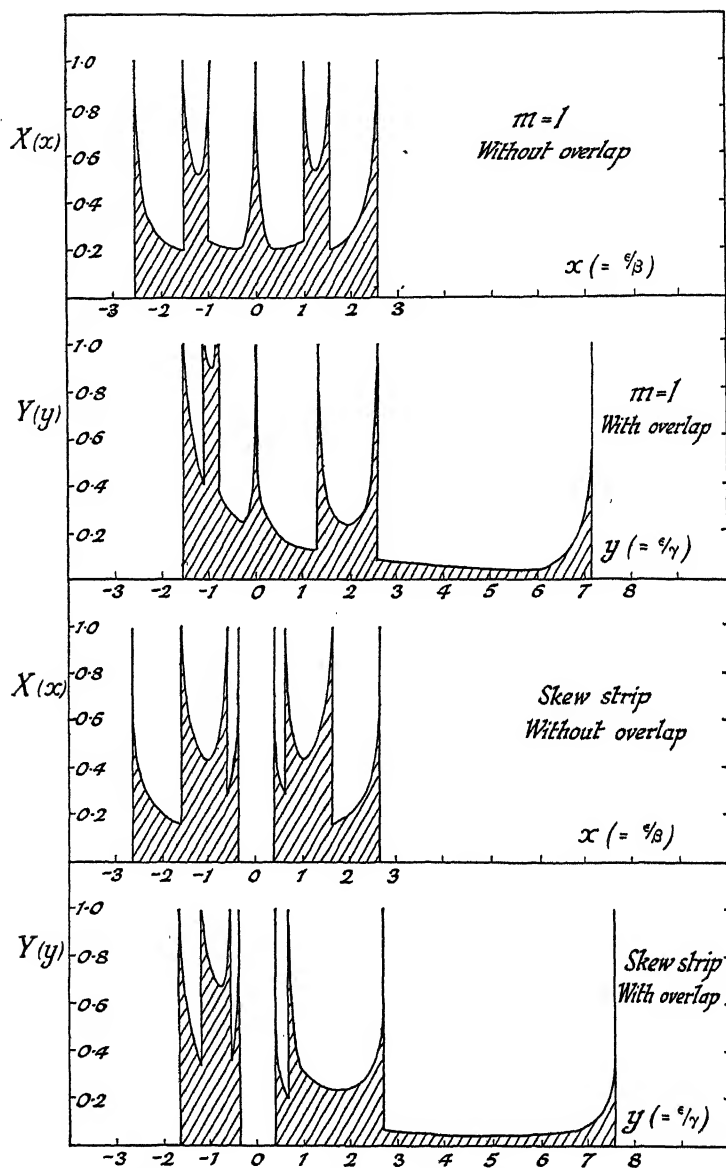


FIG. 2b.—Density distribution of energy levels for the infinite polyacene-type strips.  
(For further details see legend under fig. 2a.)

## 3. THE EFFECT OF THE OVERLAP INTEGRAL

So far we have calculated the energy levels of the mobile electrons on the simplifying assumption that the overlap integral  $S$  can be neglected. This assumption seems always to have been made hitherto in the "tight binding theory" of metals (see Mott and Jones, 1936). It remains to investigate the validity of such an assumption and, if necessary, to modify our results accordingly. We shall, in fact, find that a quite considerable modification of the energy bands is necessary when we include the effect of the overlap integral.

We have already shown (I, § 5) that if  $x$  is a root of the determinantal equation I (10), then the binding energy of a mobile electron, associated with this root  $x$ , is given by

$$\epsilon = \gamma\gamma,$$

where

$$\gamma = \frac{x}{1 - xS} \quad (3)$$

and  $\gamma$  is a constant given by

$$\gamma = \beta - E_0 S.$$

In our previous sections we have neglected  $S$  and have therefore dealt with the approximation  $\epsilon = x\beta$ . Equations (3) show us how to "correct" our present energies when we do not ignore the overlap integral—which we shall again assume to have the value of 0.25.

We notice first that there is no longer symmetry about the energy  $\epsilon = 0$ . While the roots  $x$ , being unchanged by the introduction of  $S$ , still occur in positive and negative pairs, this is no longer true of the corresponding values of  $\gamma$ , and consequently no longer true of the energy levels  $\epsilon$ . Now both  $\beta$  and  $\gamma$  are algebraically negative, and  $\epsilon$  is positive for a binding state. Thus the binding states correspond to negative values of  $x$  (and therefore also of  $\gamma$ ), while positive values of  $x$  (and  $\gamma$ ) correspond to "anti-bonding" states. We shall refer to the half-band for which  $x$  (or  $\gamma$ ) is negative as the "lower" half of the band; then, when the system of mobile electrons is in its ground state, every level in the lower half-band is doubly occupied.

We can now describe the convention adopted for the normalisation of the energy bands of figs. 2a, 2b. Working in terms of  $x$  or of  $\gamma$ , we have densities which we can call  $X(x)$  and  $Y(\gamma)$ . The definition of  $X(x)$ , for example, is that  $X(x)dx$  is the probability that the value of  $x$  lies between  $x$  and  $x + dx$ . The normalisation is such that when all the binding levels are filled with two electrons (having opposed spins) the total probability is 1. This means that for each half-band

$$\int X(x)dx = \int Y(\gamma)d\gamma = 1. \quad (4)$$

In other words, if  $N$  is the number of carbon atoms in the strip, and  $N(\epsilon)$  is the distribution function just determined, we plot the function obtained by letting  $n$  or  $m$  tend to infinity in the expression  $N(\epsilon)/N$ .

In the strips for which  $m$  tends to infinity there are concentrated levels at  $x = 1$ . In the normalisation scheme used above this gives a concentrated function  $1/(2n + 1)$  in such cases. There is no concentrated level in the strips  $n \rightarrow \infty$ . When the concentrated level exists it is included in the normalisation; for example, if  $n = 2$ , the curve of  $X(x)$  gives, for each half-band, an area  $4/5$  which, with the concentrated level, makes a total of unity.

The normalisation of the  $X(x)$  curves now follows at once either from their explicit equations, *e.g.* (2), or, when they are derived numerically, on the basis of (4). The transformation from  $X(x)$  to  $Y(\gamma)$  is surprisingly easily obtained by combining (3) and (4). It is

$$Y(\gamma) = \frac{x^2}{\gamma^2} X(x).$$

In figs. 2a, 2b we show certain of these presumably more accurate energy bands for comparison with those calculated when the overlap integral is ignored. It is obvious at once that the upper (*i.e.* unoccupied) parts of the bands are very considerably changed, being now much wider than before, though even with the inclusion of the overlap integral our underlying assumptions in the molecular orbital method probably fail before we reach the very top of a band. The occupied half-bands, however, are comparatively unchanged, *i.e.* number of



peaks and existence or otherwise of an energy gap between the occupied and unoccupied halves. Further, the ordinate  $\epsilon=0$  still divides the total area below the X and Y curves into two equal parts, so that in each case an exactly half-filled band stretches up to this value.

We must next ask whether the less severe distortions of the occupied half-bands significantly affect the mean binding energies of the mobile electrons. In those cases in which we have an explicit algebraic expression for the allowed values of  $x$  (e.g. equation (1)) we can of course use (3) to find the "corrected" mean energy. But this leads to a third order elliptic integral, and it has seemed more straightforward in every case to "correct" the energy levels at regularly spaced values of the continuous parameter  $\phi$  or  $\theta$ , and then perform a numerical integration, as in case (b) above, to find the mean energy. The results of such calculations are in Table I. We see that the mean value of  $y$  is in an almost constant ratio to the mean value of  $x$ ; so that inclusion of the overlap integral has no effect on the relative stabilities of the structures. A similar conclusion, it will be remembered, was reached in I for large, but not infinitely large, systems; and, for still smaller systems, by Wheland (1942).

An inspection of the density curves in fig. 2 shows that in all cases there are one or more infinite values of the ordinate, though of course the total area below the curve is necessarily finite. At first sight this may appear somewhat surprising; but there are three comments that we should like to make with regard to it. In the first place, a similar infinity occurs in the density curve for a body-centred cubic crystal (Mott and Jones, 1936, p. 85) and for a linear chain of atoms in  $s$  states (unpublished work). In the second place, any distortion or irregularity in the crystal, as will necessarily occur in real crystals, will destroy the infinity and replace it by a large peak. And finally, in the more accurate Wigner-Seitz calculations on face-centred iron by Greene and Manning (1943; see fig. 4 on p. 207), the density function shows four reasonably sharp peaks, two of which are extremely narrow. Indeed their curve quite closely resembles some of our curves in its general pattern. Evidently the existence of sharp maxima in the density curve is a not uncommon feature of periodic lattices.

#### 4. DIVISION OF THE STRIPS INTO "CONDUCTORS" AND "INSULATORS"

The energy bands, the densities of whose levels are shown in fig. 2, divide into two distinct classes. In the case of strips for which  $m$  is finite and  $n$  tends to infinity (polyacene type) there is no energy gap between the occupied and unoccupied states; so we may refer to such strips as "conductors". On the other hand, for the skew array and for most of the strips of polyphenyl type, when  $m$  tends to infinity, there is a finite energy gap between the occupied and the unoccupied levels of the mobile electrons. By analogy with three-dimensional metals, we must call such strips "insulators". This gap decreases as  $m$  increases, so that the infinite graphite plane will be conducting in both directions in the plane.

The exceptional strips of polyphenyl type occur when  $n=2, 5, 8 \dots$ , and, rather unexpectedly, will behave as conductors. Equation (2) shows at once how strips of width  $n=3p-1$  are exceptional, compared with strips of width  $3p$  or  $3p+1$ . For (2) shows that the lower half of the band  $k$  extends from  $\epsilon/\beta = -(1+z_k)$  to  $\epsilon/\beta = -(1-z_k)$ , where  $z_k = 2 \cos \frac{k\pi}{2(n+1)}$ , and the allowed values of  $k$  are  $k=1, 2, \dots, n$ . Thus the energy gap between the two halves of the band  $k$  is  $-2\beta(1-z_k)$ , and it vanishes if, and only if,  $z_k$  is allowed to have the value 1. This implies that  $n$  is of the form  $n=3p-1$ .

The conductive behaviour of these strips is unexpected in that it does not follow, in the way that the conductive or non-conductive properties of the other strips do, from simple considerations based on possible alternative canonical bond diagrams, involving double and single bonds. For all strips of the polyacene type we can write down alternative canonical structures resonance between which will provide long circuits for the mobile electrons, extending the whole length of the strips. Indeed it is possible to find a Kekule-type structure in which any chosen bond is double. It might therefore be surmised that resonance among all these structures would allow the migration of electrons over the whole system. Such a situation does not obtain, as trial will soon show, for any strip of the polyphenyl type  $m=\infty$ , where there are certain bonds which always appear as single bonds in all Kekule structures that can be drawn. At first sight, then, it is a little surprising that certain of these strips, viz. those

for which  $n = 2, 5, 8, \dots$ , should behave as conductors. It would take us too far afield to discuss this matter in detail; suffice it to indicate here an essential limitation of any argument based solely on the existence of alternative unexcited canonical structures.

## SUMMARY

The shapes of the energy bands have been determined for the mobile electrons in graphitic strips of infinite length but finite width, using as a basis the approximation of tight binding discussed for finite crystal layers in the previous paper. It appears that the effect of including overlap between the orbitals of adjacent atoms, whose incorporation in this type of calculation has hitherto been neglected, is to widen the top half of the band by a factor of the order of 2 or 3, the lower half of the band not being greatly affected. Some of these strips may be classed as conductors, the others as insulators; but the distinction between the two may not be made on the basis of any simple chemical bond diagrams.

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## APPENDIX

## ROOTS OF THE SECULAR EQUATION FOR THE INFINITE SKEW STRIP

Fig. 3 (a) shows a finite skew array. Its secular determinant, however, is very difficult to deal with in general terms owing to the different number of centres in successive rows of the array (in this case 6, 7, 7, 6). But the equations become manageable if we add two more terminal centres, as in fig. 3 (b), so that there are now the same number in every row. This

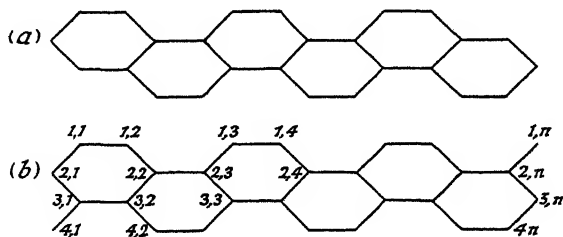


FIG. 3.—(a) An infinite skew strip.  
 (b) The skew strip, with numbering system, for which calculations were made.

will appreciably affect the roots of the secular determinant for any finite array, but as the number of hexagons in the strip increases, the modification of the roots due to the two extra terminal centres will decrease in importance and eventually cease to have any significance whatever (Ledermann, *Proc. Roy. Soc. London*, A, CLXXXII, 362, 1944). Thus the roots of the secular determinant for an infinite strip of type 3 (a) are indistinguishable from those of an infinite strip of type 3 (b), and to find the former we can develop the theory for strips of the latter type. This theory is somewhat similar to the analysis developed by Rutherford (*loc. cit.*) when evaluating the determinants which occurred in I. It is not, however, included in Rutherford's work, and seems worth reporting separately.

Numbering the centres as in fig. 3 (*b*) we obtain (*cf.* paper I) the secular determinant

$$\Delta = \begin{vmatrix} A & I & & \\ I & B & I & \\ & I & A & I \\ & & I & B \end{vmatrix},$$

where  $I$  is the unit matrix having  $n$  rows and columns, and  $A$  and  $B$  are the matrices

$$A = \begin{bmatrix} x & I & & & \\ I & x & & & \\ & & x & I & \\ & & I & x & \\ & & & & \ddots \\ & & & & & x & I \\ & & & & & I & x \\ & & & & & & x \end{bmatrix}_{(n)}, \quad B = \begin{bmatrix} x & & & & \\ & x & I & & \\ & I & x & & \\ & & & x & I \\ & & & I & x \\ & & & & \ddots \\ & & & & & x & I \\ & & & & & I & x \end{bmatrix}_{(n)}.$$

Here  $n$  is the number of centres in each row of 3 (*b*) and is necessarily odd. Rearranging, we obtain

$$\Delta = \begin{vmatrix} A & . & I & . \\ . & A & I & I \\ I & I & B & . \\ . & I & . & B \end{vmatrix},$$

which factorises (Rutherford, *loc. cit.*) into

$$\Delta = \prod_{k=1}^2 \begin{vmatrix} A & z_k I \\ z_k I & B \end{vmatrix},$$

where

$$z_k = 2 \cos \frac{k\pi}{5}.$$

Again rearranging the rows and columns in  $\begin{vmatrix} A & zI \\ zI & B \end{vmatrix}$ , we obtain

$$\begin{vmatrix} xI & zI + J' \\ zI + J & xI \end{vmatrix},$$

a determinant of a type also dealt with by Rutherford, who shows its value to be given by

$$\frac{z^{n-1} \{z \sin (n+1)\phi + \sin n\phi\}}{\sin \phi},$$

where

$$2z \cos \phi = x^2 - z^2 - 1, \quad \text{as in I (9).}$$

Consequently to find the roots ( $x$ -values) of  $\Delta = 0$ , we

- (i) write down the values of  $z_k = 2 \cos k\pi/5$ , to find  $z_1$  and  $z_2$ ;
- (ii) for each  $z_k$  solve  $z_k \sin (n+1)\phi_{k,r} + \sin n\phi_{k,r} = 0$ ,  $k=1, 2$  and  $r=1, 2, \dots, n$  to give us the values of  $\phi_{k,r}$  ( $0 < \phi_{k,r} < \pi$ );
- (iii) evaluate  $x$  from  $x = \pm \sqrt{1 + z_k^2 + 2z_k \cos \phi_{k,r}}$ .

We thus find  $2n$  pairs of roots. Of each corresponding pair of energy levels one is doubly filled and the other unoccupied. To find the mean energy of the mobile electrons for the infinite strip we therefore require

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \left( \sum_{r=1}^n \sqrt{1 + z_1^2 + 2z_1 \cos \phi_{1,r}} + \sum_{r=1}^n \sqrt{1 + z_2^2 + 2z_2 \cos \phi_{2,r}} \right).$$

Since, as  $n \rightarrow \infty$ , the values of  $\phi_{k,r}$  become uniformly spaced over the range  $0 \leq \phi_{k,r} \leq \pi$ , we can write the limit as

$$\bar{x} = \frac{1}{2} \int_0^1 \sqrt{1 + z_1^2 + 2z_1 \cos \pi x} dx + \frac{1}{2} \int_0^1 \sqrt{1 + z_2^2 + 2z_2 \cos \pi x} dx.$$

On simple transformation this leads to

$$\bar{x} = \frac{2 + \sqrt{5}}{\pi} E\left(a, \frac{\pi}{2}\right),$$

where

$$\sin^2 a = 4(\sqrt{5} - 2),$$

and  $E\left(a, \frac{\pi}{2}\right)$  is a complete elliptic integral.

(Issued separately September 22, 1948)

## XXXVII.—The Van der Waals Force between a Proton and a Hydrogen Atom.

II. Excited States. By C. A. Coulson (King's College, London) and Miss C. M. Gillam (University College, Dundee). (With One Text-figure.)

(MS. received October 24, 1946. Read February 3, 1947)

## I. INTRODUCTION

IN a previous paper (Coulson, 1941, referred to hereafter as I) a calculation was made of the energy of interaction between a bare proton and a hydrogen atom in its ground state. In particular it was shown that if the separation of the two nuclei was  $R$ , this interaction energy, which gives rise to the Van der Waals force, could be expressed in the form

$$E(R) = \frac{1}{R} E_1 + \frac{1}{R^2} E_2 + \dots \quad (1)$$

and the coefficients  $E_1, E_2, \dots$  could be determined absolutely. The calculation was carried through as far as the term  $1/R^{10}$ .

But this analysis only applied to the hydrogen atom in its ground state. And the problem has recently acquired a new importance in astrophysics; for one explanation of the broadening of certain lines in stellar spectra is that hydrogen atoms, which are very abundant in these atmospheres, are being perturbed, at the moment of emission or absorption, by the presence of neighbouring protons. It becomes desirable, therefore, to calculate the interaction energy between a proton and a hydrogen atom, not only when the latter is in its ground state (paper I), but also when it is in a general excited state. It is the purpose of the present paper to make this calculation.

A partial solution, to which we shall refer later, has been given by Mrs Krogdahl (1944). It must suffice for the moment to say that although Mrs Krogdahl found the first three non-vanishing coefficients in (1) for several excited states of the hydrogen atom, her method was not such as to give a general formula applicable to all states. We shall provide such a formula, which may be used to determine the first four non-vanishing terms in (1), *i.e.* up to and including  $E_6/R^5$ , for any initial state of the hydrogen atom.

## 2. CHOICE OF CO-ORDINATES

We use the notation of I, which may be seen from fig. 1. P is the electron with polar co-ordinates  $r, \theta, \phi$  relative to its nucleus A, B is the disturbing proton and AB is the polar

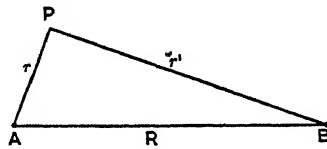


FIG. 1.

axis. Then if  $H$  is the Hamiltonian for the isolated hydrogen atom around A, so that (in atomic units)

$$H = -\frac{1}{2}\nabla^2 - \frac{1}{r}, \quad (2)$$

the problem consists in solving the equation

$$(H - E + V_{\text{pert}})\psi = 0, \quad (3)$$

where  $E$  is the energy, and

$$V_{\text{pert}} = \frac{1}{R} - \frac{1}{r'}. \quad (4)$$

As in I, we expand  $V_{\text{pert}}$  in powers of  $1/R$ , and we write

$$E = E_0 + \frac{1}{R}E_1 + \frac{1}{R^2}E_2 + \dots, \quad (5)$$

so that the interaction energy is

$$E(R) = E - E_0. \quad (6)$$

Next we expand

$$\psi = \psi_0 + \frac{1}{R}\psi_1 + \frac{1}{R^2}\psi_2 + \dots, \quad (7)$$

where  $\psi_0$  is the wave function of the unperturbed hydrogen atom in any one of its allowed quantum states.

Substituting all the expansions in (3) and equating to zero the coefficients of all the various powers of  $1/R$ , we obtain the set of equations ((5) in I):

$$(H - E_0)\psi_0 = 0, \quad (8a)$$

$$(H - E_0)\psi_1 = E_1\psi_0, \quad (8b)$$

$$(H - E_0)\psi_2 - rP_1(\cos\theta)\psi_0 = E_1\psi_1 + E_2\psi_0, \quad (8c)$$

$$(H - E_0)\psi_3 - r^2P_2(\cos\theta)\psi_0 - rP_1(\cos\theta)\psi_1 = E_1\psi_2 + E_2\psi_1 + E_3\psi_0, \quad (8d)$$

etc.

The  $(n+1)$ th of these equations enables us to find  $\psi_n$  if we have previously calculated  $\psi_0, \psi_1, \dots, \psi_{n-1}$ . The condition that  $\psi_n$  is everywhere finite enables us to determine  $E_n$ . The situation so far is very similar to that in I. But it differs quite fundamentally in the following respect.

The first equation (8a) shows that  $\psi_0$  is the wave function and  $E_0$  is the energy of the unperturbed hydrogen atom. But there is degeneracy, for if the total quantum number is  $n$ , so that  $E_0 = -1/2n^2$ , there are no less than  $n^2$  wave functions  $\psi_0$ . In I, where  $n=1$ , there was no degeneracy and therefore no difficulty of this kind. But now there is a difficulty, for the perturbation will remove the degeneracy and its nature will determine the correct zero-order combinations of the  $n^2$  degenerate wave functions. It is easy to see that the symmetry of the problem around AB implies that the magnetic quantum number  $m$  is still an exact quantum number, so that we need only consider at one time linear combinations for which  $m$  has the same value. This reduces the difficulty, but still leaves a degeneracy of  $n - |m|$ . If  $n$  is not large, it is possible to set up the usual secular determinant, using the familiar  $r, \theta, \phi$  wave functions, and in this way we can fix the appropriate zero-order combinations. This indeed is the technique of Mrs Krogdahl, but it is open to two objections. On the one hand it makes the calculations rather cumbersome, since  $\psi_0$ , which repeatedly occurs in (8), is a linear combination of  $n - |m|$  terms with rather untidy coefficients (see Krogdahl, Table I). For this reason Krogdahl only considered values of  $n - |m|$  lying between 1 and 5, and, except for  $n - |m| = 1$ , she only calculated as far as  $E_3$ : in the latter case  $E_4$  was also determined. The second objection is that, by the nature of the calculation, no general formula valid for all  $n$  and  $m$  could be obtained.

The problem is made much simpler by recognising that the first term in  $V_{\text{pert}}$  (see eq. (4)) is  $\frac{r}{R^2}P_1(\cos\theta)$ , i.e.  $\frac{z}{R^2}$ . This represents the interaction between the hydrogen atom and a uniform electric field. Indeed it will be recognized that (8a) and (8c) are identical with the equations for the Stark effect of a hydrogen atom, so that the correct zero-order wave functions in our present problem, being determined by the first terms of the perturbation, are identical with those encountered in the Stark effect. In this latter problem, as Epstein (1926), Schrödinger (1926) and others have shown, great simplification results from working in parabolic co-ordinates  $\xi=r+z, \eta=r-z$ , and  $\phi$  instead of polar co-ordinates  $r, \theta, \phi$ . The wave equation is separable in these new co-ordinates, and—what is important for us—these solutions are themselves already the correct ones for use in the ordinary perturbation treatment. For they represent the correct linear combinations of the original  $n - |m|$  degenerate wave functions. The same must also be true for our problem, and we shall henceforth deal with the set of equations (8) entirely on this basis, giving  $\psi_0$  any one of the values found in parabolic co-ordinates, and paying no further attention to the question of degeneracy.

## 3. METHOD OF CALCULATION

According to the reasoning above we are to solve the set of equations (8) in which

$$E_0 = -\frac{1}{2n^2}, \quad (9)$$

and

$$\psi_0 = f_{k_1} \left( \frac{\xi}{n} \right) f_{k_2} \left( \frac{\eta}{n} \right) e^{im\phi}, \quad (10)$$

where

$$f_k(x) = \frac{e^{-\frac{x}{2} \frac{|m|}{2}} L_{k+|m|}^{(|m|)}(x)}{(|k| + |m|)!} \quad (11)$$

(see Schrödinger, 1926).

The parameters  $n$ ,  $k_1$ ,  $k_2$ , which may be regarded as the three defining quantum numbers, are necessarily positive integers, and are related to the magnetic quantum number  $m$  by the equation

$$k_1 + k_2 + |m| + 1 = n. \quad (12)$$

In practice we have found it easier to work in terms of  $k_1$ ,  $k_2$  and  $n$  rather than any other combination (e.g.  $n$ ,  $m$ ,  $k_1 - k_2$ ) which involves  $m$  directly.

We shall find it convenient to introduce the notation

$$\{r, s\} = f_{k_1+r} \left( \frac{\xi}{n} \right) f_{k_2+s} \left( \frac{\eta}{n} \right), \quad (13)$$

so that (10) may be written in the form

$$\psi_0 = \{0, 0\} e^{im\phi}. \quad (14)$$

Our earlier comment about  $m$  being an exact quantum number implies that each function  $\psi_\rho$  in (8) is of the form

$$\psi_\rho = X_\rho e^{im\phi}, \quad (15)$$

in which  $X_\rho$  is a function of  $\xi$  and  $\eta$  only. If we put

$$x = \frac{\xi}{n}, \quad y = \frac{\eta}{n}, \quad (16)$$

the equation which  $X_\rho$  must satisfy is easily found to be

$$\left[ \frac{\partial}{\partial x} \left( x \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left( y \frac{\partial}{\partial y} \right) - \frac{n^2}{4} \left( \frac{1}{x} + \frac{1}{y} \right) - \frac{x+y}{4} + n \right] X_\rho = -\frac{n^2}{2} (x+y) \left[ r^{\rho-1} P_{\rho-1}(\cos \theta) X_0 \right. \\ \left. + r^{\rho-2} P_{\rho-2}(\cos \theta) X_1 + \dots + r P_1(\cos \theta) X_{\rho-2} + E_1 X_{\rho-1} + E_2 X_{\rho-2} + \dots + E_\rho X_0 \right]. \quad (17)$$

Let the differential operator on the left be denoted by  $L$ . Thus we require to solve the set of equations

$$L(X_1) = -\frac{n^2}{2} (x+y) E_1 X_0, \quad (18a)$$

$$L(X_2) = -\frac{n^2}{2} (x+y) [r P_1(\cos \theta) X_0 + E_1 X_1 + E_2 X_0], \quad (18b)$$

$$L(X_3) = -\frac{n^2}{2} (x+y) [r^2 P_2(\cos \theta) X_0 + r P_1(\cos \theta) X_1 + E_1 X_2 + E_2 X_1 + E_3 X_0], \quad (18c)$$

etc.,

and we are given that

$$X_0 = \{0, 0\}. \quad (19)$$

The importance of the functions  $\{r, s\}$  introduced in (13) lies in the differential equation which they satisfy. Thus, since the equation

$$(xv)' + \left( \frac{1 - |m|}{2} - \frac{x}{4} - \frac{m^2}{4x} \right) v + (|m| + k_1 + r)v = 0$$

is satisfied (Courant and Hilbert, p. 285) by

$$v = f_{k_1+r}(x),$$

it follows from the definitions of  $\{r, s\}$  and from (12) that

$$L\{r, s\} = -(r+s)\{r, s\}. \quad (20)$$

This simple result suggests that we should seek the solution of our fundamental equations (18) in terms of the functions  $\{r, s\}$ , which form a complete orthogonal set. In order to do this we require to know the expansion of  $x^p\{r, s\}$  in terms of the members of this set. It is shown in the Appendix that we may write

$$x^p\{r, s\} = \sum_{i=-p}^p a_i\{r+i, s\},$$

where

$$a_i = \frac{(k_1+r+i)!}{(k_1+r+i+|m|)!} p!^2 (-1)^i \sum_c \frac{(p+|m|+k_1+r-c)!}{(k_1+r-c)! c! (p-c)! (c+i)! (p-c-i)!}. \quad (21)$$

Similarly

$$y^p\{r, s\} = \sum_{i=-p}^p b_i\{r, s+i\},$$

where  $b_i$  is obtained from  $a_i$  by replacing  $k_1$  by  $k_2$  and  $r$  by  $s$ . The summation in (21) is over all integral values of  $c$  which make every one of the factorials non-negative. This formula is only required for fairly small values of  $p$ , and it is convenient, in order to avoid confusion, to introduce the following notation:—

$$\begin{aligned} x\{r, s\} &= \sum_{i=-1}^1 A_i^{k_1+r}\{r+i, s\}, \\ y\{r, s\} &= \sum_{i=-1}^1 A_i^{k_2+s}\{r, s+i\}, \\ x^2\{r, s\} &= \sum_{i=-2}^2 B_i^{k_1+r}\{r+i, s\}, \\ x^3\{r, s\} &= \sum_{i=-3}^3 C_i^{k_1+r}\{r+i, s\}, \\ &\text{etc.} \end{aligned} \quad (22)$$

The numerical values of the constants  $A_i, B_i, \dots$  are found as particular examples of the general formula (21).

We are now in a position to solve the equations (18) one by one. Consider first (18a). By using the formulæ for  $x\{r, s\}$  and  $y\{r, s\}$ , it may be written

$$L(X_1) = -\frac{n^2}{2} E_1 \sum_{i=-1}^1 [A_i^{k_1}\{i, 0\} + A_i^{k_2}\{0, i\}].$$

So let us put

$$X_1 = \sum_{r,s} a_{r,s}\{r, s\}. \quad (23)$$

Then according to (20) we get the equation

$$\sum_{r,s} (r+s) a_{r,s}\{r, s\} = \frac{n^2}{2} E_1 \sum_{i=-1}^1 [A_i^{k_1}\{i, 0\} + A_i^{k_2}\{0, i\}]. \quad (24)$$



Since the functions  $\{r, s\}$  are an orthogonal set we may equate coefficients of  $\{r, s\}$  on both sides. In particular, the coefficient of  $\{0, 0\}$  on the left-hand side is zero, while that on the right is

$$\frac{n^2}{2}E_1[A_0^{k_1} + A_0^{k_2}] = n^2E_1.$$

Therefore

$$E_1 = 0. \quad (25)$$

Equation (24) now reduces to

$$\sum_{r,s} (r+s)a_{r,s}\{r, s\} = 0.$$

This shows that every  $a_{r,s}$  is zero except possibly those for which  $r+s=0$ , which are at present indeterminate. We can take  $a_{0,0}=0$ , as this only affects the normalisation (see I).

Instead of (23) we can now write

$$X_1 = \sum_j a_{j,-j}\{j, -j\}, \quad (26)$$

where the  $a_{j,-j}$  will have to be determined later (from a detailed study of  $X_3$ : see equation (34)).

We can now proceed to equation (18b). Thus, to calculate  $X_2$  and  $E_2$ , we put

$$X_2 = \sum_{r,s} b_{r,s}\{r, s\}, \quad \text{with } b_{0,0}=0, \quad (27)$$

in this equation. It becomes

$$\sum_{r,s} (r+s)b_{r,s}\{r, s\} = \frac{n^2}{2}(x+y)[rP_1(\cos \theta)X_0 + E_2X_0]. \quad (28)$$

Now

$$rP_1(\cos \theta) = z = \frac{n}{2}(x-y).$$

Thus

$$\begin{aligned} (x+y)rP_1(\cos \theta)X_0 &= \frac{n}{2}(x^2 - y^2)\{0, 0\} \\ &= \frac{n}{2} \sum_{i=-2}^2 [B_i^{k_1}\{i, 0\} - B_i^{k_2}\{0, i\}], \end{aligned}$$

and

$$\begin{aligned} (x+y)E_2X_0 &= E_2(x+y)\{0, 0\} \\ &= E_2 \sum_{i=-1}^1 [A_i^{k_1}\{i, 0\} + A_i^{k_2}\{0, i\}]. \end{aligned}$$

Substituting these results in (28) and equating the coefficients of  $\{0, 0\}$  on either side of the equation gives

$$0 = \frac{n}{2}[B_0^{k_1} - B_0^{k_2}] + E_2[A_0^{k_1} + A_0^{k_2}].$$

But

$$B_0^{k_1} - B_0^{k_2} = 6n(k_1 - k_2),$$

and

$$A_0^{k_1} + A_0^{k_2} = 2n,$$

so that

$$E_2 = -\frac{3n}{2}(k_1 - k_2). \quad (29)$$

By considering the coefficients of the remaining  $\{r, s\}$  which occur on the right-hand side of (28) we obtain the results

$$\begin{aligned}
 b_{-2,0} &= -\frac{n^3}{8}B_{-2}^{k_1}, & b_{-1,0} &= -\frac{n^3}{4}B_{-1}^{k_1} - \frac{n^2}{2}E_2A_{-1}^{k_1}, \\
 b_{2,0} &= \frac{n^3}{8}B_2^{k_1}, & b_{1,0} &= \frac{n^3}{4}B_1^{k_1} + \frac{n^2}{2}E_2A_1^{k_1}, \\
 b_{0,-2} &= \frac{n^3}{8}B_{-2}^{k_2}, & b_{0,-1} &= \frac{n^3}{4}B_{-1}^{k_2} - \frac{n^2}{2}E_2A_{-1}^{k_2}, \\
 b_{0,2} &= -\frac{n^3}{8}B_2^{k_2}, & b_{0,1} &= -\frac{n^3}{4}B_1^{k_2} + \frac{n^2}{2}E_2A_1^{k_2}.
 \end{aligned} \tag{30}$$

The remaining  $b_{r,s}$  are zero except possibly those of the form  $b_{i,-i}$ , which are indeterminate at present.

The solutions of the remaining equations (18c), (18d), . . . for  $X_3, X_4, \dots$  follow in a similar way. But there is one new factor which makes it desirable to work through the particular case of  $X_3$  in rather more detail. For each of these equations determines not only the corresponding energy  $E_3, E_4, \dots$  and most of the coefficients in the wave functions  $X_3, X_4, \dots$  of the same order, but it also completes the calculation of the wave functions whose order is two less than that of the equation itself. Thus the equation for  $X_3$  completes  $X_1$ , and that for  $X_4$  completes  $X_2$ , etc. This is a rather unusual situation, but we can illustrate it by the case of  $X_3$  as follows. In equation (18c) put

$$X_3 = \sum_{r,s} c_{r,s} \{r, s\}, \quad \text{with } c_{0,0} = 0, \tag{31}$$

obtaining

$$\sum_{r,s} (r+s) c_{r,s} \{r, s\} = \frac{n^2}{2} (x+y) [r^2 P_2(\cos \theta) X_0 + r P_1(\cos \theta) X_1 + E_2 X_1 + E_3 X_0]. \tag{32}$$

Expanding the terms on the right-hand side we get

$$\begin{aligned}
 (x+y) r^2 P_2(\cos \theta) X_0 &= \frac{n^2}{4} (x^3 - 3x^2 y - 3xy^2 + y^3) \{0, 0\} \\
 &= \frac{n^2}{4} \left[ \sum_{i=-3}^3 C_i^{k_1} \{i, 0\} - 3 \sum_{i=-2}^2 \sum_{j=-1}^1 B_i^{k_1} A_j^{k_1} \{i, j\}, \right. \\
 &\quad \left. - 3 \sum_{i=-2}^2 \sum_{j=-1}^1 B_i^{k_2} A_j^{k_2} \{j, i\} + \sum_{i=-3}^3 C_i^{k_2} \{0, i\} \right],
 \end{aligned}$$

$$\begin{aligned}
 (x+y) r P_1(\cos \theta) X_1 &= \frac{n}{2} (x^2 - y^2) \sum_j a_{j, -j} \{j, -j\} \\
 &= \frac{n}{2} \sum_j \sum_{i=-2}^2 a_{j, -j} [B_i^{k_1+j} \{j+i, -j\} - B_i^{k_2-j} \{j, i-j\}],
 \end{aligned}$$

$$\begin{aligned}
 (x+y) E_2 X_1 &= E_2 (x+y) \sum_j a_{j, -j} \{j, -j\} \\
 &= E_2 \sum_j \sum_{i=-1}^1 a_{j, -j} [A_i^{k_1+j} \{j+i, -j\} + A_i^{k_2-j} \{j, i-j\}],
 \end{aligned}$$

$$\begin{aligned}
 (x+y) E_3 X_0 &= E_3 (x+y) \{0, 0\} \\
 &= E_3 \sum_{i=-1}^1 [A_i^{k_1} \{i, 0\} + A_i^{k_2} \{0, i\}].
 \end{aligned}$$

Using these expressions in (32) and equating coefficients of  $\{0, 0\}$  we find

$$0 = \frac{n^2}{4} [C_0^{k_1} - 3B_0^{k_1}A_0^{k_2} - 3B_0^{k_2}A_0^{k_1} + C_0^{k_2}] + E_3[A_0^{k_1} + A_0^{k_2}].$$

On simplifying, this gives

$$E_3 = \frac{n^2}{2} [n^2 - 6(k_1 - k_2)^2 - 1]. \quad (33)$$

The coefficients  $c_{r,s}$  in the expansion (31) of  $X_3$  are similarly found from (32), except those for which  $r+s=0$ . Twenty-six of them are non-zero, but only a few of these are needed in the calculations of  $E_4$  and  $E_5$ , so that it does not seem necessary to tabulate their values here. Terms for which  $r+s=0$  do not appear on the left of (32), but they do appear on the right in a form which involves the hitherto unknown coefficients in  $X_1$ . Considering, for example, the coefficients of  $\{1, -1\}$ , we see that

$$0 = \frac{n^2}{4} [-3B_1^{k_1}A_{-1}^{k_2} - 3B_{-1}^{k_2}A_1^{k_1}] + \frac{n}{2} [B_0^{k_1+1} - B_0^{k_2-1}]a_{1,-1} + E_2[A_0^{k_1+1} + A_0^{k_2-1}]a_{1,-1},$$

from which we obtain

$$a_{1,-1} = \frac{n}{2}(k_1+1)(n-k_1-1).$$

Similarly

$$a_{-1,1} = -\frac{n}{2}(k_2+1)(n-k_2-1),$$

and the remaining  $a_{i,-i}$  are zero. In this way we complete the calculation of  $X_1$ , which takes the form

$$X_1 = \frac{n}{2} [(k_1+1)(n-k_1-1)\{1, -1\} - (k_2+1)(n-k_2-1)\{-1, 1\}]. \quad (34)$$

It is rather surprising that the precise form of  $X_1$  is only determined by later considerations involved in the discussion not of  $X_2$  but of  $X_3$ .

In the differential equation for  $X_4$  we put

$$X_4 = \sum_{r,s} d_{r,s} \{r, s\}, \quad \text{with } d_{0,0} = 0, \quad (35)$$

and obtain

$$\sum_{r,s} (r+s) d_{r,s} \{r, s\} = \frac{n^2}{2} (x+y) [\tau^3 P_3(\cos \theta) X_0 + \tau^2 P_2(\cos \theta) X_1 + \tau P_1(\cos \theta) X_2 + E_2 X_2 + E_3 X_1 + E_4 X_0]. \quad (36)$$

This determines  $E_4$  and, if necessary,  $b_{i,-i}$  and  $d_{r,s} (r+s \neq 0)$ . In fact

$$E_4 = -\frac{n^4}{8} [4n^2 + 9n(k_1+k_2+1) - 6(k_1^2+k_2^2) - 6k_1k_2 - 9(k_1+k_2) + 5] - \frac{n^3}{8} (k_1-k_2) [-24n^2 + 9n(k_1+k_2+1) + 50(k_1^2+k_2^2) - 118k_1k_2 - 9(k_1+k_2) + 25]. \quad (37)$$

Likewise from the next equation for  $X_5$  we get the final result:

$$E_5 = -\frac{3n^4}{8} \{n^4 + n^2 [-31(k_1^2+k_2^2) + 64k_1k_2 + k_1+k_2 - 5] + n(k_1+k_2+1) [16(k_1^2+k_2^2) - 34k_1k_2 - k_1-k_2] + 35(k_1^4+k_2^4) - 172k_1k_2(k_1^2+k_2^2) + 276k_1^2k_2^2 - 16(k_1^3+k_2^3) + 18k_1k_2(k_1+k_2) + 41(k_1^2+k_2^2) - 80k_1k_2 + 4\} - \frac{3n^5}{16} (k_1-k_2) [22n^2 + 63n(k_1+k_2+1) - 30(k_1^2+k_2^2) - 66k_1k_2 - 63(k_1+k_2) + 65]. \quad (38)$$

## 4. CONCLUSION

Equations (25), (29), (33), (37) and (38) give the coefficients  $E_1 \dots E_5$  of the desired expansion (1). It will be noticed that they exhibit considerable similarity in their dependence on  $k_1$  and  $k_2$ ; and they may be regarded as the appropriate generalisation of the result in paper I. For some purposes it is more convenient to show the dependence of the perturbation energy upon  $n$ ,  $m$ , and  $k_1 - k_2$ . This can be done by using (12) to eliminate  $k_1 + k_2$ . The results are:

$$E_2 = -\frac{3n}{2}(k_1 - k_2), \quad (39a)$$

$$E_3 = -\frac{n^2}{2}[n^2 - 6(k_1 - k_2)^2 - 1], \quad (39b)$$

$$E_4 = -\frac{n^4}{16}[17n^2 - 9m^2 - 3(k_1 - k_2)^2 + 19] - \frac{n^3}{16}(k_1 - k_2)[-39n^2 - 9m^2 + 109(k_1 - k_2)^2 + 59], \quad (39c)$$

$$E_5 = -\frac{3n^4}{64}[345(k_1 - k_2)^4 + 6(k_1 - k_2)^2(-31n^2 - 11m^2 + 65) + 9n^4 + m^4 - 2n^2m^2 - 42n^2 - 2m^2 + 33] \\ - \frac{3n^5}{32}(k_1 - k_2)[107n^2 - 63m^2 + 3(k_1 - k_2)^2 + 193]. \quad (39d)$$

It will be noticed that  $E_2$  is the first order Stark effect, and the first term of  $E_4$  is the second order Stark effect. It is an easy matter to calculate the coefficients for any assigned state of the hydrogen atom; indeed we have not thought it worth while to include any of these here, because Krogdahl gives several of them in her work previously referred to, and, so far as they go, her results agree with ours.

There is only one final comment to make. In I, where  $n = 1$ , it was noticed that successive coefficients  $E_1, E_2, \dots$  increased steadily, but slowly, up to  $E_{10}$ , which was the last one calculated. A similar increase occurs here, but on account of the high powers of  $n$  involved ( $n^8$  appears in the expression for  $E_5$ ) the increase is much greater. This is an expression of the fact that the Van der Waals forces for an excited atom are much greater than those for an atom in its ground state; it implies that the original expansion (1) becomes invalid, through non-convergence, at much larger values of  $R$  than in I. For smaller values of  $R$  other expressions for the interaction energy than the expansion (1) must be employed.

## 5. SUMMARY

The interaction energy, or Van der Waals force, between a proton and a hydrogen atom in any one of its allowed quantum states is calculated in terms of the internuclear distance  $R$  by an expansion of the form

$$E(R) = \frac{1}{R}E_1 + \frac{1}{R^2}E_2 + \dots$$

All the coefficients up to and including  $E_5$  are obtained in closed form. For values of  $R$  for which the expansion is valid, the coefficients are determined absolutely, no approximations being introduced.

## APPENDIX

To express  $x^p \{r, s\}$  in the form  $\sum_i a_i \{r + i, s\}$ . For small values of  $p$  this expansion can be obtained by repeated application of the recurrence formula for the Laguerre polynomials

$$x\{r, s\} = -(|m| + k_1 + r)\{r - 1, s\} + (|m| + 2k_1 + 2r + 1)\{r, s\} - (k_1 + r + 1)\{r + 1, s\}.$$

But the general result is best obtained by the use of generating functions. Thus, let

$$x^p \{r, s\} = \sum_{i=-p}^p a_i \{r+i, s\},$$

$$\text{i.e.} \quad x^p f_{k_1+r}(x) = \sum_{i=-p}^p a_i f_{k_1+r+i}(x). \quad (40)$$

We know that

$$\int_0^\infty f_{n_1}(x) f_{n_2}(x) dx = \begin{cases} 0 & n_1 \neq n_2 \\ \frac{(n+|m|)!}{n!} & n = n_1 = n_2. \end{cases} \quad (41)$$

Therefore, multiplying each side of (40) by  $f_{k_1+r+i}(x)$  and integrating gives

$$\begin{aligned} \frac{(k_1+r+i+|m|)!}{(k_1+r+i)!} a_i &= \int_0^\infty x^p f_{k_1+r}(x) f_{k_1+r+i}(x) dx \\ &= \int_0^\infty \frac{x^{p+|m|} e^{-x} L_{k_1+r+|m|}^{(m)}(x) L_{k_1+r+i+|m|}^{(m)}(x) dx}{(k_1+r+|m|)! (k_1+r+i+|m|)!}. \end{aligned}$$

Schrödinger (1926) has shown that this integral may be evaluated by the use of the generating function:

$$\sum_n \frac{z^{n-|m|} L_n^{(m)}(x)}{n!} = (-1)^{|m|} \frac{e^{-\frac{xz}{1-z}}}{(1-z)^{|m|+1}}.$$

In fact,

$$\begin{aligned} & \frac{(k_1+r+i+|m|)!}{(k_1+r+i)!} a_i \\ &= \text{coefficient of } z^{k_1+r} u^{k_1+r+i} \text{ in } \int_0^\infty \frac{x^{p+|m|} e^{-x} \left[ \frac{z}{1-z} + \frac{u}{1-u} \right]}{(1-z)^{|m|+1} (1-u)^{|m|+1}} dx \\ &= \text{,, ,, ,, ,, } (p+|m|)! (1-z)^p (1-u)^p (1-uz)^{-(p+|m|+1)} \\ &= (p+|m|)! \sum_n \frac{(p+|m|+n)! (-1)^{k_1+r-n} p! (-1)^{k_1+r+i-n} p!}{n! (p+|m|)! (k_1+r-n)! (p-k_1-r+n)! (k_1+r+i-n)! (p-k_1-r-i+n)!}. \end{aligned}$$

Therefore, putting  $k_1+r-n=c$ ,

$$a_i = \frac{(k_1+r+i)!}{(k_1+r+i+|m|)!} p!^2 (-1)^i \sum_c \frac{(p+|m|+k_1+r-c)!}{(k_1+r-c)! c! (p-c)! (c+i)! (p-c-i)!}. \quad (42)$$

This gives the required expansion (21).

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XXXVIII.—On the Estimation of Many Statistical Parameters. By A. C. Aitken,  
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I. INTRODUCTORY

IN an earlier paper (Aitken and Silverstone, 1941) the problem of estimating from sample a parameter  $\theta$  of unknown value was treated by adopting two postulates for the estimating function: (i) that it should be unbiased in the linear sense; (ii) that its sampling variance should be minimal. Specifically, in the case of a probability function  $\phi = \phi(x; \theta)$  involving a parameter  $\theta$ , if a sample of  $N$  independent values of  $x$  has provided the vector

$$x = \{x_1 x_2 \dots x_N\}, \quad (1.1)$$

then under certain conditions an estimating function

$$t = t(x_1, x_2, \dots, x_N) \quad (1.2)$$

exists such that

$$\begin{aligned} \int t \Phi(x) dx &= \theta, \\ \int (t - \theta)^2 \Phi(x) dx &= \text{minimum}, \\ \int \Phi(x) dx &= 1, \end{aligned} \quad (1.3)$$

where  $\Phi(x)$  is the probability density of the sample vector  $x$ , and  $dx = dx_1 dx_2 \dots dx_N$ . The classes of function  $\phi$  amenable to the method are limited. An important sub-class is that of functions  $\phi$  satisfying the differential equation

$$\frac{\partial}{\partial \theta} \log \Phi = (t - \theta) / \lambda(\theta), \quad (1.4)$$

where  $\Phi$  is the product of  $N$  similar functions  $\phi$ , and  $\lambda(\theta)$  is independent of  $x$ . It was proved that the sampling variance of  $t(x)$  is then equal to  $\lambda(\theta)$ , and that the function  $\phi$  admitting estimation of  $\theta$  under the postulates (1.3) is of the type

$$\phi(x; \theta) = \exp \{f(x) + F(\theta) + t(x)\gamma(\theta)\}, \quad (1.5)$$

known (Koopman, 1936) to be the only type admitting sufficient statistics. It was also proved that in many cases, where  $\theta$  itself does not admit an estimating function  $t$ , some function  $\tau(\theta)$  does so, where

$$\tau(\theta) = - \frac{\partial F}{\partial \theta} / \frac{\partial \gamma}{\partial \theta}. \quad (1.6)$$

The above procedure was extended (Solomon, 1944) to the case of two parameters  $\theta_1, \theta_2$ , but to avoid an *impasse* the second postulate had to be modified. It was now proposed to minimize, not the two separate sampling variances of the estimators  $t_1(x)$  and  $t_2(x)$ , but the determinant  $|V|$  of their variance matrix  $V = [v_{ij}]$ , where

$$v_{ij} = \int (t_i - \theta_i)(t_j - \theta_j) \Phi dx, \quad i, j = 1, 2. \quad (1.7)$$

The determinant  $|V|$  is in fact the *generalized variance* of Wilks (Wilks, 1932). This approach to the problem of multiple estimation is from a direction opposite to that taken by R. C. Geary, who proved (Geary, 1942) that when many parameters are simultaneously estimated from a sample of  $N$  values by the method of maximum likelihood the generalized variance, for fixed but indefinitely large  $N$ , tends to a minimum in an asymptotic sense, while at the same time (as was first shown by R. A. Fisher, 1934) the joint distribution of the  $t_i(x)$  tends to the correlated normal type.

Analogues in two parameters were found by Dr Solomon for the various theorems of Silverstone in one parameter, and such cases as the simultaneous estimation of mean and variance from a normal sample were studied in detail. The results, and their generalizations, will be the subject of remark in §§ 3, 5 and 6.

A recent paper (Rao, 1945) treats the estimation of many parameters from a standpoint resembling the present one, but centring around the matrix

$$I(\phi, \theta) = E \left[ - \frac{\partial^2 \log \Phi}{\partial \theta_i \partial \theta_j} \right], \quad (1.8)$$

which is fundamental in R. A. Fisher's theory (Fisher, 1921) of the "amount of information" latent in a sample.

Our purpose is now to apply the postulates of unbiased estimate and minimal determinant of variance to the case of any number  $k$  of parameters, and to examine in detail the case of a multivariate normal sample.

## 2. POSTULATES FOR ESTIMATION

For conciseness we adopt vector and matrix notation. The probability density of the vector variate,  $\phi(x_1, x_2, \dots, x_n; \theta_1, \theta_2, \dots, \theta_k)$ , is denoted by  $\phi(x; \theta)$ , where  $\theta = \{\theta_1, \theta_2, \dots, \theta_k\}$ ; that of a sample of  $N$  such vectors will be denoted by  $\Phi(X; \theta)$ , where  $X$  is the sample vector of  $Nn$  elements, imagined written out in  $N$  sets of  $n$ . In the cases considered  $\Phi$  will be the product of  $N$  similar factors  $\phi$ , so that  $\log \Phi = \sum \log \phi$ , and in fact the results for  $\phi$  and  $\Phi$  will differ merely by some power of  $N$ . We shall assume that  $\phi$  possesses first and second derivatives with respect to the  $\theta_i$ .

The postulates for the estimators  $t_i = t_i(x)$  are now:

$$(i) \quad \int t_i \Phi dx = \theta_i, \quad i = 1, 2, \dots, k,$$

$$(ii) \quad |V| = |v_{ij}| = \text{minimum},$$

where

$$v_{ij} = \int (t_i - \theta_i)(t_j - \theta_j) \Phi dx, \quad (2.1)$$

$$(iii) \quad \int \Phi dx = 1.$$

We proceed to show that a class of functions  $\Phi$  admitting the method is given by the solutions of

$$(t - \theta)\Phi = \Lambda \Phi_\theta, \quad (2.2)$$

where  $t$  is the vector of estimating functions,  $\Phi_\theta$  is the vector of partial derivatives of  $\Phi$  with respect to the elements of  $\theta$ , and  $\Lambda = [\lambda_{ij}(\theta)]$  is a matrix of Lagrange functions independent of  $x$  but depending on  $\theta$ . For from (2.1) we have

$$\int \Phi_\theta dx = 0, \quad [\int t_i \Phi_\theta dx] = I, \quad (2.3)$$

where  $I$  is the unit matrix. Hence

$$\begin{aligned} \int (t_i - \theta_i)(t_j - \theta_j) \Phi dx &= \int (t_i - \theta_i) \left( \sum_{\nu=1}^k \lambda_{j\nu} \Phi_{\theta_\nu} \right) \Phi dx \\ &= \lambda_{ij} = \lambda_{ji}, \quad \text{by (2.3).} \end{aligned} \quad (2.4)$$

Hence under the conditions (2.1) we have

$$|V| = |\lambda_{ij}| = |\Lambda|. \quad (2.5)$$

We must now prove that  $|\Lambda|$  is the minimal value of  $|V|$  attainable by any set of  $k$  unbiased functions estimating the  $\theta_i$ . Let us consider any vector  $u$  composed of such functions. Then

$$\begin{aligned} \int u_i \Phi dx &= \theta_i, & [\int u_i \Phi_{\theta_j} dx] &= I, \\ \int (u_i - t_i) \Phi_{\theta_i} dx &= \int (u_i - t_i) \Phi_{\theta_j} dx = 0. \end{aligned} \quad (2.6)$$

Hence

$$\int (u_i - \theta_i)(u_j - \theta_j) \Phi dx \quad (2.7)$$

$$\begin{aligned} &= \int (u_i - t_i)(u_j - t_j) \Phi dx + \int (u_i - t_i)(t_j - \theta_j) \Phi dx \\ &+ \int (u_j - t_j)(t_i - \theta_i) \Phi dx + \int (t_i - \theta_i)(t_j - \theta_j) \Phi dx \end{aligned} \quad (2.8)$$

$$= \int (u_i - t_i)(u_j - t_j) \Phi dx + \lambda_{ij}, \quad (2.9)$$

by substituting from (2.2) in the second and third integrals of (2.8) and referring to (2.3). Writing (2.9) as  $\mu_{ij} + \lambda_{ij}$ , we have to prove that

$$|\lambda_{ij} + \mu_{ij}| = |\Lambda + M| > |\Lambda|. \quad (2.10)$$

It has been assumed that  $|\Lambda| \neq 0$ , that is, that the derivatives  $\Phi_{\theta}$  are linearly independent in the functional sense. Thus  $\Lambda$  is symmetric and also positive definite of full rank  $k$ , while  $M$  from its mode of derivation is positive definite, or possibly non-negative definite of rank less than  $k$ . In any case a real non-singular matrix  $H$  exists such that

$$H' \Lambda H = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_k \end{bmatrix}, \quad H' M H = \begin{bmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_k \end{bmatrix}, \quad (2.11)$$

where  $\alpha_i > 0$ ,  $\beta_i > 0$ . Hence, taking determinants and cancelling  $|H|^2$ , we have to prove that

$$(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) \dots (\alpha_k + \beta_k) > \alpha_1 \alpha_2 \dots \alpha_k, \quad (2.12)$$

and this is evidently true, since  $\alpha_i + \beta_i > \alpha_i > 0$ . Further, equality occurs only when all  $\beta_i$  are zero, in which case  $M=0$  and so, since  $\Phi$  is positive over the range of  $x$ , each  $(u_i - t_i)(u_j - t_j) = 0$ , and so finally

$$u_i = t_i, \quad i = 1, 2, \dots, k. \quad (2.13)$$

Thus the functions  $t_i(x)$ , provided that  $\phi(x)$ , and therefore  $\Phi(X)$ , satisfies the conditions (2.1), uniquely ensure for  $|V|$  the minimum value  $|\Lambda|$ .

### 3. ESTIMATION OF MEAN AND VARIANCE FROM NORMAL SAMPLE

The following example (Solomon, 1944) brings out in relief some of the special features of the problem of multiple estimation. Let us consider the univariate normal probability function

$$\phi(x; \theta) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2}(x - \mu)^2/\sigma^2 \right\}, \quad (3.1)$$

and let it be desired to estimate, from a sample  $x = \{x_1, x_2, \dots, x_N\}$ , the mean  $\mu$  and the variance  $\sigma^2$ . We examine first  $\partial \log \Phi / \partial \mu$  and  $\partial \log \Phi / \partial \sigma^2$  to see whether these can be expressed in vector form

$$\partial \log \Phi / \partial \theta = \Lambda^{-1}(t - \theta), \quad (3.2)$$

that is to say, the reciprocal set of the equations (2.2) of estimation. It is found, however, that the  $t_i$  and the  $\theta_i$  cannot be extricated from each other in this explicit way. The fact is that  $\mu$  and  $\sigma^2$  are not the parameters to estimate by the method, and on further examination. (Solomon, 1944) it appears that the basic parameters are really  $\theta_1 = \mu$ ,  $\theta_2 = \sigma^2 + \mu^2$ . This revised choice leads to

$$\begin{bmatrix} \partial \log \Phi / \partial \theta_1 \\ \partial \log \Phi / \partial \theta_2 \end{bmatrix} = \frac{N}{(\theta_2 - \theta_1^2)^2} \begin{bmatrix} \theta_2 + \theta_1^2 & -\theta_1 \\ -\theta_1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{N} \sum x - \theta_1 \\ \frac{1}{N} \sum x^2 - \theta_2 \end{bmatrix}, \quad (3.3)$$



which is now a relation of the desired form, and by inspection the estimating functions for  $\mu$  and  $\sigma^2 + \mu^2$  are

$$t_1 = \Sigma x/N, \quad t_2 = \Sigma x^2/N, \quad (3.4)$$

namely the first and second moments about the origin of measurement, which may be arbitrary. At the same time the sampling variance matrix of  $t_1$  and  $t_2$  is the reciprocal of the premultiplying matrix on the right of (3.3), and so is

$$\Lambda = \frac{\sigma^2}{N} \begin{bmatrix} 1 & 2\mu \\ 2\mu & 2(\sigma^2 + 2\mu^2) \end{bmatrix}. \quad (3.5)$$

The sampling variances and product moment of  $t_1$  and  $t_2$  are the respective diagonal and non-diagonal elements of  $\Lambda$ , and by inspection the minimum value of  $|V|$  is

$$|\Lambda| = 2\sigma^6/N^2. \quad (3.6)$$

These results may be contrasted (Solomon, 1944) with those arrived at by the customary process of successive estimation, according to which the estimates of  $\mu$  and  $\sigma^2$  are

$$\bar{x} = \Sigma x/N, \quad s^2 = \Sigma (x - \bar{x})^2 / (N - 1), \quad (3.7)$$

these being unbiased estimates, independent of each other and having the variance matrix

$$V = \begin{bmatrix} \sigma^2/N & . \\ . & 2\sigma^4/(N-1) \end{bmatrix}. \quad (3.8)$$

The value of  $|V|$  for this pair of estimating functions is

$$|V| = 2\sigma^6/\{N(N-1)\}. \quad (3.9)$$

#### 4. PROBABILITY FUNCTIONS ADMITTING THE POSTULATES

By solving the simultaneous partial differential equations

$$\partial \log \Phi / \partial \theta = \Lambda^{-1} (t - \theta) \quad (4.1)$$

we may ascertain the form of  $\Phi$ , and so of  $\phi$ . The solution is of Koopman type, in the multi-parameter form

$$\phi = \exp \{f(x) + F(\theta) + \Sigma t_i(x) \gamma_i(\theta)\}, \quad (4.2)$$

where  $x$ ,  $\theta$  continue to represent vectors, and where

$$-\frac{\partial F}{\partial \theta_i} = \kappa_{i1}\theta_1 + \kappa_{i2}\theta_2 + \dots + \kappa_{ik}\theta_k, \quad \frac{\partial \gamma_i}{\partial \theta_j} = \kappa_{ij}, \quad (4.3)$$

or, briefly

$$-\{\partial F / \partial \theta\} = K\theta, \quad K = [\kappa_{ij}] = \Lambda^{-1} = [\partial \gamma / \partial \theta]. \quad (4.4)$$

The normal probability function evidently belongs to this class, and indeed the mere comparison of

$$\log \phi = -\frac{1}{2} \log (2\pi) - \frac{1}{2} \left( \log \sigma^2 + \frac{\mu^2}{\sigma^2} \right) + x \frac{\mu}{\sigma^2} - \frac{1}{2} \frac{x^2}{\sigma^2} \quad (4.5)$$

with (4.2) might have suggested what is indeed the case, namely that the basic estimating functions here involve  $\Sigma x$  and  $\Sigma x^2$  separately.

#### 5. DETERMINATION OF THE BASIC PARAMETERS

Let us suppose that the parameters admitting the procedure of estimation, the basic parameters, are not  $\theta_1, \theta_2, \dots, \theta_k$ , but certain functions  $\tau_i(\theta)$  of these with non-singular

Jacobian with respect to the  $\theta_j$ . We have then, by differentiating the logarithm of the Koopman function, the vector relation

$$\{\partial \log \Phi / \partial \tau\} = \{\partial F / \partial \tau\} + [\partial \gamma / \partial \tau] t \quad (5.1)$$

$$= [\partial \gamma / \partial \tau] \{t + \partial F / \partial \gamma\}. \quad (5.2)$$

Comparing this with the equation of estimation,

$$t - \tau = \Lambda \{\partial \log \Phi / \partial \tau\},$$

we have

$$\tau = -\{\partial F / \partial \gamma\} = -[\partial \gamma / \partial \theta]^{-1} \{\partial F / \partial \theta\}, \quad (5.3)$$

where  $[\partial \gamma / \partial \theta]$  is the Jacobian matrix of the functions  $\gamma$  with respect to the parameters  $\theta$ ; and the estimating functions, apart from a constant factor  $N$ , are the  $t_i(x)$ .

It is instructive to determine the five basic parameters and their estimating functions in the case of a bivariate normal sample. We have then

$$\log \phi = -\log(2\pi) - \frac{1}{2}(\log |V| + \mu' V^{-1} \mu) + x' V^{-1} \mu - \frac{1}{2} x' V^{-1} x, \quad (5.4)$$

where

$$x = \{x_1 \ x_2\}, \quad \mu = \{\mu_{10} \ \mu_{01}\}, \quad V = \begin{bmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{bmatrix}, \quad (5.5)$$

in the usual notation of bivariate moments. Here  $V$  is the variance matrix of  $x = \{x_1 \ x_2\}$ , the pair of correlated normal variates. The complete sample vector  $X$  would thus be of  $2N$  elements. We might presume by inspection that the estimating functions would be the means, and the mean squares and products, derived from the  $N$  sample pairs. Referring to the Koopman form, we have

$$F = -\frac{1}{2}(\log |V| + \mu' V^{-1} \mu), \quad (5.6)$$

$$t' \gamma = x' V^{-1} \mu - \frac{1}{2} x' V^{-1} x, \quad (5.7)$$

and we begin by taking

$$\theta = \{\mu_{10} \ \mu_{01} \ \mu_{20} \ \mu_{11} \ \mu_{02}\}. \quad (5.8)$$

This gives

$$\frac{\partial F}{\partial \theta} = \begin{bmatrix} |V|^{-1} \begin{bmatrix} \mu_{02} & -\mu_{11} \\ -\mu_{11} & \mu_{20} \end{bmatrix} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & |V|^{-2} \begin{bmatrix} \frac{1}{2}\mu_{02}^2 & -\mu_{02}\mu_{11} & \frac{1}{2}\mu_{11}^2 \\ -\mu_{02}\mu_{11} & \mu_{11}^2 + \mu_{02}\mu_{20} & -\mu_{20}\mu_{11} \\ \frac{1}{2}\mu_{11}^2 & -\mu_{20}\mu_{11} & \frac{1}{2}\mu_{20}^2 \end{bmatrix} & \begin{bmatrix} \mu_{10} \\ \mu_{01} \\ \mu_{20} - \mu_{10}^2 \\ \mu_{11} - \mu_{10}\mu_{01} \\ \mu_{02} - \mu_{01}^2 \end{bmatrix} \end{bmatrix} \quad (5.9)$$

Denoting by  $U$  the premultiplying matrix on the right of (5.9), we find that  $[\partial \gamma / \partial \theta]$  is the following matrix:—

$$U \begin{bmatrix} \text{I} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \text{I} & \cdot & \cdot & \cdot \\ -2\mu_{10} & \cdot & \text{I} & \cdot & \cdot \\ -\mu_{01} & -\mu_{10} & \cdot & \text{I} & \cdot \\ \cdot & -2\mu_{01} & \cdot & \cdot & \text{I} \end{bmatrix} = U \begin{bmatrix} \text{I} & \cdot \\ -\frac{\partial \mu^{[2]}}{\partial \mu} & \text{I} \end{bmatrix}, \quad (5.10)$$

where

$$\mu^{[2]} = \{\mu_{10}^2 \ \mu_{10}\mu_{01} \ \mu_{01}^2\}. \quad (5.11)$$

Applying now the rule (5.3), we find that  $U$  and  $U^{-1}$  annihilate each other and leave

$$\tau = \begin{bmatrix} \text{I} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \text{I} & \cdot & \cdot & \cdot \\ -2\mu_{10} & \cdot & \text{I} & \cdot & \cdot \\ -\mu_{01} & -\mu_{10} & \cdot & \text{I} & \cdot \\ \cdot & -2\mu_{01} & \cdot & \cdot & \text{I} \end{bmatrix}^{-1} \begin{bmatrix} \mu_{10} \\ \mu_{01} \\ \mu_{20} - \mu_{10}^2 \\ \mu_{11} - \mu_{10}\mu_{01} \\ \mu_{02} - \mu_{01}^2 \end{bmatrix} \quad (5.12)$$

$$= \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ 2\mu_{10} & . & 1 & . & . \\ \mu_{01} & \mu_{10} & . & 1 & . \\ . & 2\mu_{01} & . & . & 1 \end{bmatrix} \begin{bmatrix} \mu_{10} \\ \mu_{01} \\ \mu_{20} - \mu_{10}^2 \\ \mu_{11} - \mu_{10}\mu_{01} \\ \mu_{02} - \mu_{01}^2 \end{bmatrix} = \begin{bmatrix} \mu_{10} \\ \mu_{01} \\ \mu_{20} + \mu_{10}^2 \\ \mu_{11} + \mu_{10}\mu_{01} \\ \mu_{02} + \mu_{01}^2 \end{bmatrix} \quad (5.13)$$

Thus the functions  $\tau(\theta)$  are indeed the moments of first and second order, and those of second order are taken about the origins of measurement, not the means of sample. It follows also that the variance matrix of the estimators is

$$[\partial\gamma/\partial\tau]^{-1} = \begin{bmatrix} I & . \\ \frac{\partial\mu^{[2]}}{\partial\mu} & I \end{bmatrix} U^{-1} \begin{bmatrix} I & \frac{\partial\mu^{[2]}}{\partial\mu'} \\ . & I \end{bmatrix}, \quad (5.14)$$

which is readily evaluated, and in § 6 is given in a general form.

From (5.9) and (5.14) the generalized variance is  $|\partial\gamma/\partial\tau|^{-1} = 8 |V|^4$ , or, for a sample of  $N$  values,  $8 |V|^4/N^5$ .

## 6. THE MULTIVARIATE NORMAL CASE

The argument of § 5 can be extended by matrix notation to the case of a multivariate normal sample. First, the matrix of (5.9) is of a recognizable type. It is seen to be partitioned into two isolated principal submatrices, of which the leading one of order  $2 \times 2$  is evidently  $V^{-1}$ , and the other, of order  $3 \times 3$ , is  $\frac{1}{2}(V^{[2]})^{-1}$ , where  $V^{[2]}$  is the second induced or Schläfian matrix, defined as that matrix which, when  $V$  transforms an arbitrary vector  $z$ , transforms the vector  $z^{[2]}$  having for elements (as in (5.11)) the squares and products of elements of  $z$ , in due priority. The properties of  $V^{[2]}$ , as of the more general  $V^{[k]}$ , are well known in the literature.

To show that the results are of the conjectured form in the general multivariate case we introduce operators of vector and matrix differentiation and examine their properties. Let us denote by

$$\mu = \{\mu^{(1)} \mu^{(2)} \dots \mu^{(n)}\}, \quad V = [v_{ij}] \quad (6.1)$$

the  $n \times 1$  vector of means and the  $n \times n$  matrix of variances of the  $n$  normal variates  $x = \{x_1 x_2 \dots x_n\}$ . Let us also write

$$\partial/\partial\mu, \quad \partial/\partial\mu' \quad \text{and} \quad \Omega = [\epsilon_{ij} \partial/\partial v_{ij}], \quad (6.2)$$

where  $\epsilon_{ii} = 1$ ,  $\epsilon_{ij} = \frac{1}{2}$ ,  $i \neq j$ , for the vector and matrix operations of differentiation associated with the elements of  $\mu$  and of  $V$ , the operands being scalar functions of these elements occurring as elements of vectors or matrices. Then we find without difficulty,

$$\frac{\partial}{\partial\mu'}(\mu' V^{-1}\mu) = 2 V^{-1}\mu, \quad \frac{\partial}{\partial\mu}(V^{-1}\mu) = V^{-1}, \quad \Omega \log |V| = V^{-1}, \quad (6.3)$$

$$\Omega(\mu' V^{-1}\mu) = (V^{[2]})^{-1}\mu^{[2]}, \quad \Omega(V^{-1}\mu) = (V^{[2]})^{-1} \frac{\partial\mu^{[2]}}{\partial\mu'}, \quad (6.4)$$

where  $\mu^{[2]}$  is the vector which has for elements the squares and products of the elements of  $\mu$ , duly arranged. It is to be noted that the operator  $\Omega$  differs from the Cayley operator  $[\partial/\partial a_{ij}]$  in that the latter (*cf.* Turnbull, 1927) refers to a *general* matrix  $A = [a_{ij}]$  of  $n^2$  independent elements, whereas in the present case  $V$  is symmetric and has only  $\frac{1}{2}n(n+1)$  independent elements.

By way of illustration of (6.4), let us obtain  $\Omega\mu' V^{-1}\mu$  by making a census of various types of result; for example, if  $|V_{ij}|$  denotes the cofactor of  $v_{ij}$  in  $|V|$ , then we have

$$\left[ \frac{\partial}{\partial v_{ij}} (|V_{kl}| / |V|) \right] = (|V_{ik}| |V_{jl}| + |V_{il}| |V_{jk}|) / |V|^2, \quad (6.5)$$

the right-hand member being a typical element of  $(V^{-1})^{[2]} = (V^{[2]})^{-1}$ , and so

$$\Omega \mu' V^{-1} \mu = (V^{-1})^{[2]} \mu^{[2]} = (V^{[2]})^{-1} \mu^{[2]}. \quad (6.6)$$

At the same time,

$$V^{-1} = [ | V_{ij} | | | V | ] = [ | V_{ij} | | V | | | V |^2 ] \quad (6.7)$$

$$= \left[ \sum_{k,l} v_{kl} ( | V_{ij} | | V_{kl} | + | V_{il} | | V_{jk} | ) / | V |^2 \right], \quad k < l, \quad (6.8)$$

many redundant terms vanishing in the summation because  $\sum_i v_{ki} V_{ii} = 0$ ,  $i \neq k$ .

Thus we obtain a vector form, useful for our purpose, of the relation of  $V^{-1}$  to  $V$ , namely that if all the  $\frac{1}{2}n(n+1)$  elements in and above the diagonal of  $V^{-1}$  be written in order, row after row, as elements of a vector, then this vector is

$$(V^{[2]})^{-1} v, \quad (6.9)$$

where  $v$  is the corresponding vector containing, in the same order, the  $\frac{1}{2}n(n+1)$  elements of  $V$ .

Hence finally, gathering all these results together and writing the partitioned vector  $\{\mu | v\}$  as  $\theta$ , we have

$$-\{\partial F / \partial \theta\} = \begin{bmatrix} V^{-1} & \\ & \frac{1}{2}(V^{[2]})^{-1} \end{bmatrix} \begin{bmatrix} \mu \\ v - \mu^{[2]} \end{bmatrix}, \quad (6.10)$$

while

$$\begin{aligned} [\partial \gamma / \partial \theta] &= \begin{bmatrix} V^{-1} & \\ -\frac{1}{2}(V^{[2]})^{-1} \frac{\partial \mu^{[2]}}{\partial \mu} & \frac{1}{2}(V^{[2]})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} V^{-1} & \\ & \frac{1}{2}(V^{[2]})^{-1} \end{bmatrix} \begin{bmatrix} I & \\ -\frac{\partial \mu^{[2]}}{\partial \mu} & I \end{bmatrix}. \end{aligned} \quad (6.11)$$

As for the submatrix  $[\partial \mu^{[2]} / \partial \mu]$ , we shall use the easily proved result

$$[\partial \mu^{[2]} / \partial \mu] \mu = 2 \mu^{[2]}, \quad (6.12)$$

which is the vector analogue of the scalar relation  $z \cdot \frac{d}{dz} z^2 = 2z^2$ . For example, in the case  $n=2$  we have

$$\mu = [\mu_{10} \quad \mu_{01}], \quad \mu^{[2]} = \{\mu_{10}^2 \quad \mu_{10}\mu_{01} \quad \mu_{01}^2\}, \quad (6.13)$$

$$[\partial \mu^{[2]} / \partial \mu] \mu = \begin{bmatrix} 2\mu_{10} & & \\ \mu_{02} & \mu_{10} & \\ & & 2\mu_{01} \end{bmatrix} \begin{bmatrix} \mu_{10} \\ \mu_{01} \end{bmatrix} = 2 \begin{bmatrix} \mu_{10}^2 \\ \mu_{10}\mu_{01} \\ \mu_{01}^2 \end{bmatrix}, \quad (6.14)$$

and similarly in general.

From (5.3) the vector of parameters to be estimated is

$$-[\partial \gamma / \partial \theta]^{-1} \{\partial F / \partial \theta\} = \begin{bmatrix} I & \\ \frac{\partial \mu^{[2]}}{\partial \mu} & I \end{bmatrix} \begin{bmatrix} \mu \\ v - \mu^{[2]} \end{bmatrix} = \begin{bmatrix} \mu \\ v + \mu^{[2]} \end{bmatrix}. \quad (6.15)$$

The final outcome is that we estimate the means  $\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n)}$  and the quadratic moments about the origin of measurement. At the same time the sampling variance matrix of the estimators is

$$\left( \begin{bmatrix} \frac{\partial \gamma}{\partial \theta} & \frac{\partial \theta}{\partial \tau} \end{bmatrix} \right)^{-1} = N^{-1} \begin{bmatrix} I & \\ \frac{\partial \mu^{[2]}}{\partial \mu} & I \end{bmatrix} \begin{bmatrix} V & \\ & 2V^{[2]} \end{bmatrix} \begin{bmatrix} I & \frac{\partial \mu^{[2]}}{\partial \mu'} \\ & I \end{bmatrix}, \quad (6.16)$$

that is,

$$\Lambda = N^{-1} \begin{bmatrix} V & V \frac{\partial \mu^{[2]}}{\partial \mu'} \\ \frac{\partial \mu^{[2]}}{\partial \mu} V & \frac{\partial \mu^{[2]}}{\partial \mu} V \frac{\partial \mu^{[2]}}{\partial \mu'} \end{bmatrix}, \quad (6.17)$$

from which all the variances and covariances may be readily found. This is the matrix generalization of Solomon's result of (3.5). By known facts regarding the determinant of induced matrices, the value of  $|\Lambda|$  is found from (6.16) to be

$$2^{\frac{1}{2}n(n+1)} N^{-\frac{1}{2}n(n+3)} |V| |V|^{n+1} = 2^{-n} (2/N)^{\frac{1}{2}n(n+3)} |V|^{n+2} \quad (6.18)$$

in agreement with the values already found,  $2\sigma^6/N^2$  for  $n=1$  and  $|V|^4/N^5$  for  $n=2$ .

At this point some comments may be made. The guiding postulates, like any others that might have been adopted, are arbitrary, and have led to formulæ of estimation differing slightly from the customary ones. It is usual, in a multivariate normal sample, first to estimate the  $n$  means in the usual way, then to estimate quadratic moments about the sample means as follows:—

$$\hat{v}_{ij} = \sum_{i,j} (x_i - \bar{x}_i)(x_j - \bar{x}_j) / (N-1). \quad (6.19)$$

The justification for the divisor  $N-1$  is that with this divisor the estimate  $\hat{v}_{ij}$  is indeed unbiased, in the sense that its mean value over all possible samples is  $v_{ij}$ . When  $n$  exceeds 1, however, the estimating functions  $\hat{v}_{ij}$  of the  $v_{ij}$  are not all independent. The  $\hat{v}_{ii}$  and  $\hat{v}_{jj}$  are independent of one another, but the  $\hat{v}_{ij}$ ,  $i \neq j$  are dependent on the  $\hat{v}_{ii}$  and  $\hat{v}_{jj}$ . The variance-determinant of these estimates and those of the means is

$$\{2/(N-1)\}^{\frac{1}{2}n(n+1)} N^{-n} |V|^{n+2}, \quad (6.20)$$

which stands in contrast with (6.18) above.

Such divergencies do not affect procedures of estimation and tests of significance adversely. If estimating functions are computed from sample values according to a logical prescription, and if tables of probability are prepared in the light of this prescription and no other, then the tests based upon them have their own validity.

## 7. THE CASE OF ORDINARY LINEAR LEAST SQUARES

The point of interest arises whether, in the classical case of Least Squares applied to linear observational equations, the adoption of the postulate of unbiased solutions and minimal determinant of variance will produce a result differing from that given by the normal equations of Gauss. The answer to this question has been given (Aitken, 1934) almost explicitly; namely the results are exactly the same. In fact, let the observational equations be the prepared set  $Ax = u$ . Then the usual normal equations are  $A'Ax = A'u$ . Now, adapting the present postulates to linear estimation, let us take the solution to be  $x = Bu$ , where

- (i)  $Bu$  is unbiased, which in this context means that  $BA = I$ ;
- (ii) the variance determinant  $|BB'|$  is minimal.

Now

$$|BB'| = (BB')^{(n)}, \quad (7.1)$$

namely the  $n$ th compound of  $BB'$ , while the  $n$ th compound of  $BA = I$  gives, by the multiplicative property of compound matrices, the compound condition of unbiasedness,

$$(BA)^{(n)} = B^{(n)}A^{(n)} = I. \quad (7.2)$$

Thus we are minimizing the trace of  $(BB')^{(n)}$  subject to (7.2), and this is equivalent (Aitken, 1934) to one form of the accepted postulates of Least Squares, but at the level of  $n$ th compounds, and therefore leading to the solution

$$B^{(n)} = [(A'A)^{(n)}]^{-1} (A^{(n)})'. \quad (7.3)$$

Now this is the  $n$ th compound relation corresponding to

$$B = (A'A)^{-1}A', \text{ that is, to } A'Ax = A'u. \quad (7.4)$$

Thus the solutions of the classical normal equations minimize at the same time the determinant  $|BB'|$ ; and so the solutions according to our postulates, provided only that they are unique, coincide with those given by the classical procedure. An exactly similar argument holds for the case where the observations of the linear functions are affected by correlated errors.

This discussion suggests a variety of alternatives to the postulates of minimum determinant of variance. It suggests that we might minimize the trace, or the trace of any compound, that is, the sum of principal minors of a given order  $m$ . It is likely that the results would be exactly the same as we have found.

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#### CORRIGENDUM

In the paper on single-parameter estimation (Aitken and Silverstone, 1941) it is stated (§ 5) that the mean can be estimated, in the sense of the postulates, as a basic parameter in a series of Type A. Closer examination (Solomon, 1944) shows that this possibility is ruled out by the condition of total probability. The normal curve is the only curve of Type A for which the statement is true.

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## XXXIX.—Transformations of Hypergeometric Functions of Two Variables.

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1. THERE are several methods for obtaining transformations of hypergeometric functions of two variables.

Firstly, by transformation of the hypergeometric series. When the double series is rewritten as an infinite sum of hypergeometric functions of one variable, the known transformation theory of such functions can be applied to each term. This method is quite simple and, in a limited range, very effective for discovering transformations as well as proving them.

Secondly, by transformation of the systems of partial differential equations satisfied by the hypergeometric functions. This method, though simple in theory, is rather laborious in practice and not very useful for discovering new transformations.

Thirdly, by transformation of contour integrals representing the hypergeometric functions. This third method will be applied here to prove a number of transformations. Some of these transformations have previously been obtained, by other writers, by the first method; and others could be so obtained. There are, however, transformations, such as (9.1), which it would have been difficult to discover by either of the first two methods.

Only complete hypergeometric functions of order two will be considered. In the notation of Horn\* these are the functions for which  $p=p'=q=q'=2$ , and which are denoted by the symbols  $F_1, \dots, F_4, G_1, \dots, G_3, H_1, \dots, H_7$ . (The definition of these functions is given in Horn's paper.) Transformations of confluent functions can be deduced from those of complete functions by limiting processes.

Even with this restriction it would be tiresome to list *all* transformations. A selection has been made so as to exhibit a property of hypergeometric functions the discovery of which caused me to develop, three years ago, the present transformation theory. It will be seen that with three possible exceptions all complete hypergeometric functions of two variables and of order two can be expressed in terms of Appell's series  $F_2$ , the basic importance of which is thus revealed.

2. The familiar abbreviation

$$(\epsilon)_n = \frac{\Gamma(\epsilon+n)}{\Gamma(\epsilon)}$$

will be used.

The contours of integration will be Pochhammer double loops and  $[a_1, \dots, a_m; b_1, \dots, b_n]$  will be the notation for the double loop which contains  $a_1, \dots, a_m$  within one loop and  $b_1, \dots, b_n$  within the other. It is understood that all other singularities of the integrand are outside the contour of integration.  $z^\alpha$  is interpreted as  $\exp(\alpha \log z)$ , where  $\log z$  is real when  $z$  is positive, and continuous on the contour of integration. With these conventions

$$\int_{[0;1]} (-z)^{\alpha-1} (z-1)^{\beta-1} dz = \frac{(2\pi i)^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}, \quad (2.1)$$

and all our contour integrals will be based on this formula.

It will be assumed that  $x$  and  $y$  have such values that the infinite series occurring in the analysis converge, and that the grouping of singularities indicated in the symbol for the double loop is possible. Exceptional values of the parameters which would make some of the gamma functions become infinite are tacitly excluded. The general validity of the results follows by analytic continuation.

Hypergeometric functions of two variables,  $x$  and  $y$ , will be represented by integrals of the form

$$\int (-t)^{\mu-1} (t-1)^{\nu-1} f(u) g(v) dt,$$

where  $u$  is a function of  $x$  and  $t$ , and  $v$  of  $y$  and  $t$ . We then say that the Euler transformation *factorises* our hypergeometric function, *i.e.* maps it into a product of two functions,  $f$  and  $g$ , each of which satisfies an ordinary differential equation.

### LINEAR TRANSFORMATIONS

3. From the definition of  $F_2$  the integral representation

$$F_2(\rho + \rho' - 1, \beta, \beta', \gamma, \gamma'; x, y) = (2\pi i)^{-2} \Gamma(\rho) \Gamma(\rho') \Gamma(2 - \rho - \rho') \\ \times \int (-t)^{-\rho} (t-1)^{-\rho'} F\left(\rho, \beta; \gamma; \frac{x}{t}\right) F\left(\rho', \beta'; \gamma'; \frac{y}{1-t}\right) dt, \quad (3.1)$$

with  $[0, x; 1, 1-y]$  as contour of integration immediately follows. This is the factorised form of  $F_2$ ; the general solution of the system of partial differential equations of  $F_2$  is obtained by replacing each of the Gauss series in the integrand by the general solution of the hypergeometric equation, and the contour by any contour closed on the Riemann surface of the integrand.

Applying Euler's transformation \*

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \quad (3.2)$$

to the first hypergeometric function in the integral in (3.1) and introducing  $(x-t)/(x-1)$  as a new variable of integration, the transformation †

$$F_2(a, \beta, \beta'; \gamma, \gamma'; x, y) = (1-x)^{-a} F_2\left(a, \gamma-\beta, \beta', \gamma, \gamma'; \frac{x}{x-1}, \frac{y}{1-x}\right) \quad (3.3)$$

of  $F_2$  is obtained. Applying (3.2) to the second or to both hypergeometric functions in the integral in (3.1), two similar transformations ‡ follow.

4. Another important transformation flows from the relation connecting different branches of Riemann's P function. It will be convenient to write this relation in the form §

$$F(a, b; c; z) = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(c-a) \Gamma(b)} (1-z)^{-a} F\left(a, c-b; 1+a-b; \frac{1}{1-z}\right) + a \rightleftharpoons b, \quad (4.1)$$

where the symbol  $+a \rightleftharpoons b$  indicates that on the right-hand side a second term must be added which originates from the first one by interchanging  $a$  and  $b$ .

The transformation alluded to expresses a certain combination of two functions  $F_2$  as an  $H_2$ , and we shall therefore start with the integral representation

$$H_2(\rho - \sigma, \beta, \gamma, \delta, \epsilon; x, y) = (2\pi i)^{-2} \Gamma(\rho) \Gamma(1 - \sigma) \Gamma(1 - \rho + \sigma) \\ \times \int_{[0, x; 1]} (-t)^{-\rho} (t-1)^{\sigma-1} F\left(\rho, \beta; \epsilon; \frac{x}{t}\right) F(\gamma, \delta; \sigma; (t-1)y) dt \quad (4.2)$$

which follows from the definition, and is the factorised form, of  $H_2$ . For the present purpose it is more convenient to use the factorisation

$$H_2 = (2\pi i)^{-2} \Gamma(\rho) \Gamma(1 - \rho + \sigma) \Gamma(1 + \gamma - \sigma) \Gamma(1 + \delta - \sigma) / \Gamma(1 + \gamma + \delta - \sigma) \\ \times \int (-t)^{-\rho} (t-1)^{\sigma-1} F\left(\rho, \beta; \epsilon; \frac{x}{t}\right) F(\gamma, \delta; 1 + \gamma + \delta - \sigma; 1 - (t-1)y) dt. \quad (4.3)$$

\* (1), p. 3, equation (6). † (1), p. 25, equation (31').

‡ (1), p. 25, equations (31), (32).

§ (1), p. 7, equation (10'): note the relations  $y_8 = y_{11}$  and  $y_6 = y_{12}$ .



The proof of this integral representation with  $[0, x; 1]$  as contour follows from the remark that the difference of the right-hand sides of (4.3) and (4.2) is the integral over  $[0, x; 1]$  of a function which is regular at  $z=1$ : this difference therefore vanishes.

The integrand of (4.3) is regular at  $z=1+y^{-1}$  and therefore the contour  $[0, x; 1]$  may be replaced by  $[0, x; 1, 1+y^{-1}]$ . Using this latter contour and applying (4.1) to the second hypergeometric function in the integral (4.3), this integral breaks up into two, each of the form (3.1): the result is the transformation

$$H_2(\alpha, \beta, \gamma, \delta, \epsilon; x, y) = \frac{\Gamma(1-\alpha)\Gamma(\delta-\gamma)}{\Gamma(\delta)\Gamma(1-\alpha-\gamma)} y^{-\gamma} F_2\left(\alpha+\gamma, \beta, \gamma, \epsilon, 1+\gamma-\delta; x, -\frac{1}{y}\right) + \gamma \Longleftrightarrow \delta. \quad (4.4)$$

At first this result may seem to suggest that conversely  $F_2$  might be expressible as a linear combination of two functions  $H_2$ ; this, however, is not the case. Indeed,  $F_2\left(x, -\frac{1}{y}\right)$  can be expressed as a combination of two hypergeometric series in  $x, y$ : one of these series is an  $H_2(x, y)$ , the other, however, is a series which is not contained in Horn's list\* and, in fact, according to the accepted classification would rank as a hypergeometric series of order *three* (while  $F_2$  and  $H_2$  are of order two). This appears to be an indication of the inadequacy of the classification of hypergeometric series of two variables accepted at present.

5. Similarly we obtain a transformation of  $F_3$  into two  $H_2$ . From the definition of  $F_3$ ,

$$F_3(\alpha, \alpha', \beta, \beta', \rho + \rho'; x, y) = (2\pi i)^{-2} \Gamma(1-\rho) \Gamma(1-\rho') \Gamma(\rho + \rho') \\ \times \int_{[0; 1]} (-t)^{\rho-1} (t-1)^{\rho'-1} F(\alpha, \beta; \rho; tx) F(\alpha', \beta'; \rho'; (1-t)y) dt. \quad (5.1)$$

As in the previous case, this representation is equivalent to

$$F_3 = (2\pi i)^{-2} \Gamma(1-\rho') \Gamma(\rho + \rho') \Gamma(1+\alpha-\rho) \Gamma(1+\beta-\rho) / \Gamma(1+\alpha+\beta-\rho) \\ \times \int (-t)^{\rho-1} (t-1)^{\rho'-1} F(\alpha, \beta; 1+\alpha+\beta-\rho; 1-xt) F(\alpha', \beta'; \rho'; (1-t)y) dt, \quad (5.2)$$

the contour of integration being either  $[0; 1]$  or  $[0, x^{-1}; 1]$ . With the latter contour, (4.1) is applied to  $F(\dots; 1-xt)$ , thus causing the integral in (5.2) to break up in two integrals, each of the type (4.2). The transformation

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) \\ = \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-x)^{-\alpha} H_2\left(1+\alpha-\gamma, \alpha, \alpha', \beta', 1+\alpha-\beta; \frac{1}{x}, -y\right) + \alpha \Longleftrightarrow \beta \quad (5.3)$$

follows at once. There is a corresponding transformation in  $y$  of  $F_3$ . Combining (5.3) with (4.4) a transformation of  $F_3$  into four  $F_2$  results.† This transformation together with (5.3) and its analogue in  $y$  constitute the extension to  $F_3$  of (4.1).

The transformations derived in sections 3-5 and their diverse combinations are the only linear transformations between  $F_2$ ,  $F_3$ , and  $H_2$  with arbitrary values of the parameters.

6. In this section linear transformations of  $F_2$  and of the associated functions  $F_3$  and  $H_2$  will be discussed when the five parameters are connected by one relation.

First let us assume that  $\alpha - \gamma - \gamma' + 1 = 0$  in  $F_2$ . Then in (3.1) we may take  $\rho = \gamma, \rho' = \gamma'$ , and have

$$F_2(\gamma + \gamma' - 1, \beta, \beta', \gamma, \gamma'; x, y) = (2\pi i)^{-2} \Gamma(\gamma) \Gamma(\gamma') \Gamma(2 - \gamma - \gamma') \\ \times \int_{[0, x; 1, 1-y]} (-t)^{-\gamma} (t-1)^{-\gamma'} \left(1 - \frac{x}{t}\right)^{-\beta} \left(1 - \frac{y}{1-t}\right)^{-\beta'} dt. \quad (6.1)$$

\* (7), p. 383.

† (1), p. 43, equation (34).

Here we may expand

$$\left(1 - \frac{x}{t}\right)^{-\beta} = (1-x)^{-\beta} \left(1 - \frac{x}{1-x} \frac{1-t}{t}\right)^{-\beta} = (1-x)^{-\beta} \sum_{m=0}^{\infty} \frac{(\beta)_m}{m!} \left(\frac{x}{1-x} \frac{1-t}{t}\right)^m$$

and find at once

$$F_2(\gamma + \gamma' - 1, \beta, \beta', \gamma, \gamma'; x, y) = (1-x)^{-\beta} H_2\left(1 - \gamma', \beta, \gamma + \gamma' - 1, \beta', \gamma; \frac{x}{x-1}, -y\right). \quad (6.2)$$

If the factor  $\{1 - y/(1-t)\}^{-\beta'}$  in (6.1) is also expanded in a similar way, a second transformation of the same function is found, viz.

$$F_2(\gamma + \gamma' - 1, \beta, \beta', \gamma, \gamma'; x, y) = (1-x)^{-\beta} (1-y)^{-\beta'} G_2\left(\beta, \beta', 1 - \gamma, 1 - \gamma'; \frac{x}{1-x}, \frac{y}{1-y}\right). \quad (6.3)$$

On the other hand, if  $\alpha + \alpha' = \gamma$  in  $F_3$ , we may take  $\rho = \alpha$ ,  $\rho' = \alpha'$  in (5.1): writing

$$F(\alpha', \beta'; \alpha'; (1-t)y) = (1-y + yt)^{-\beta'} = (1-y)^{-\beta'} \left(1 + \frac{yt}{1-y}\right)^{-\beta'}$$

and expanding in powers of  $t$ , we immediately obtain the transformation formula \*

$$F_3(\alpha, \alpha', \beta, \beta', \alpha + \alpha'; x, y) = (1-y)^{-\beta'} F_1\left(\alpha, \beta, \beta', \alpha + \alpha'; x, \frac{y}{y-1}\right). \quad (6.4)$$

Many more transformations of the functions considered in this section are obtainable by the standard procedure of transformation of the variable of integration combined in some cases with deformation of the contour.

## QUADRATIC TRANSFORMATIONS

7. With our method, quadratic transformations of hypergeometric functions of two variables arise in two different ways. From integral representations of the type used in sections 3-5, quadratic transformations of hypergeometric functions of two variables arise as consequences of such transformations of hypergeometric functions of one variable in the integrand. For the more special functions possessing integral representations of the type used in section 6, quadratic transformations are the consequence of two of the exponents of the integrand being equal. Let us first obtain some transformations belonging to the first group. The quadratic transformation of Gauss's series will be used in the form †

$$F(\alpha, \beta; 2\beta; z) = (1 - \frac{1}{2}z)^{-\alpha} F\left(\frac{1}{2}\alpha, \frac{1}{2}\alpha + \frac{1}{2}; \beta + \frac{1}{2}; \frac{z^2}{(2-z)^2}\right). \quad (7.1)$$

If  $\gamma = 2\beta$  in (3.1), (7.1) can be applied to  $F(x/t)$  and the result is

$$F_2(\rho + \rho' - 1, \beta, \beta', 2\beta, \gamma'; x, y) = (2\pi i)^{-2} \Gamma(\rho) \Gamma(\rho') \Gamma(2 - \rho - \rho') \\ \times \int \left(\frac{1}{2}x - t\right)^{-\rho} (t-1)^{-\rho'} F\left(\frac{1}{2}\rho, \frac{1}{2}\rho + 1; \beta + \frac{1}{2}; \frac{x^2}{(2t-x)^2}\right) F\left(\rho', \beta'; \gamma'; \frac{y}{1-t}\right) dt. \quad (7.2)$$

Introducing a new variable of integration  $u$  by the substitution  $t = \frac{1}{2}x + (1 - \frac{1}{2}x)u$ , the integral in (7.2) changes into

$$(1 - \frac{1}{2}x)^{1-\rho-\rho'} \int (-u)^{-\rho} (u-1)^{-\rho'} F\left(\frac{1}{2}\rho, \frac{1}{2}\rho + 1; \beta + \frac{1}{2}; \frac{x^2}{(2-x)^2 u^2}\right) F\left(\rho', \beta'; \gamma'; \frac{y}{(1-\frac{1}{2}x)(1-u)}\right) du, \quad (7.3)$$

where the contour of integration in the  $u$ -plane is  $[0, x/(2-x); 1, 1 - y/(1 - \frac{1}{2}x)]$ . Expanding the hypergeometric functions in the integrand, the quadratic transformation

$$F_2(\alpha, \beta, \beta', 2\beta, \gamma'; x, y) = (1 - \frac{1}{2}x)^{-\alpha} H_4\left(\alpha, \beta', \beta + \frac{1}{2}, \gamma'; \frac{x^2}{4(2-x)^2}, \frac{2y}{2-x}\right) \quad (7.4)$$

\* (1), p. 25, equation (34).

† (6), equation (45).

is obtained. Similar is the proof of the transformation

$$H_2(\alpha, \beta, \gamma, \delta, 2\beta; x, y) = (1 - \frac{1}{2}x)^{-\alpha} H_7\left(\alpha, \gamma, \delta, \beta + \frac{1}{2}; \frac{x^2}{4(2-x)^2}, y(1 - \frac{1}{2}x)\right). \quad (7.5)$$

Again, if in the last integral, (7.3),  $\gamma' = 2\beta'$ , the second hypergeometric function admits of a quadratic transformation which induces the transformation

$$H_4(\alpha, \beta, \gamma, 2\beta; x, y) = (1 - \frac{1}{2}y)^{-\alpha} F_4\left(\frac{1}{2}\alpha, \frac{1}{2}\alpha + \frac{1}{2}, \gamma, \beta + \frac{1}{2}; \frac{16x}{(2-y)^2}, \frac{y^2}{(2-y)^2}\right). \quad (7.6)$$

While both (7.4) and (7.6) seem to be new, their combination, a quadratic transformation of  $F_2(\alpha, \beta, \beta', 2\beta, 2\beta'; x, y)$  into an  $F_4$ , has been proved in a different way by Bailey.\*

There is also another relation between  $H_4$  and  $F_4$ ,

$$F_4(\alpha, \alpha + \frac{1}{2} - \beta, \gamma, \beta + \frac{1}{2}; x, y) = (1 + y)^{-\alpha} H_4\left(\alpha, \alpha + \frac{1}{2} - \beta, \beta + \frac{1}{2}, \gamma; \frac{y}{(1+y)^2}, \frac{x}{1+y}\right), \quad (7.7)$$

which in connection with (7.4) yields a second transformation,†

$$F_4(\alpha, \alpha + \frac{1}{2} - \beta, \gamma, \beta + \frac{1}{2}; x, y^2) = (1 + y)^{-2\alpha} F_2\left(\alpha, \alpha + \frac{1}{2} - \beta, \beta, \gamma, 2\beta; \frac{x}{(1+y)^2}, \frac{4y}{(1+y)^2}\right), \quad (7.8)$$

between  $F_2$  and  $F_4$ .

8. The second group of quadratic transformations arises when the parameters of  $F_2$  (or  $H_2$ , or  $F_3$ ) satisfy two independent relations. If, e.g.,  $\alpha = \gamma + \gamma' - 1$  and  $\beta = \beta'$  in  $F_2$ , from (6.1)

$$F_2(\gamma + \gamma' - 1, \beta, \beta, \gamma, \gamma'; x, y) = (2\pi i)^{-2} \Gamma(\gamma) \Gamma(\gamma') \Gamma(2 - \gamma - \gamma') \\ \times \int (-t)^{-\gamma} (t - 1)^{-\gamma'} \left\{ 1 - x - y + 2xy - \gamma(1 - x) \frac{t}{1 - t} - x(1 - y) \frac{1 - t}{t} \right\}^{-\beta} dt. \quad (8.1)$$

If  $x$  and  $y$  are sufficiently small, the contour of integration,  $[0, x; 1, 1 - y]$ , may be so chosen that  $|y(1 - x)t/(1 - t)| + |x(1 - y)(1 - t)/t| < |1 - x - y + 2xy|$  at all points of the contour. This being so, the expansion

$$\{ \dots \}^{-\beta} = (1 - x - y + 2xy)^{-\beta} \sum \frac{(\beta)_{m+n}}{m! n!} \left( \frac{y(1 - x)}{1 - x - y + 2xy} \frac{t}{1 - t} \right)^m \left( \frac{x(1 - y)}{1 - x - y + 2xy} \frac{1 - t}{t} \right)^n$$

may be used in (8.1) to give

$$F_2(\gamma + \gamma' - 1, \beta, \beta, \gamma, \gamma'; x, y) \\ = (1 - x - y + 2xy)^{-\beta} G_1\left(\beta, 1 - \gamma, 1 - \gamma'; \frac{x(1 - y)}{1 - x - y + 2xy}, \frac{y(1 - x)}{1 - x - y + 2xy}\right). \quad (8.2)$$

Similar is the proof of the quadratic transformations

$$H_2(\epsilon + \gamma - 1, \beta, \gamma, 1 - \gamma, \epsilon; x, y) \\ = (1 + y)^{1 - \epsilon - \gamma} (1 + 2y)^{\epsilon - 1} H_6\left(1 - \epsilon, \epsilon + \gamma - 1, \beta; -\frac{y(1 + y)}{(1 + 2y)^2}, -\frac{x(1 + 2y)}{1 + y}\right) \quad (8.3)$$

and

$$F_3(\alpha, \alpha', 1 - \alpha, \beta', \alpha + \alpha'; x, y) \\ = (1 - x)^{\alpha + \alpha' - 1} (1 - 2x)^{-\alpha'} H_3\left(\alpha', \beta', \alpha + \alpha'; \frac{x(x - 1)}{(1 - 2x)^2}, \frac{1 - x}{1 - 2x}\right). \quad (8.4)$$

If, on the other hand, the expansion

$$\{ \dots \}^{-\beta} = \left\{ 1 - \frac{y(1 - x)}{1 - t} - \frac{x(1 - y)}{t} \right\}^{-\beta} = \sum \frac{(\beta)_{m+n}}{m! n!} \left( \frac{y(1 - x)}{1 - t} \right)^m \left( \frac{x(1 - y)}{t} \right)^n$$

\* (2), § 3.

† (1), p. 27, last formula.

is used in (8.1), the result is \*

$$F_2(\gamma + \gamma' - 1, \beta, \beta, \gamma, \gamma'; x, y) = F_4(\gamma + \gamma' - 1, \beta, \gamma, \gamma'; x(1-y), y(1-x)). \quad (8.5)$$

Similar is the proof of

$$H_2(\epsilon - \delta, \beta, \beta, \delta, \epsilon; x, y) = (1 + xy)^{-\beta} H_1\left(\epsilon - \delta, \beta, \delta, \epsilon; \frac{x(1+y)}{1+xy}, \frac{y}{1+xy}\right). \quad (8.6)$$

### A BIQUADRATIC TRANSFORMATION

9. As an example of a rational transformation of higher degree, let us consider the transformation of  $F_2(\gamma + \gamma' - 1, 2 - \gamma - \gamma', 2 - \gamma - \gamma', \gamma, \gamma'; x, y)$ . In the integral representation (8.1), with  $\beta = 2 - \gamma - \gamma'$ , introduce the new variable  $u = (1 - y + xy)\epsilon / \{1 - x + xy + (x - y)\epsilon\}$  to obtain

$$(2\pi i)^{-2} \Gamma(\gamma) \Gamma(\gamma') \Gamma(2 - \gamma - \gamma') (1 - x + xy)^{\gamma' - 1} (1 - y + xy)^{\gamma - 1} \\ \times \int (-u)^{-\gamma} (u - 1)^{-\gamma'} \left\{ 1 + \gamma(1 - x) \frac{1 - x + xy}{(1 - y + xy)^2} \frac{u^2}{u - 1} - x(1 - y) \frac{1 - y + xy}{(1 - x + xy)^2} \frac{(u - 1)^2}{u} \right\}^{\gamma + \gamma' - 2} du.$$

Expanding  $\{ \dots \}^{\gamma + \gamma' - 2}$  according to the multinomial theorem, the transformation

$$F_2(\gamma + \gamma' - 1, 2 - \gamma - \gamma', 2 - \gamma - \gamma', \gamma, \gamma'; x, y) \\ = (1 - y + xy)^{\gamma - 1} (1 - x + xy)^{\gamma' - 1} G_3\left(1 - \gamma, 1 - \gamma'; x(1 - y) \frac{1 - y + xy}{(1 - x + xy)^2}, y(1 - x) \frac{1 - x + xy}{(1 - y + xy)^2}\right) \quad (9.1)$$

immediately follows.

### APPLICATIONS

10. A first application of the transformation theory developed in this paper concerns the relations between hypergeometric functions of two variables. Leaving aside the confluent functions which are limiting cases of the complete ones, fourteen distinct functions of the second order are found in Horn's list—those denoted by the letters F, G, H. From the preceding sections it is seen that eleven of these fourteen functions can be expressed in terms of Appell's series  $F_2$ . *All functions F, G, H, with the only possible exception of  $F_4$ ,  $H_1$ , and  $H_5$ , can be expressed in terms of  $F_2$ . With  $F_4$  and  $H_1$  such an expression is still possible provided that the parameters satisfy a certain relation.* I am inclined to believe, though I have not succeeded in proving, that there is no simple connection between  $F_2$  on the one side and  $F_4$ ,  $H_1$  (with arbitrary parameters), and  $H_5$  on the other.

This reduction of all but possibly three functions F, G, H to  $F_2$  is of great importance for the integration of the hypergeometric systems of partial differential equations. Leaving aside the confluent systems which are limiting forms of complete ones, there are fourteen apparently distinct types of systems of order two, corresponding to the fourteen series F, G, H. In spite of much valuable work done, especially by Professor Horn† and Dr Borngässer,‡ the integration of these systems still presents considerable difficulties. It is therefore of interest to know that a complete theory of the system associated with Appell's series  $F_2$  would settle the problem of all but possibly three of them. As to the integration of this latter system, (3.1) suggests that it could be accomplished by integrals of the type

$$\int_C (-t)^{-\rho} (t - 1)^{-\rho'} P \left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 0 & \rho \\ 1 - \gamma & \gamma - \beta - \rho & \beta \end{matrix} \middle| \frac{x}{t} \right\} P \left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 0 & \rho \\ 1 - \gamma' & \gamma' - \beta' - \rho' & \beta \end{matrix} \middle| \frac{y}{1 - t} \right\} dt,$$

where  $\rho + \rho' = \alpha + 1$ , P is the usual symbol for Riemann's P function, and C is any contour closed on the Riemann surface of the integrand.

\* (4), equation (52).

† (7), (8).

‡ (3).

A closer scrutiny of our results shows that all those of the fourteen systems which have only *three* linearly independent integrals can be transformed into the system of  $F_1$ , and this in its turn can be integrated (as is well known) by integrals of the type

$$\int (-t)^{-\gamma} (t-1)^{-\gamma'} \left(1 - \frac{x}{t}\right)^{-\beta} \left(1 - \frac{y}{1-t}\right)^{-\beta'} dt.$$

It is even true that *all* (also confluent) hypergeometric systems of the second order with only three linearly independent integrals can be transformed into the system of  $F_1$  or limiting cases thereof: the proof of this theorem is contained in a paper which will appear in the *Acta Mathematica*.

11. Another application of the transformation theory consists in using it in connection with known expansions of  $F_2$  and thereby deriving expansions of the other functions. Merely to give an example, we use Burchall and Chaundy's expansion \*

$$F_2(\alpha, \beta, \beta, \gamma, \gamma'; x, y) = \sum_{r=0}^{\infty} (-)^r \frac{(\alpha)_{2r}(\beta)_r}{r! (\gamma)_r (\gamma')_r} x^r y^r F_4(\alpha + 2r, \beta + r, \gamma + r, \gamma' + r; x, y)$$

in conjunction with (8.5) to obtain

$$F_4(\gamma + \gamma' - 1, \beta, \gamma, \gamma'; x(1-y), y(1-x)) \\ = \sum_{r=0}^{\infty} (-)^r \frac{(\gamma + \gamma' - 1)_{2r}(\beta)_r}{r! (\gamma)_r (\gamma')_r} x^r y^r F_4(\gamma + \gamma' - 1 + 2r, \beta + r, \gamma + r, \gamma' + r; x, y). \quad (11.1)$$

In the same way known reduction formulæ for any of the functions can be utilised in order to derive other such formulæ. For instance, from the reduction formula

$$F_4(\alpha, \gamma + \gamma' - \alpha - 1, \gamma, \gamma'; \sin^2 u \cos^2 v, \cos^2 u \sin^2 v) \\ = F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma; \sin^2 u) F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma'; \sin^2 v), \quad (11.2)$$

due to Bailey and Watson, and (7.6),

$$H_4\left(\gamma + \beta - 1, \beta, \gamma, 2\beta; \frac{\sin^2 u \cos^2 v}{4(1 + \cos u \sin v)^2}, \frac{2 \cos u \sin v}{1 + \cos u \sin v}\right) \\ = (1 + \cos u \sin v)^{\gamma + \beta - 1} F\left(\frac{\gamma + \beta - 1}{2}, \frac{\gamma + \beta}{2}; \gamma; \sin^2 u\right) F\left(\frac{\gamma + \beta - 1}{2}, \frac{\gamma + \beta}{2}; \beta + \frac{1}{2}; \sin^2 v\right), \quad (11.3)$$

and from (11.2) and (7.4),

$$F_2\left(\beta + \beta' - \frac{1}{2}, \beta, \beta', 2\beta, 2\beta'; \frac{2 \sin u \cos v}{1 + \sin(u+v)}, \frac{2 \sin v \cos u}{1 + \sin(u+v)}\right) \\ = \{1 + \sin(u+v)\}^{\beta + \beta' - \frac{1}{2}} F\left(\frac{\beta + \beta'}{2} - \frac{1}{4}, \frac{\beta + \beta'}{2} + \frac{1}{4}; \beta + \frac{1}{2}; \sin^2 u\right) \\ F\left(\frac{\beta + \beta'}{2} - \frac{1}{4}, \frac{\beta + \beta'}{2} + \frac{1}{4}; \beta' + \frac{1}{2}; \sin^2 v\right) \quad (11.4)$$

immediately follow. By a quadratic transformation of the Gauss series the right-hand side of (11.4) changes into

$$\left\{ \frac{1 + \sin(u+v)}{(1 + \sin u)(1 + \sin v)} \right\}^{\beta + \beta' - \frac{1}{2}} F\left(\beta + \beta' - \frac{1}{2}, \beta; 2\beta; \frac{2 \sin u}{1 + \sin u}\right) \\ F\left(\beta + \beta' - \frac{1}{2}, \beta'; 2\beta'; \frac{2 \sin v}{1 + \sin v}\right). \quad (11.5)$$

Similar reduction formulæ can be derived from (7.7) and (7.8).

\* (4), equation (37).

## GENERALISATIONS

12. The methods used here apply also to hypergeometric functions of more than two variables. A complete factorisation by relations analogous to (3.1) would involve multiple integrals, and therefore in some cases it is better only partly to factorise the functions in question, grouping their variables in two groups. Thus properties of functions with  $n$  variables can be deduced from properties of functions with a smaller number of variables, and ultimately, by induction, from the properties of hypergeometric functions of one variable.

As an example we shall consider Lauricella's series \*

$$F_A^n(\alpha; \beta_i; \gamma_i; x_i) = \sum \frac{(\alpha)_{m_1+\dots+m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{m_1! \dots m_n! (\gamma_1)_{m_1} \dots (\gamma_n)_{m_n}} x_1^{m_1} \dots x_n^{m_n} \quad (12.1)$$

Clearly  $F_A^1 = F$  and  $F_A^2 = F_2$ . The relation corresponding to (3.1) is

$$F_A^{n+n'}(\rho+\rho'-1; \beta_i, \beta'_i; \gamma_i, \gamma'_i; x_i, y_i) = (2\pi i)^{-2} \Gamma(\rho) \Gamma(\rho') \Gamma(2-\rho-\rho') \\ \times \int (-t)^{-\rho} (t-1)^{-\rho'} F_A^n\left(\rho; \beta_i; \gamma_i; \frac{x_i}{t}\right) F_A^{n'}\left(\rho'; \beta'_i; \gamma'_i; \frac{y_i}{1-t}\right) dt, \quad (12.2)$$

where the contour of integration is a  $[0; 1]$  along which

$$|t| > |x_1| + \dots + |x_n| \quad \text{and} \quad |t-1| > |y_1| + \dots + |y_{n'}|.$$

From (12.2) a proof by induction of the linear transformations of  $F_A$  into itself † immediately follows. In the same way the expression of  $F_B$  in terms of  $2^n$  functions  $F_A$  is easily obtained. The transformations analogous to (4.4) and (5.3) are also deducible from (12.2) and the analogous relation for  $F_B$ . The place of  $H_2$  is taken by the series ‡  $H_{n,p}$ . There is a more general linear transformation representing an  $H_{n,p'}$  by a linear combination of series of the type  $H_{n,p'}$  with  $p' > p$ .

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\* (1), p. 114, equation (1).

† (1), p. 116, equation (9).

‡ (5), § 5.



# XL.—The Linear Difference-differential Equation with Constant Coefficients.

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## INTRODUCTION

1. THE general homogeneous difference-differential equation with constant coefficients is

$$\Lambda\{y(x)\} \equiv \sum_{\mu=0}^m \sum_{\nu=0}^n a_{\mu\nu} y^{(\nu)}(x+b_{\mu}) = 0, \quad (1.1)$$

where  $x$  is a real variable,  $y^{(0)}(x) \equiv y(x)$ ,  $0 = b_0 < b_1 < \dots < b_m$ , and the  $a_{\mu\nu}$  are any (real or complex) numbers independent of  $x$ . We suppose that  $m > 1$ ,  $n > 1$ , and that each of the sets

$$a_{0\nu} \quad (0 \leq \nu \leq n), \quad a_{m\nu} \quad (0 \leq \nu \leq n), \quad a_{\mu 0} \quad (0 \leq \mu \leq m), \quad a_{\mu n} \quad (0 \leq \mu \leq m)$$

contains at least one non-zero member. These conditions exclude pure differential and pure difference equations (whose theory is sufficiently well known) but do not otherwise restrict generality.

As "boundary conditions" we suppose assigned the values of  $y^{(\nu)}(0)$  for  $0 \leq \nu \leq n$  and of  $y^{(n)}(x)$  for  $0 < x < b_m$ ,  $y^{(n)}(x)$  being integrable (in the Lebesgue sense) in this interval. It is convenient to regard (1.1) as an equation in the unknown function  $y^{(n)}(x)$  and to define  $y^{(\nu)}(x)$  ( $\nu \leq n-1$ ) by

$$y^{(\nu)}(x) = y^{(\nu)}(0) + \int_0^x y^{(\nu+1)}(\xi) d\xi.$$

Thus (1.1) is essentially a difference-integral equation.

The equation (1.1) is satisfied by

$$y(x) = \sum P_r e^{s_r x}, \quad (1.2)$$

where  $s_r$  is any zero of order  $l+1 > 1$  of the associated transcendental equation

$$\tau(s) \equiv \sum_{\mu} \sum_{\nu} a_{\mu\nu} s^{\nu} e^{b_{\mu} s} = 0, \quad (1.3)$$

$P_r$  is any polynomial in  $x$  of degree not greater than  $l$ , and the sum in (1.2) is finite or infinite with suitable convergence conditions. Our main object is to show that, under suitable conditions, (1.2) is the most general solution of (1.1) and to determine the coefficients in the  $P_r$  in terms of the boundary conditions. The latter problem is analogous to that of determining the "arbitrary constants" in the solution of a differential equation in terms of the boundary conditions.

If  $y_1(x)$  is the general solution of (1.1) and  $y_2(x)$  any particular solution of

$$\Lambda\{y(x)\} = v(x), \quad (1.4)$$

where  $v(x)$  is a known function, the general solution of (1.4) is  $y_1(x) + y_2(x)$ . I show how to find a particular solution of (1.4) in Wright (1948*b*) which deals with a more general equation.

In what follows,  $\mu, \nu, \rho$  are whole numbers satisfying  $0 \leq \mu \leq m$ ,  $0 \leq \nu \leq n$ ,  $0 \leq \rho \leq n$ . Any statement involving  $\mu, \nu$  or  $\rho$  holds for all these values of  $\mu, \nu$  or  $\rho$  unless the contrary is expressly stated;  $\sum_{\mu}, \sum_{\nu}, \sum_{\rho}$  denote summation over these values of  $\mu, \nu, \rho$ . The numbers  $x$  and  $T$  are real,  $X$  is real and positive,  $s = \sigma + it$  is complex, and  $M$  is a positive



whole number. The number  $C$ , not always the same at each occurrence, is a positive number independent of  $x$ ,  $s$ ,  $T$  and  $M$  but possibly depending on any other relevant parameters. The numbers  $C_1, C_2, \dots$  are of the type  $C$ , but  $C_1$ , for example, always retains the same value at each occurrence. The  $O(\ )$  and  $o(\ )$  notations refer to the passage of  $x$ ,  $|x|$ ,  $T$  or  $M$  to  $\infty$ , as may be stated; the constant implied is of the type  $C$ .

It follows from Lemma 2 (i) in the sequel that we can arrange the zeroes  $s_r$  of  $\tau(s)$  in a sequence such that  $|\mathbf{I}(s_r)| < |\mathbf{I}(s_{r+1})|$  and that there are numbers  $C_1, C_2$  and a sequence  $\{T_M\}$  satisfying

$$|T_M - MC_1| < C \quad (1.5)$$

$$\text{and} \quad |\mathbf{I}(s_{R(M)})| + C_2 < T_M < |\mathbf{I}(s_{R(M)+1})| - C_2 \quad (1.6)$$

for some  $R(M)$ . We shall also see that  $R(M)/M$  tends to a finite limit as  $M \rightarrow \infty$ .

We write

$$H_1(s) = \sum_{\mu=1}^m \sum_{\nu} a_{\mu\nu} e^{b_{\mu}s} \int_0^{b_{\mu}} y^{(\nu)}(u) e^{-su} du, \quad (1.7)$$

$$H(s) = H_1(s) + \sum_{\mu} \sum_{\nu=1}^n a_{\mu\nu} e^{b_{\mu}s} \sum_{\lambda=0}^{\nu-1} s^{\nu-1-\lambda} y^{(\lambda)}(0), \quad (1.8)$$

and  $e^{s_r x} P(r, \nu, x)$  for the residue of  $s^{\nu} e^{s x} H(s) / \tau(s)$  at the pole  $s = s_r$ , so that

$$P(r, \nu, x) = e^{s_r x} \frac{d^{\nu}}{dx^{\nu}} \{P(r, 0, x) e^{s_r x}\}$$

is a polynomial of degree  $l$  (at most) in  $x$  if  $s_r$  is a zero of  $\tau(s)$  of order  $l+1$ . Finally we write

$$S(M, \nu, x) = \sum_{r=1}^{R(M)} P(r, \nu, x) e^{s_r x}.$$

We shall prove

**THEOREM 1.**—If  $a_{0n} \neq 0$ ,  $b$  is the least value of  $b_{\mu}$  such that  $a_{\mu n} \neq 0$  and (1.1) is satisfied for  $x > 0$ , then

$$y^{(\nu)}(x) = \lim_{M \rightarrow \infty} S(M, \nu, x) \quad (1.9)$$

for  $x > b$  and  $\nu \leq n-1$ . The convergence is uniform in any finite interval  $b+\delta < x < C$  ( $\delta > 0$ ). If  $y^{(n)}(x)$  is continuous and of bounded variation for  $0 \leq x \leq b_m$ , then so it is for all  $x > 0$  and (1.9) holds for  $x > b$  and  $\nu = n$ .

A trivial change of variable enables us to deduce

**THEOREM 2.**—If  $a_{0n} \neq 0$ ,  $b'$  is the greatest value of  $b_{\mu}$  such that  $a_{\mu n} \neq 0$  and (1.1) is satisfied for  $x \leq 0$ , then (1.9) is true for  $x < b'$  and  $\nu \leq n-1$ . The convergence is uniform in any finite interval  $-C < x < b' - \delta$  ( $\delta > 0$ ). If  $y^{(n)}(x)$  is continuous and of bounded variation for  $0 \leq x \leq b_m$ , then so it is for all  $x \leq 0$  and (1.9) holds for  $x < b'$  and  $\nu = n$ .

From these two theorems we have

**THEOREM 3.**—If  $a_{0n} \neq 0$ ,  $a_{mn} \neq 0$  and (1.1) is satisfied for all  $x$ , then (1.9) is true for all  $x$  and  $\nu \leq n-1$ . The convergence is uniform in any finite interval. If  $y^{(n)}(x)$  is continuous and of bounded variation for  $0 \leq x \leq b_m$ , so it is for all  $x$  and (1.9) holds for all  $x$  and  $\nu = n$ .

## PREVIOUS WORK

2. Schmidt (1911), Bochner (1932) and Titchmarsh (1937, 1939) discussed particular cases of (1.4), and Pitt (1944) dealt with an integro-differential equation of which (1.1) is a special case. Titchmarsh (1937) and Pitt make the hypothesis that  $|y^{(\nu)}(x)| < C e^{C|x|}$  for all  $x$ , while Schmidt and Bochner impose severer restrictions; the object in every case is to justify the use of transforms. Titchmarsh (1939) has sketched a means of avoiding this assumption, given the necessary differentiability.

Hilb (1918) used Cauchy's theorem to obtain the expansion (1.9) of  $y(x)$  provided (i) that neither  $a_{mn}$  nor  $a_{0n}$  is zero and  $y^{(v)}(x)$  exists for all  $v \leq n$  and all  $x$ , or (ii) that one of  $a_{mn}$ ,  $a_{0n}$  does not vanish and that  $y(x)$  has continuous derivatives of all orders for all  $x$ .

Elsewhere (Wright, 1948*a*) I have proved (by comparatively elementary methods) a result for the equation

$$\sum_{\mu} \sum_{\nu} A_{\mu\nu}(x) y^{(\nu)}(x+b_{\mu}) = v(x)$$

of which the following is a very special case.

LEMMA 1.—If  $a_{mn} \neq 0$ , if  $y(x)$  is a solution of (1.1) and if  $y^{(n)}(x)$  is integrable for  $0 \leq x \leq b_m$ , then  $y^{(n)}(x)$  is integrable in any finite interval  $(0, X)$ ,  $y^{(v)}(x) = O(e^{cx})$  for all  $v \leq n-1$  and

$$\int_0^x |y^{(n)}(\xi)| d\xi = O(e^{cx})$$

as  $x \rightarrow \infty$ . If, in addition,  $y^{(n)}(x)$  is continuous and of bounded variation throughout the interval  $(0, b_m)$ , it is both throughout any finite interval  $(0, X)$ .

This lemma and the corresponding result for negative  $x$  when  $a_{0n} \neq 0$  enable us to deduce the behaviour of  $y^{(v)}(x)$  at infinity from that of  $y^{(n)}(x)$  in the interval  $0 \leq x \leq b_m$ , i.e. from the boundary conditions. Professor Pitt, to whom I communicated my results as soon as my attention was drawn to his paper, was kind enough to let me see the manuscript of a sequel (Pitt, 1947). In this he proves a theorem effectively equivalent to Lemma 1. When we combine this with the result of Pitt (1944) my Theorem 3 follows. Theorems 1 and 2, however, do not follow from his results, and I prove them here by a method differing in important particulars from his. These differences are necessary to cover the case in which one of  $a_{mn}$ ,  $a_{0n}$  is zero, since, as we shall see in the next section, the behaviour of  $\tau(s)$  and the distribution of its zeroes are much more complicated in this case than when neither of  $a_{mn}$ ,  $a_{0n}$  is zero. The former case seems to be important for applications (see, for example, Callender, etc., 1936; Hartree, etc., 1937; Sievert, 1941; van der Werff, 1942) and so worthy of separate attention.

#### PROPERTIES OF $\tau(s)$ AND THE LOCATION OF ITS ZEROES

3. We write

$$\beta(s) = \max_{\mu, \nu} |a_{\mu\nu} s^{\nu} e^{b_{\mu}s}|$$

and require the following

LEMMA 2.—(i) The number of zeroes of  $\tau(s)$  for which  $|t - T| < C$  is bounded for all  $T$ .

(ii) For any  $C_3$  and suitable  $C_4$ ,  $\tau(s)$  has no zeroes in the region

$$|\arg(-s)| < \frac{1}{2}\pi - C_3, \quad |s| > C_4.$$

(iii) If  $a_{mn} \neq 0$ , all the zeroes of  $\tau(s)$  are to the left of the line  $\sigma = C_5$  for some  $C_5$ .

(iv) If we surround every zero of  $\tau(s)$  by a circle of radius any  $C_6$  we have  $|\tau(s)| > C\beta(s)$  for all  $s$  outside these circles.

Of these, (i), (ii) and (iv) follow at once from Langer (1929, p. 844),\* while (iii) is trivial.

Although Lemma 2 is all we need for the proof of Theorem 1, it will make matters clearer if we describe the location of the zeroes in more detail. We omit proofs, since our statements follow from Langer's results.

If  $a_{0n} \neq 0$  and  $a_{mn} \neq 0$ , the zeroes of  $\tau(s)$  are confined to a strip  $\sigma_1 < \sigma < \sigma_2$  and approach asymptotically the zeroes of

$$\phi(s) \equiv \sum_{\mu=0}^m a_{\mu n} e^{b_{\mu}s} \quad (3.1)$$

\* Langer states that  $\tau(s)$  is uniformly bounded from zero if  $s$  is uniformly distant from the zeroes of  $\tau(s)$ . This is not quite correct; what is correct and what follows at once from his argument is that the ratio of  $\tau(s)$  to its term of maximum modulus is uniformly bounded from zero when  $s$  is restricted as stated.

for large  $|t|$ . The number of zeroes of  $\tau(s)$  for which  $T < t < T + C_7$  lies between

$$\frac{C_7 b_m}{2\pi} \pm m$$

for large enough  $|T|$ . Langer's Theorem B gives  $\pm(m+1)$  in place of  $\pm m$  in this result, but Wilder (1917), from whom Langer quotes the theorem, proves that  $\pm m$  is sufficient.

Now let us suppose that, for example,  $a_{0n} = 0$ . We plot the points  $(b_\mu, \nu)$  for every non-zero term of  $\tau(s)$ . The result is an incomplete array of points arranged in columns and rows for which  $0 < b_\mu < b_m$ ,  $0 < \nu < n$ . Since  $a_{0n} = 0$ , the left-hand top corner point is certainly missing.

Through  $P_0(b_0, \nu_0)$ , the point at the top of the left-hand column, we draw the line  $L_1$  with least slope which has no plotted point *above* it.  $L_1$  must pass through at least one plotted point other than  $P_0$ . Through  $P_1(b_{\mu_1}, \nu_1)$ , the point on  $L_1$  furthest to the right, we draw the line  $L_2$  with least slope which has no plotted point *above* it, and so on. For some  $j > 1$ , the point  $P_j$  will have  $\nu_j = n$  and the process ends. Let  $d_j$  be the slope of  $L_j$  and  $(b_{\mu_j}, \nu_j)$  the co-ordinates of  $P_j$ . We observe that

$$\nu - d_j b_\mu$$

is the same (say  $c_j$ ) for all points on  $L_j$  and less than  $c_j$  (by a finite amount) for every other plotted point.

The zeroes of  $\tau(s)$  to the left of  $\sigma = -C$ , for large enough  $C$ , are confined to strips  $\mathcal{S}_j$  of finite width enclosing the curves

$$\sigma = -d_j \log |t|,$$

or, what is asymptotically the same,

$$\mathbf{R}(s + d_j \log s) = 0.$$

In each such strip the zeroes approach asymptotically those of

$$\phi_j(s) \equiv s^{-c_j} \sum a_{\mu\nu} s^\nu e^{b_\mu s} = \sum a_{\mu\nu} (s^{d_j e^s})^{b_\mu},$$

the summation extending over every  $\mu, \nu$  such that the corresponding plotted point lies on  $L_j$ . If there are only two such points the zeroes of  $\phi_j(s)$  are asymptotically determinate.

For any  $C_7$  and large enough  $|T|$ , the number of zeroes in the strip  $\mathcal{S}_j$  with  $T < t < T + C_7$  lies between

$$\frac{C_7}{2\pi} (b_{\mu_j} - b_{\mu_{j-1}}) \pm h_j,$$

where  $h_j$  is the number of terms in  $\phi_j(s)$ .

For suitable  $C$  the zeroes of  $\tau(s)$  for which  $|\sigma| < C$  approach asymptotically those of

$$\phi(s) = \sum_{\mu=0}^m a_{\mu n} e^{b_\mu s} = \sum_{b < b_\mu < b'} a_{\mu n} e^{b_\mu s},$$

where  $b, b'$  are the numbers of Theorems 1 and 2. If  $b' \neq b$ , the number of these zeroes with  $t$  in the interval  $(T, T + C_7)$  lies between

$$\frac{1}{2\pi} (b' - b) C_7 \pm h$$

for large enough  $|T|$ , where  $h$  is the number of non-zero terms in  $\phi(s)$ . If  $b' = b$ ,  $\phi(s)$  contains only one term and  $\tau(s)$  has only a finite number of zeroes in any strip  $\sigma_1 < \sigma < \sigma_2$ .

If  $a_{mn} \neq 0$ , then  $b' = b_m$  and  $\tau(s)$  has no zeroes to the right of  $\sigma = C$  for suitable  $C$ . If  $a_{mn} = 0$ , we can use the same diagram as before and determine strips containing the remaining zeroes in the obvious way.

## PROOF OF THEOREM I

4. We now suppose the conditions of Theorem 1 satisfied. Then, by Lemma 1,  $y^{(\nu)}(x) = O(e^{C_8 x})$  for  $\nu < n-1$  and  $\int_0^x |y^{(n)}(\xi)| d\xi = O(e^{C_8 x})$  as  $x \rightarrow \infty$ . Hence

$$Y_\nu(s) = \int_0^\infty \{y^{(\nu)}(x) - y^{(\nu)}(0)\} e^{-sx} dx$$

is a regular function of  $s$  for  $\sigma > C_8$ . We have

$$Y_\nu(s) = s Y_{\nu-1}(s) - s^{-1} y^{(\nu)}(0)$$

by integration by parts. Applying this  $\nu$  times in succession we obtain

$$Y_\nu(s) = s^\nu Y_0(s) - \sum_{\lambda=1}^{\nu} s^{\nu-\lambda-1} y^{(\lambda)}(0). \quad (4.1)$$

If we multiply (4.1) by  $e^{-sx}$ , integrate with respect to  $x$  from 0 to  $\infty$  and use (4.1), we have

$$\tau(s) Y_0(s) = H(s) - \frac{\tau(s) y^{(0)}(0)}{s}$$

and

$$\frac{s^\nu H(s)}{\tau(s)} = Y_\nu(s) + \sum_{\lambda=0}^{\nu} s^{\nu-\lambda-1} y^{(\lambda)}(0). \quad (4.2)$$

This provides the analytic continuation of  $Y_\nu(s)$  for all values of  $s$  except the origin and the zeroes of  $\tau(s)$  which are, in general, poles of  $Y_\nu(s)$ .

We now take the sequence  $\{T_M\}$  of (1.5) and (1.6); all our statements contain the implied condition "for  $M$  greater than a suitable  $C$ ". By Lemma 2 (iv) and (1.6),

$$|\tau(\sigma \pm iT_M)| > C\beta(\sigma \pm iT_M)$$

for all  $\sigma$ . We choose  $C_9 > \max(C_5, C_8)$  and consider the contour  $\Gamma(M)$  formed of the four lines

$$\begin{aligned} \Gamma_1(M) & (\sigma = C_9, -T_M < t < T_M), \\ \Gamma_2(M) & (t = T_M, -T_M < \sigma < C_9), \\ \Gamma_3(M) & (\sigma = -T_M, -T_M < t < T_M), \\ \Gamma_4(M) & (t = -T_M, -T_M < \sigma < C_9). \end{aligned}$$

By (4.2) and Cauchy's Theorem,

$$2\pi i S(M, \nu, x) = \int_{\Gamma(M)} \frac{s^\nu H(s) e^{sx}}{\tau(s)} ds = \int_{\Gamma(M)} Y_\nu(s) e^{sx} ds + 2\pi i y^{(\nu)}(0). \quad (4.3)$$

By (4.2) and the definition of  $H(s)$  we have

$$\begin{aligned} \tau(s) Y_\nu(s) - s^\nu H_1(s) &= s^\nu \{H(s) - H_1(s)\} - \tau(s) \sum_{\lambda=0}^{\nu} s^{\nu-\lambda-1} y^{(\lambda)}(0) \\ &= \sum_{\mu=0}^m \left\{ \sum_{\rho=1}^n \sum_{\lambda=0}^{\rho-1} - \sum_{\rho=0}^n \sum_{\lambda=0}^{\rho} \right\} a_{\mu\rho} e^{b_\mu s} s^{\nu+\rho-\lambda-1} y^{(\lambda)}(0). \end{aligned}$$

Hence, for  $\sigma < C$  and  $|s| > C$ ,

$$\tau(s) Y_n(s) - s^n H_1(s) = - \sum_{\mu=0}^m \sum_{\rho=0}^n \sum_{\lambda=\rho}^n a_{\mu\rho} e^{b_\mu s} s^{n+\rho-\lambda-1} y^{(\lambda)}(0) = O(s^{n-1}),$$

and, if  $\nu < n-1$ ,

$$\tau(s)Y_\nu(s) - s^\nu H_1(s) = \sum_{\mu=0}^m \left\{ \sum_{\rho=\nu+2}^n \sum_{\lambda=\nu+1}^{\rho-1} - \sum_{\rho=0}^{\nu} \sum_{\lambda=\rho}^{\nu} \right\} a_{\mu\rho} e^{b_\mu s} s^{\nu+\rho-\lambda-1} y^{(\lambda)}(0) = O(s^{n-2}) \quad (4.4)$$

as  $|t| \rightarrow \infty$ , where  $\sum_{\rho=n+1}^n$  denotes an empty sum.

By an obvious change of variable

$$e^{b_\mu s} \int_0^{b_\mu} y^{(\nu)}(u) e^{-su} du = \int_0^{b_\mu} y^{(\nu)}(b_\mu - v) e^{sv} dv$$

and

$$H_1(s) = \sum_{\mu=1}^m \sum_{\nu=0}^n a_{\mu\nu} \int_0^{b_\mu} y^{(\nu)}(b_\mu - v) e^{sv} dv.$$

Hence, for  $\sigma < C$ ,  $|H_1(s)| < C$  and  $H_1(s) \rightarrow 0$  uniformly in  $\sigma$  as  $|t| \rightarrow \infty$ . Also, by Titchmarsh (1937, p. 70, Theorem 49),

$$\int_{-\infty}^{\infty} |H_1(C_9 + it)|^2 dt \quad (4.5)$$

converges.

We now suppose that  $b + \delta < x < C$ . On  $\Gamma_3(M)$ ,  $|\tau(s)| > C\beta(s)$  by Lemma 2. But there is at least one non-zero term with  $\mu=0$  in  $\tau(s)$  and so  $\beta(s) > C$  for  $|s| > C$ . Hence, on  $\Gamma_3(M)$ ,

$$|Y_\nu(s)| < \left| \frac{s^\nu H_1(s)}{\tau(s)} \right| + C \left| \frac{s^{n-1}}{\tau(s)} \right| < C|s^n| < CM^n,$$

and so

$$\left| \int_{\Gamma_3(M)} Y_\nu(s) e^{sx} ds \right| < CM^{n+1} e^{-\sigma M \delta} \rightarrow 0 \quad (4.6)$$

uniformly in  $x$  as  $M \rightarrow \infty$ .

On  $\Gamma_2(M)$  we have

$$|\tau(s)| > C\beta(s) > C|s^n e^{bs}|,$$

$$|Y_\nu(s) e^{bs}| < \left| \frac{s^\nu e^{bs} H_1(s)}{\tau(s)} \right| + C \left| \frac{s^{n-1} e^{bs}}{\tau(s)} \right| < C \left( |H_1(s)| + \frac{1}{|s|} \right) \rightarrow 0$$

uniformly in  $\sigma$  as  $M \rightarrow \infty$ . Hence

$$\left| \int_{\Gamma_2(M)} Y_\nu(s) e^{sx} ds \right| < o(1) \int_{-T_M}^0 e^{\sigma(x-b)} d\sigma < C e^{\sigma} o(1) \rightarrow 0 \quad (4.7)$$

uniformly in  $x$  as  $M \rightarrow \infty$ . Similarly for  $\Gamma_4(M)$ .

Combining (4.3), (4.6) and (4.7) we see that

$$2\pi i S(M, \nu, x) - \int_{\Gamma_1(M)} Y_\nu(s) e^{sx} ds \rightarrow 2\pi i y^{(\nu)}(0) \quad (4.8)$$

uniformly in  $x$  as  $M \rightarrow \infty$ . By a well-known result in the theory of the Laplace transform (see, for example, Widder, 1941, p. 66, Theorem 7.3)

$$\int_{\Gamma_1(M)} Y_\nu(s) e^{sx} ds \rightarrow 2\pi i \{y^{(\nu)}(x) - y^{(\nu)}(0)\} \quad (4.9)$$

as  $M \rightarrow \infty$ , provided  $y^{(\nu)}(x)$  is continuous and of bounded variation in the neighbourhood of  $x$ . For  $\nu < n-1$ , this is always true since  $y^{(\nu)}(x)$  is an integral; for  $\nu=n$ , it is true by Lemma 1 if  $y^{(n)}(x)$  is continuous and of bounded variation for  $0 < x < b_m$ . Hence the first and last parts of Theorem 1 follow from (4.8) and (4.9).

To prove the second part of Theorem 1 it only remains to show that the convergence in (4.9) is uniform in  $x$  for  $\nu \leq n-1$ . For this it is enough to show that

$$\int_{-\infty}^{\infty} |Y_{\nu}(C_9 + it)| dt \quad (4.10)$$

converges. By (4.4),

$$|Y_{\nu}(C_9 + it)| < C \left| \frac{H_1(C_9 + it)}{C_9 + it} \right| + \frac{C}{C^2 + t^2}$$

and

$$\left\{ \int_{-T}^T \left| \frac{H_1(C_9 + it)}{C_9 + it} \right| dt \right\}^2 < \int_{-T}^T |H_1(C_9 + it)|^2 dt \int_{-T}^T \frac{dt}{C_9^2 + t^2} < C$$

by (4.5). Hence (4.10) converges.

# SUMMARY

Under the condition that one at least of the leading coefficients  $a_{mn}$ ,  $a_{0n}$  differs from zero, the equation

$$\sum_{\mu=0}^m \sum_{\nu=0}^n a_{\mu\nu} y^{(\nu)}(x + b_{\mu}) = 0$$

has as solution a series convergent for all  $x$  greater (or all  $x$  less) than a fixed number. The coefficients of the various terms in the series are expressed in terms of the arbitrary values of the solution and its first  $n$  derivatives in an initial interval of appropriate length.

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XLI.—Problems in Factor Analysis.\* By D. N. Lawley, M.A., D.Sc.

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1. IN cases where a large sample has been drawn from a multivariate population it is possible to apply tests of the hypotheses made by factor analysts regarding the number of common factors on which the variables depend (Lawley, 1940, 1941). There still remain, however, certain points which have not yet been discussed, and it is the purpose of the present paper to deal with these. For convenience it would seem desirable to summarise briefly results already obtained.

We suppose that there are  $n$  variables, denoted by  $x_i$  ( $i=1, 2, \dots, n$ ), which obey a multivariate normal distribution; and, without loss of generality, we may assume that all means are zero. We shall denote by  $x$  the column vector of which  $x_i$  is the typical element. It is supposed that the  $x_i$  depend upon  $m$  common factors, represented by the column vector  $f_0$ , in addition to the  $n$  specific factors, represented by  $f_1$ . We may then write

$$x = Kf_0 + Tf_1,$$

where  $K$  is a matrix having  $n$  rows and  $m$  columns, and  $T$  is an  $n \times n$  diagonal matrix. The typical element  $\lambda_{ir}$  of  $K$  represents the "loading" of  $x_i$  in the  $r$ th common factor; while  $\tau_i$ , the typical element of  $T$ , represents the specific loading of  $x_i$ .

Since the factors are assumed to be distributed independently with unit variances, the matrix  $C$  whose elements are the variances and covariances  $\{c_{ij}\}$  of the  $x_i$  satisfies the equation

$$C = KK' + T^2.$$

It will also be found useful to express the reciprocal of the variance matrix in the form

$$C^{-1} = T^{-2} - T^{-2}K(I+J)^{-1}K'T^{-2}, \quad (1)$$

where

$$J = K'T^{-2}K.$$

Owing to the fact that any orthogonal transformation of the common factors leaves the variance matrix  $C$  unaltered, we impose the condition that  $J$  should be a diagonal matrix; this condition is sufficient to determine  $f_0$  and  $K$  uniquely.

2. Now suppose that a sample of size  $N$  is drawn from the multivariate population; Then if  $A$  is a matrix representing the sample variances and covariances of the  $x_i$ , the typical element of  $A$  is given by

$$a_{ij} = \frac{1}{(N-1)} S(x_i - \bar{x}_i)(x_j - \bar{x}_j),$$

where  $S$  denotes summation over the sample and where  $\bar{x}_i$  is the sample mean of  $x_i$ .

The joint distribution of the sample variances and covariances, first found by Wishart (1928), takes the form

$$L \prod_{i < j} (da_{ij}),$$

where

$$L \propto |A|^{\frac{1}{2}(N-n-2)} |C|^{-\frac{1}{2}(N-1)} \exp \left\{ -\frac{(N-1)}{1} \text{tr} (AC^{-1}) \right\},$$

the notation  $\text{tr} (Z)$  being used to denote the trace, or sum of diagonal elements, of  $Z$ .

An approximate expression, when  $N$  is large, may be obtained for  $\log L$  by expanding in powers of the quantities  $\{a_{ij} - c_{ij}\}$ , these being of order  $1/\sqrt{N}$ . In doing this it must

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be remembered that  $(a_{ij} - c_{ij})$  is identically equal to  $(a_{ji} - c_{ji})$ . Neglecting terms of order  $1/\sqrt{N}$  or less, we find that

$$\log L = \alpha - \frac{N}{4} \operatorname{tr} \{ (A - C)C^{-1}(A - C)C^{-1} \}, \quad (2)$$

where  $\alpha$  is a constant.

It will be seen that the above expression represents a quadratic form in the quantities  $\{a_{ij} - c_{ij}\}$ , which we are, in effect, assuming to be approximately normally distributed. Their sampling variances and covariances have been found by Wishart and may be expressed by the formula (E denoting the expectation or mean value)

$$E\{(a_{ij} - c_{ij})(a_{hk} - c_{hk})\} = \frac{1}{N}(c_{ih}c_{jk} + c_{ik}c_{jh}).$$

In order to estimate the unknown parameters  $\lambda_{ir}$  we may apply the maximum likelihood method and choose our estimates so as to maximise the expression (2) with respect to the  $\lambda_{ir}$ . This leads to the equation (quantities of order  $1/\sqrt{N}$  once more being neglected)

$$\overset{v}{K} T^{-2} (A - \overset{v}{C}) = 0, \quad (3)$$

where  $\overset{v}{K}$  is the matrix of estimated loadings  $\lambda_{ir}$ , and

$$\overset{v}{C} = \overset{v}{K} \overset{v}{K}' + T^2.$$

Equation (3) is the same as that previously obtained when the exact expression for  $\log L$  was used. We are assuming, however, that the elements of  $T$ , *i.e.* the specific loadings, are known.

3. Let  $U$  be an orthogonal matrix whose first  $m$  columns are those of the matrix

$$U_0 = T^{-1} K J^{-\frac{1}{2}}.$$

We may then partition  $U$  and write

$$U = [U_0 \mid U_1].$$

Now define a matrix  $D$  by the equation

$$D = U' T^{-1} (A - C) T^{-1} U,$$

so that the elements of  $D$  are linear functions of the quantities  $\{a_{ij} - c_{ij}\}$ . Then, using (2), we may write

$$\begin{aligned} \log L &= \alpha - \frac{N}{4} \operatorname{tr} \{ (T U D U' T) C^{-1} (T U D U' T) C^{-1} \} \\ &= \alpha - \frac{N}{4} \operatorname{tr} (D B^{-1} D B^{-1}), \end{aligned} \quad (4)$$

where

$$B^{-1} = U' T C^{-1} T U;$$

thus

$$\begin{aligned} B &= U' T^{-1} C T^{-1} U \\ &= U' (I + T^{-1} K K' T^{-1}) U \\ &= U' U + U' U_0 J U_0' U \\ &= \begin{bmatrix} I_m + J & 0 \\ 0 & I_{n-m} \end{bmatrix}, \end{aligned}$$

denoting by  $I_r$  the  $r \times r$  unit matrix.

The expression (4) gives us the joint distribution of the elements  $\{d_{ij}\}$  of  $D$ , which are clearly uncorrelated, in view of the fact that  $B$  is a diagonal matrix. If we let  $b_{ij}$  represent the typical element of  $B$ , the variance of  $d_{ij}$  can be expressed as

$$(b_{ii} b_{jj} + b_{ij}^2).$$

The equation of estimation (3) may be regarded as imposing a number of linear restrictions on the elements of  $(A - C)$ . These restrictions are equivalent to putting

$$d_{ij} = 0 \quad (i, j < m),$$

where  $d_{ij}$  is the typical element of

$$D = U' T^{-1} (A - C) T^{-1} U.$$

Now let  $\bar{B}$  denote the matrix

$$\begin{bmatrix} \circ & \circ \\ \circ & I_{n-m} \end{bmatrix},$$

formed from  $B$  by substituting zeros for the elements of the top left-hand quadrant; and let  $\bar{C}$  be a matrix bearing the same relation to  $\bar{B}$  as  $C$  does to  $B$ . Thus

$$\begin{aligned} \bar{C} &= T U \bar{B} U' T \\ &= T (U U' - U_0 U_0') T \\ &= T^2 - K J^{-1} K'. \end{aligned}$$

Then the variances and covariances of the elements of  $(A - C)$  are given by the formula

$$E\{(a_{ij} - c_{ij})(a_{hk} - c_{hk})\} = \frac{1}{N} (\bar{c}_{ih} \bar{c}_{jk} + \bar{c}_{ik} \bar{c}_{jh}), \quad (5)$$

where  $\bar{c}_{ij}$  is the typical element of  $\bar{C}$ .

It has been shown previously that the expression

$$N \sum_{i < j} \frac{(a_{ij} - c_{ij})^2}{\tau_i^2 \tau_j^2}$$

is distributed approximately as  $\chi^2$  with  $\frac{1}{2}\{(n-m)^2 - n - m\}$  degrees of freedom; we are thus able to perform a test of significance on the residual covariances taken collectively. We have now derived expressions for the sampling variances of the residuals  $\{a_{ij} - c_{ij}\}$ , considered separately. It must, however, be realised that the expression for  $\bar{C}$  involves the matrix  $K$ , whose elements consist of the unknown population parameters  $\lambda_{ir}$ . These parameters have in practice to be replaced by their sample estimates  $l_{ir}$ . This emphasises the fact that the above treatment is only appropriate for large samples. Owing to the difficulty of the problem a treatment suitable for small samples would appear to be virtually impossible.

Another fact which must be taken into account is that we have assumed the specific loadings  $\tau_i$  to be known; or, what comes to the same thing, we have neglected the errors of estimation of these quantities. The errors could be allowed for, but at the cost of greater complexity in the formulæ. It has already been shown that the estimation of the parameters  $\tau_i$  is equivalent to the imposition of the further restrictions

$$a_{ii} - c_{ii} = 0 \quad (i = 1, 2, \dots, n).$$

It would therefore be necessary to find the partial variances and covariances of the quantities  $\{a_{ij} - c_{ij}\}$  under the above restrictions. This would involve, amongst other things, calculating the reciprocal of the  $n \times n$  matrix representing the variances and covariances of the diagonal residuals  $\{a_{ii} - c_{ii}\}$ . The calculation would, however, scarcely be worth making, especially since when  $n$ , the number of variables, is large the effect of errors in estimation of the parameters  $\tau_i$  is, in general, small.

To illustrate the use of formula (5) let us consider the case where two factors have been fitted. We shall then have

$$\begin{aligned} \bar{c}_{ii} &= \tau_i^2 - \frac{\lambda_{i1}^2}{\gamma_1} - \frac{\lambda_{i2}^2}{\gamma_2}, \\ \bar{c}_{ij} &= -\frac{\lambda_{i1}\lambda_{j1}}{\gamma_1} - \frac{\lambda_{i2}\lambda_{j2}}{\gamma_2} \quad (i \neq j), \end{aligned}$$

where

$$\gamma_1 = \sum_i (\lambda_{i1}^2 / \tau_i^2),$$

$$\gamma_2 = \sum_i (\lambda_{i2}^2 / \tau_i^2);$$

and the typical residual  $(a_{ij} - \xi_{ij})$  may be written as

$$a_{ij} - l_{i1}l_{j1} - l_{i2}l_{j2} \quad (i \neq j).$$

4. Let us now consider the problem of determining the sampling variances and co-variances of the estimated loadings  $l_{ir}$ . For large  $N$  the joint distribution of these quantities may be obtained by finding

$$\log (L/L_1),$$

where  $\log L_1$  is the result of replacing  $\lambda_{ir}$  by  $l_{ir}$ , for all  $i$  and  $r$ , in the expression for  $\log L$ . Thus, to the usual approximation,

$$\log (L/L_1) = -\frac{N}{2} \sum \{a_{ir, js} (l_{ir} - \lambda_{ir})(l_{js} - \lambda_{js})\},$$

where

$$a_{ir, js} = -\frac{1}{N} \frac{\partial^2 (\log L)}{\partial \lambda_{ir} \partial \lambda_{js}},$$

and where the summation is over all possible values of the suffices. Neglecting quantities of order  $1/\sqrt{N}$ , the constants  $a_{ir, js}$  may be expressed in the form

$$a_{ir, js} = [K' C^{-1} K]_{rs} [C^{-1}]_{ij} + [K' C^{-1}]_{is} [K' C^{-1}]_{jr},$$

where, for example,  $[K' C^{-1} K]_{rs}$  denotes the element in the  $r$ th row and  $s$ th column of the matrix  $K' C^{-1} K$ .

It is convenient to use a set of linear functions  $m_{ir}$  of the quantities  $(l_{ir} - \lambda_{ir})$  given by the elements of the matrix

$$M = U' T^{-1} (K - K) J^{\frac{1}{2}}$$

$$= [T^{-2} K J^{-\frac{1}{2}} : T^{-1} U_1]' (K - K) J^{\frac{1}{2}}.$$

We must remember at this point the conditions which are imposed on the loadings  $\lambda_{ir}$  and on their estimates  $l_{ir}$  in order to obtain a unique solution. Inspection of the form of  $M$  indicates that these conditions are equivalent to putting

$$m_{ir} = 0 \quad (i < r).$$

We may now write

$$\log (L/L_1) = -\frac{N}{2} \sum_{i > r} \sum_{j \geq s} (\beta_{ir, js} m_{ir} m_{js}),$$

where

$$\beta_{ir, js} = [J^{-\frac{1}{2}} K' C^{-1} K J^{-\frac{1}{2}}]_{rs} [U' T C^{-1} T U]_{ij} + [J^{-\frac{1}{2}} K' C^{-1} T U]_{is} [J^{-\frac{1}{2}} K' C^{-1} T U]_{jr}$$

$$= [B^{-1}]_{rs} [B^{-1}]_{ij} + [B^{-1}]_{is} [B^{-1}]_{jr},$$

the matrix  $B$  being as previously defined. Since  $B$  is a diagonal matrix it is clear that, for  $i > r$  and  $j > s$ ,  $\beta_{ir, js} = 0$  unless  $i = j$  and  $r = s$ . Hence

$$\log (L/L_1) = -\frac{N}{2} \left\{ \sum_{i > r} \left( \frac{m_{ir}^2}{b_{ii} b_{rr}} \right) + 2 \sum_r \left( \frac{m_{rr}^2}{b_{rr}^2} \right) \right\}.$$

The form of the above expression shows that, for large  $N$ , the quantities  $m_{ir}$  are distributed independently. The variance of  $m_{rr}$  is  $b_{rr}^2/2N$ , while for  $i > r$  the variance of  $m_{ir}$  is  $b_{ii} b_{rr}/N$ .

Since the quantities  $(l_{ir} - \lambda_{ir})$  are linear functions of the  $m_{ir}$ , being given by the equation

$$K - K = T U M J^{-\frac{1}{2}},$$

it is now a simple matter to find their variances and covariances. If we denote by  $G$  the diagonal matrix whose elements  $g_{ii}$  are given by

$$\begin{aligned} g_{ii} &= b_{ii} & (i < r), \\ g_{ii} &= 0 & (i > r), \\ g_{rr} &= \frac{1}{2}b_{rr}, \end{aligned}$$

the variance of  $m_{ir}$  for all  $i$  and  $r$  may be written as

$$\frac{1}{N}b_{rr}(b_{ii} - g_{ii}).$$

The covariance between  $l_{ir}$  and  $l_{jr}$  is therefore

$$\frac{b_{rr}}{N}[J^{-1}]_{rr}[TU(B-G)U'T]_{ij}.$$

If we now put

$$\gamma_r = [J]_{rr} = \sum_h (\lambda_{hr}^2 / \tau_h^2),$$

$$b_{rr} = [I + J]_{rr} = 1 + \gamma_r \quad (r \leq m),$$

$$\theta_r = 1 + 1/\gamma_r,$$

the above expression becomes

$$\begin{aligned} & \frac{\theta_r}{N}[TU(B-G)U'T]_{ij} \\ &= \frac{\theta_r}{N}[C - TUGU'T]_{ij}. \end{aligned}$$

Hence the covariance between  $l_{ir}$  and  $l_{jr}$  is

$$\frac{\theta_r}{N} \left\{ c_{ij} - \sum_{s=1}^r (\theta_s \lambda_{is} \lambda_{js}) + \frac{1}{2} \theta_r \lambda_{ir} \lambda_{jr} \right\}. \quad (6)$$

The variance of  $l_{ir}$  is obtained by putting  $i=j$  in the above expression. On the other hand, when  $r \neq s$  the covariance between  $l_{ir}$  and  $l_{js}$  is zero for all values of  $i$  and  $j$ . This is a consequence of choosing our factors in such a way that  $J$  is a diagonal matrix. The property will no longer hold if the factors are subsequently rotated, *i.e.* replaced by an orthogonal transformation of themselves. Since, however, the rotated loadings will merely be linear functions of the original ones, their sampling variances and covariances may easily be determined.

Now the estimated covariance  $\check{c}_{ij}$  may be expressed as

$$\sum_{r=1}^m (l_{ir} l_{jr});$$

so that its sampling variance is given by

$$\sum_r \{ \lambda_{ir}^2 \text{var}(l_{jr}) + \lambda_{jr}^2 \text{var}(l_{ir}) + 2 \lambda_{ir} \lambda_{jr} \text{cov}(l_{ir}, l_{jr}) \}.$$

This expression may be evaluated by using the results of (6). We thus find that the variance of  $\check{c}_{ij}$  is

$$\begin{aligned} & \frac{1}{N} \left\{ (c_{ii} c_{ij} + c_{ij}^2) - \left( c_{ii} - \sum_r \theta_r \lambda_{ir}^2 \right) \left( c_{jj} - \sum_r \theta_r \lambda_{jr}^2 \right) - \left( c_{ij} - \sum_r \theta_r \lambda_{ir} \lambda_{jr} \right)^2 \right\} \\ &= \frac{1}{N} \{ c_{ii} c_{jj} + c_{ij}^2 - (\bar{c}_{ii} \bar{c}_{jj} + \bar{c}_{ij}^2) \}, \end{aligned}$$

where  $\bar{c}_{ij}$  is as previously defined. This provides a verification that

$$E(a_{ij} - c_{ij})^2 = E(a_{ij} - \check{c}_{ij})^2 + E(\check{c}_{ij} - c_{ij})^2,$$

a result which is a consequence of the fact that the sets of quantities  $\{a_{ij} - \epsilon_{ij}\}$  and  $\{\epsilon_{ij}\}$  are distributed independently of each other.

#### SUMMARY

A set of variables is assumed to depend upon a number of common factors and specifics. Formulæ are then derived for the sampling variances and covariances of the residual covariances obtained by removing the effect of the factors. The variances and covariances of the set of estimated loadings are also found. It must, however, be noted that the results obtained are valid only when an efficient method of estimation is used.

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XLII.—The Nature of Scientific Philosophy. By Professor Herbert Dingle,  
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(*The James Scott Lecture delivered on July 5, 1948*)

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THE title I have chosen for this lecture contains an implication which perhaps will not be generally accepted; there was a time when I would not have accepted it myself. The implication is that a particular kind of philosophy is possible which may be called scientific in contrast with other kinds which cannot be so called. I would go further and identify this scientific philosophy with what is generally called science, and this implies that the distinction that is often assumed to exist between science and philosophy is a false one. For this view I believe there is historical evidence. Science, as a separate, self-contained study, dates from the seventeenth century. Before that time, such consideration as was given to the subject-matter of present-day science was given it by philosophers and regarded as a part of their philosophising, and when in the seventeenth century a new kind of procedure was introduced, it was looked upon by its pioneers not as an attack on a new problem but as a new attack on an old problem. The science of that time was the "new philosophy", faintly adumbrated by some mediæval philosophers, struggling for expression in Francis Bacon, and coming to full recognition in Galileo. Only later, when it had made such progress in certain limited fields of study that a new body of investigators was called into being who confined themselves to those fields, was the new philosophy transformed into a non-philosophy and called generally by the name "science".

I believe this to have been unfortunate in more ways than one. Not only has it created at least a coldness between scientists and philosophers—*i.e.* between those who limit their studies to the phenomena readily amenable to treatment by the new philosophy and those interested in the no less important phenomena not so readily amenable—but also it has fostered the belief that science is by its nature limited to those problems in which it has been immediately successful. Thus there has grown up a view of the nature of science—as I shall continue for brevity to call what is more properly denominated scientific philosophy—which I believe to originate more in the accidents than in the substance of its being. Thus we are told that the "scientific method" is applicable only to the world of sensation, or to the measurable, or to the material, and so on, and, so far as I have been able to gather, such statements are never based on any fundamental consideration of the possibilities inherent in the "scientific method" but rather on its actual achievements up to now, just as one might in an earlier time have asserted that men would never be able to fly because the principles of navigation were applicable only to land and sea travel.

It seems worth while, therefore, to try to get beneath the surface of modern science and see if, from an examination of its roots, we can form a more trustworthy idea of its essential nature and possibilities. To do that I think the first step must be to understand as clearly as possible what happened in the seventeenth century, when science as we now know it may be said to have begun. I know, of course, that in one sense the beginnings of science lie much further back, and I shall not ignore pre-Galilean thought; but when all due respect has been paid to that, one outstanding fact stares us in the face. In the 2000 years or more during which men had been philosophising before the seventeenth century began, the amount of what we now call science which was discovered or created was almost negligible, and continuous progress, if any, is hardly discernible. In the 250 years since the seventeenth century ended almost the whole of our scientific knowledge has been obtained, and conspicuous progress has been not only continuous but continuously accelerated. In that short space of 100 years something was introduced which started a movement unknown before, and what I want to do

is to identify that something and delineate it as clearly as possible. I do not say that by itself it would necessarily, in any circumstances whatever, have given rise to science. The ground had to be prepared before it could operate. But I do say that this new element which we are to seek is the *seed* of science, and that it has pre-eminent claims to be regarded as the factor which, above all others, made science possible.

I believe that this factor is to be found in the work of Galileo. There have, of course, been other claims. Canon Raven,\* for example, has argued cogently for the recognition of a fact of history which he thinks has been unduly neglected, namely the recovery, mainly in the seventeenth century, of the habit of looking at nature without presuppositions, which was possessed by the Greeks but denied to the thinkers of the Middle Ages through the influence of the mythology of the bestiaries and other fanciful legends. He attributes modern science to the recovery of this habit, which is seen more prominently in biology than in mechanics, and he accordingly considers that the significance of Galileo and his followers has been over-emphasised. I do not wish in the least degree to underestimate the importance of the unprejudiced outlook, but I must point out that no matter how completely it may have been recovered, it is not the vital factor which we are seeking; it is at best the soil and not the seed of science. The Greeks possessed this outlook, but they did not create science; we do not produce something new merely by returning to something old. I cannot find an adequate explanation of the remarkable contrast between science before and after the seventeenth century in a mere revival, whether of learning or of anything else. There must be something new, and something new, I believe, is to be found in the work of Galileo.

I want, therefore, first of all to direct your attention to the thought and practice of Galileo, but not merely in order to recover his outlook. Our object is to *interpret* it in the manner calculated to give us the best understanding of its nature and potentialities. That will not necessarily coincide with Galileo's own view of what he was doing. No human being, I think, could have foretold that on the foundations laid down by his simple experiments with falling bodies, a structure would have been reared that in 300 years would have presented the amazing façade of modern physics in all its intricacy and comprehensiveness, and we shall hardly expect Galileo's view of the new philosophy to be adequate to include all that it has produced and may yet produce. He builded better than he knew. From our more advantageous viewpoint we should be able to form a truer estimate of his achievement, and in order to do that we must see it not only as he saw it—as a new method of answering old problems—but rather as a radically new departure in philosophy in which the very problems themselves were changed and only the fundamental human impulse to philosophise remained as a common basis for the new and the old.

What is this fundamental human impulse to philosophise? I think it can be described as the desire to make sense of our experience, to see it as an ordered rational system instead of a succession of fortuitous happenings. As soon as we become conscious we become aware that things happen: we see sights, hear noises, feel hot or cold, experience pains and pleasures, desires and satisfactions, and a host of other things too numerous to be mentioned. We feel also a need to find some order in this chaos, and we are conscious of some innate principle of order, if such a phrase may be used, which we call reason. To philosophise is ultimately to see all the elements of our experience in rational relations with one another, and all philosophies are at bottom attempts to do that as completely as possible.

Now unless some order is found pretty quickly survival is impossible. If one has no idea at all of what is going to happen next—if he does not know that when he feels hungry the feeling may be removed by eating; that certain things may be eaten with safety and others not—if one knows nothing of all this he will not continue to experience for very long. Even the lower animals have achieved this degree of organisation of experience, and so may be said to be philosophers of a very rudimentary kind. What kind of philosophy directs their actions, and leads among other things to the laying of eggs in the right places, to bird migration, and so on, we do not know, but we can form a fairly good idea of the philosophy of early man because it is not so very different from the philosophy we automatically adopt to-day when we are not consciously philosophising at all but merely meeting the ordinary necessities of life. (By

\* *Synthetic Philosophy in the Seventeenth Century* (Herbert Spencer Lecture for 1945), Basil Blackwell, Oxford.



"early man" I mean here the inhabitants of the most civilised parts of the earth just before conscious philosophy began—say one or two thousand years before Christ.) To explain the things he experienced, early man supposed that there was a world around him consisting of numerous bodies which occupied different positions in space and persisted for greater or shorter lengths of time. These bodies could move about in space or lie still, and one of them was identified by each man in a peculiar manner with himself and called his *body*: we will call it the *sensitive body* to distinguish it. The other objects caused experiences when they impinged in some way on the sensitive body. Thus, if they made contact with it there came the experience of touch; if they sent certain emanations towards it, or perhaps if the sensitive body put out feelers to reach them, there came the experience of sight; and so on. One could therefore in large measure predict what he would experience by following the course of the objects constituting the world, noticing their continuity of behaviour, and drawing deductions on the basis of previous experience. Some experiences, such as certain pains and pleasures, hopes and fears, were not readily attributable to the world of material bodies, and so were assigned to invisible and intangible spirits which usually shared the time and space but not the material character of the world responsible for sensations. This exceedingly rough description of our primitive ideas is, of course, not intended to be a precise and complete statement of the outlook of early man, but only an outline of the sort of very naïve realism which in most of its features was so admirably adapted to the elementary needs of life that we still retain it. The description serves its purpose if it indicates to you what you know already, and the only point I wish to make here is that, in its limited way, it is a philosophy of a kind, so that everyone who has survived the hours of entire dependence on others is a philosopher in embryo.

What distinguishes the philosopher properly so called from the ordinary man is that he desires not only to make such rationalisation of his experience as will meet his elementary needs, but to rationalise the *whole* of his experience. Primitive man needed to know that some plants were good to eat and some bad, but he did not need to know why some were green and some red, and he felt no grievance against his philosophy because it did not supply this superfluous information. Philosophy in the full sense of the word began when *all* experiences, without discrimination, became the object of study. It is often said that the characteristic of true philosophising is that it is disinterested, and this of course is true, but I do not think it is the best expression of the truth. It is rather the desire to find a place in our system for everything, whether small or large, useful or useless, past or present, without assigning originally any degrees of relative importance, that makes a man a true philosopher, and although this necessarily directs his interest to things to which he is not attracted for any other reason, that is ultimately an accident. A man may devote his life to the study of the social habits of ants, with no ulterior motive whatever, but he is not a philosopher so long as he does not grant every other phenomenon the same *a priori* significance; and, on the other hand, a man who, after due consideration, finds that the social habits of human beings demand more of his thought than those of ants, does not fail to be a philosopher because his material comfort promises thereby to be better served.

Philosophy proper, then, may be said to have begun with the first man who took the whole of experience as his field of study and tried to see it as a related set of manifestations of some common unifying principle. In the Western world that man is generally considered to have been Thales of Miletus, who lived some 600 years before Christ, and the fact that his unifying principle was something now so unacceptable as water is as nothing beside the immeasurably important fact that he caught the first glimpse of the goal which philosophers have ever since been striving to reach. I need not recount the various alternative principles which successive philosophers substituted for the water of Thales: air, number, eternal flux, and the rest—they are familiar enough, and it is not our purpose now to compare their merits. But the point which I wish to emphasise as strongly as possible is this: that when men began to improve on the partial and purely practical rationalisation of experience, which satisfied their ancestors, by attempting to create a universal philosophy, they took over that partial rationalisation as though it were a necessary, inescapable nucleus, and tried to complete the rationalisation of all experience by extending it to realms previously ignored. Thales and his successors right up to the seventeenth century took for granted, as though it were directly given them in experience itself, the world of material bodies moving in time and space and causing experiences by

impinging on the sensitive body. They overlooked the fact that it was experience they had to rationalise, and thought they had to explain the material world.

A single example must suffice. Early man knew that the grey metal, lead, was heavy, but he did not concern himself with the idle question why a substance which was heavy should also be grey. This was of no practical importance, but it became of great theoretical importance when a universal philosophy was in view, and alchemists gave much attention to it. But it never occurred to the alchemists or anyone else that they could ignore the objective piece of lead in front of them and consider greyness and weight as original elementary experiences. To them these phenomena were by nature associated in the given piece of lead, and to understand lead you had to understand that association. And so the whole of prescientific philosophy was built on to the practical philosophy of an entirely utilitarian age, and no one considered that any other basis was possible.

There is no time to demonstrate this by tracing the course of philosophy from Thales to Galileo, but one or two examples might be given. I have mentioned the alchemists, but the naïve acceptance of the primitive realism characterised their opponents no less than themselves. Thus, Avicenna, in attempting to refute their claims, maintained that although they might succeed in changing all the properties of lead into those of gold, the essential nature of the substance still remained that of lead. The metaphysical notion that an object, or a type of object, had an "essential nature" which existed independently of its manifest properties was not questioned by either party, and this entirely supposititious "essential nature" had displaced experience as the object of the philosopher's investigation. Again, in the great controversy between the nominalists and the realists, what was in dispute was the credentials of "universals"—of lead, for example, but not of a particular piece of lead. That was accepted by both sides as an undeniable and indivisible unit of the material world which had to be taken as given. And so throughout the whole of ancient and mediæval philosophy, the problem was to interpret the world of material and other objects in space and time. No attempt was made to get beneath that world and, if necessary, shatter it to bits and build a world nearer to the mind's desire.

And yet a very simple consideration shows that such a world is very unlikely to lend itself to a complete solution of the philosophical problem. No one even now can dispute its efficacy—indeed its indispensability—for its purpose of enabling us to carry on the practical business of living, but for the complete rationalisation of experience something other than partial efficiency is needed. The first requirement for such a purpose is clearly that the basis of one's philosophy shall be essentially rational, and in this the practical philosophy fails lamentably. For in any rational argument one must proceed from premisses—from evidence—to conclusion, and not from conclusion to premisses, and in the practical, commonsense philosophy it is the latter course that is taken. Experience is interpreted as a consequence of the action of the world of material objects on the sensitive body. But what we know immediately is experience; the world of material objects is what we (rightly or wrongly) infer from it, and a scheme which attributes the data to the action of an inference from the data thereby declares itself as essentially irrational. In the same breath we say there must be a world of matter because we have experiences, and we have experiences because there is a world of matter which causes them. On such a basis it is not to be expected that great systems of rational thought will be erected, and the history of science before the seventeenth century exhibits about as much progress as we have a right to expect.

The undying glory of Galileo's contribution to thought is that, though only half consciously, he discarded this everyday, commonsense world as a philosophical necessity. He paid no attention to material objects as such, but analysed the experiences they were supposed to induce in us into their elementary constituents and reassembled them differently. Rejecting, for example, the supposition that the motion of a body had any direct connection with its weight, so that a heavy body necessarily moved differently from a light one, he detached the motions of all bodies from the properties associated with them in those bodies and sought for laws of *motion* in itself. To him the primary affinity was not between motion and lightness or motion and heaviness, but between motion and motion. The unity of the material object was, in effect, denied (or, more exactly, the material object was not inferred as the necessary first step in the rationalisation of experience), but instead various examples of a particular kind

of experience—that of motion—were grouped together, represented by appropriately chosen concepts, and described by means of general laws.

It is that return to a starting-point more fundamental than that of Thales, or even of the semi-savages who preceded him, that I believe to be the new element that made modern science possible. Galileo's treatment of motion was quickly imitated by the treatment of the other phenomena of physics—temperature, visibility, electric and magnetic actions, and the rest. Each of them formed its own group of phenomena and was studied quite apart from the other so-called properties of the bodies which displayed it. Concepts were chosen for each set of phenomena regardless of what any other set demanded. The fact that a body was in such and such a position in space, for example, was important in the consideration of its motion because space was a concept employed in the description of motion, but it had no importance for the consideration of the temperature of the body because temperature was described in non-spatial terms. It mattered not at all that the *body* had a unique position in space, for the body as such was left out of account.

Of course, all this was not realised immediately; it is not generally realised even yet. The new philosophy was practised faithfully by men who still thought in terms of the old. They still assumed that the material object was a necessary datum, and regarded themselves as performing an act of violation on it by abstracting qualities which really had no right to a separate existence. They could only somewhat unconvincingly defend their action against philosophers of the traditional order by pointing to its success. It would be of great interest to look at the history of modern science, bearing in mind that we are watching the actions of men who are working in accordance with one philosophy and viewing their work in terms of another. Lack of time, however, makes this impossible, but we cannot leave Galileo without taking a glance at his own view of what he had done, for although he did not fully understand his achievement he approached as near to a full understanding as was humanly possible to any scientist before the present century.

By good fortune Galileo was on one occasion persuaded to pause in his pioneer labours and reflect on the metaphysics which they connoted. This is what he wrote: "I feel myself impelled by the necessity, as soon as I conceive a piece of matter or corporeal substance, of conceiving that in its own nature it is bounded and figured in such and such a figure, that in relation to others it is large or small, that it is in this or that place, in this or that time, that it is in motion or remains at rest, that it touches or does not touch another body, that it is single, few, or many; in short, by no imagination can a body be separated from such conditions; but that it must be white or red, bitter or sweet, sounding or mute, of a pleasant or unpleasant odour, I do not perceive my mind forced to acknowledge it necessarily accompanied by such conditions; so if the senses were not the escorts, perhaps the reason or the imagination by itself would never have arrived at them. Hence I think that these tastes, odours, colours, etc., on the side of the object in which they seem to exist, are nothing else than mere names, but hold their residence solely in the sensitive body; so that if the animal were removed, every such quality would be abolished and annihilated. Nevertheless, as soon as we have imposed names on them, particular and different from those of the other primary and real accidents, we induce ourselves to believe that they also exist just as truly and really as the latter."

It is clear from this, I think, that Galileo took the existence of material bodies for granted; so far he spoke in traditional terms. But it is clear also that, whatever he may have supposed them to be in themselves, it was his *conception* of them that occupied his interest. "As soon as I *conceive* a piece of matter", he begins, and henceforth discusses his conceptions, not some possibly inconceivable essence that might be supposed to reside in the piece of matter. Furthermore, when he speaks of the supposed properties of colour, taste, and so on, he implies that they were arrived at by reason or imagination, not given as initial data, and he indicates further that reason or imagination has been led to postulate them in order to account for our sensations; it is the senses that are the escorts. All this is quite in keeping with what I have described as the new outlook which was necessary to bring modern physics into being. Where I think he does not quite attain to the view now possible is in the fundamental distinction he sees between what we now call the mechanical properties of bodies and their other properties. The former he seems to have regarded as having in some way a right to exist independently of experience, and to be attainable by reason or imagination without the escort of the senses. In the light of

later experience it is difficult to maintain this distinction. No doubt the fact that he had been able himself to arrive at far-reaching laws of motion, whereas the other physical phenomena had not even begun to be organised, had something to do with the unique status he assigned to mechanical qualities, but in truth the magnitude of his achievement was so overwhelming that we need not linger on the reasons why it was not even greater.

We do not find the same degree of penetration in his successors. They indeed started the new sciences of heat, optics, electricity and the rest on the authentic lines he had laid down, but they lagged far behind him in their understanding of what they were doing. Instead of regarding "heat" and "temperature" as conceptions useful for representing experiences of warmth and cold (extended later to include thermometer readings), which alone are the fundamental data, they put them back into the reinstated material bodies as inherent *properties* of those bodies. Instead of regarding light and the space and time it was supposed to move in as conceptions useful for representing experiences of vision, which alone are the fundamental data, they gave light the status of a material object and automatically identified the space and time of optics with the space and time of mechanics as objectively existing entities which were given them for study. They did the same thing with other conceptions, putting them back in thought one by one into the supposedly given material world, while all the time unconsciously using them as they should be used, as conceptions to be moulded and changed at the dictate of experience. Only when the inevitable incompatibilities at last force themselves on our attention does it become possible for us to realise the true nature of what has been done. Our "light", which we thought a constituent of the material world, turns out to have contradictory properties. But what can we expect if we put into the material world something which by its very origin cannot belong there? We postulate material objects and to explain how they reveal themselves to our senses we postulate light. But something whose basic function it is to reveal material objects cannot itself be a material object, or we should want another something to reveal *it*, and so on *ad infinitum*. The confusion arises not from the practice of physicists but from the illegitimate metaphysical idea that has lurked behind that practice. Again, space and time were introduced into mechanics in order to describe motion, and into optics in order to describe optical phenomena. It is not necessary that the same space and time that serve the one purpose shall serve the other, and, in fact, physicists worked with them quite independently, in the true scientific manner, while all the time thinking that they were of necessity the same objective things and obscuring their differences by filling them with different ethers. The test came in 1919, when the deflection of light in the gravitational field of the Sun was observed. The result of that test showed that the space and time that met the needs of optics were the same space and time, so far as existing knowledge went, as that which met the needs of mechanics. The observation was not generally regarded in that light, but that is in fact what it amounted to. For suppose the test had been made when Einstein first suggested it in 1911. At that time he still thought that Euclidean space would suffice for mechanics, and predicted a deflection of half the actual amount. The experiment would then have revealed that light suffered twice the calculated deflection, and the most satisfactory explanation so far as I can see—supposing anyone had been bold enough to think of it—would have been that light travelled in a non-Euclidean space and matter in a Euclidean space. The independence of the concepts of time used in the various departments of physics is, I think, still more important, but I have dealt with that elsewhere \* and a brief discussion of it is impossible. But it is not necessary to follow the course of physics since Galileo's time to realise how completely at variance its practice has been with its metaphysical assumptions: we need only look at its conclusions. The culmination of 300 years of the ostensible study of material bodies has been the production of the equations of the electro-magnetic field, the field equations of relativity, the wave equation of the electron, and the laws of thermodynamics. Where in that magnificent epitome of knowledge does one find the least indication that it is a world of material objects that is being described?

The peculiar characteristic of scientific philosophy, then, may be expressed in this way. Like all philosophy, its aim is to organise the whole of experience into a rationally connected system, but, unlike all previous philosophies, it does not accept the world of material objects, located and moving in a unique space and time, as a necessary starting-point, but goes back

\* *Proc. Aristot. Soc.*, XLVIII, 1948, 153; *Phil. Mag.*, xxxv, 1944, 499.

to the original experiences that led to the conception of that world for practical ends, and groups them differently. Instead of regarding the greyness and heaviness of lead as indissolubly associated in the piece of lead, and the yellowness and lightness of sulphur as indissolubly associated in the piece of sulphur, it seeks first a relation between greyness and yellowness on the one hand and heaviness and lightness on the other, and only when its work is wellnigh complete will it arrive at a connection, almost inexpressibly indirect, between the colour and density of any of the pieces of matter which to the commonsense view are indivisible units. It is an historical fact that this is the procedure which, from the time of Galileo onwards, scientific men have practised, though largely unconsciously, and I do not think it can be questioned by anyone who compares the science of post-Galilean with that of pre-Galilean times, that it is the mainspring of the remarkable impetus which the former reveals. Let us now look at one or two consequences which I think have been largely overlooked because of our unconsciousness of the true nature of scientific philosophy.

The first is that, if the view I have advanced is correct, there is no realm of experience which is excluded from scientific treatment. When the world of material objects is taken as forming the primary data of science, a natural distinction arises between those experiences which can be directly traced to that world—*i.e.* in the main the experiences obtained through the five senses—and those others, including religious and æsthetic experiences, which appear to have other sources. A sharp difference of opinion has thus arisen, among scientists as well as others, as to how these latter experiences are to be regarded. On the one hand there is the materialistic view that they are mere “illusions”, ultimately traceable, when we have sufficient knowledge, to physiological peculiarities and therefore to be ignored except in so far as the advance of medical science may show us how to control them. On the other hand there is the “idealistic” view (to choose one of the meanings of an overworked word) that such experiences are worth more than mere sensations and come from a world just as “real” as the material world, if not more so. Each view claims to be scientific—more, I think, because to be scientific is now regarded as an honour than because of any clear conception of what being scientific means.

It is evident, however, that if not the world of material objects but experiences themselves are the fundamental data of science, then there is no reason whatever to grant any experience initial priority over any other; all are alike submitted for consideration, and our rationalisation is incomplete so long as any are excluded. Sensations are not in the least degree more “scientific” than emotions. The problem that faces us now is not “How can we account for these apparently causeless experiences?” but rather “Why is it that science has made so much progress with sensations and so little with emotions?” And the answer, I think, is that it is because, in the state of knowledge in the seventeenth century, sensations—or, more precisely, those experiences which form the subject-matter of the physical sciences—were the only experiences for which the necessary rational machinery existed.

Consider the problem. You have to represent your experiences by concepts defined in such a way that the system of relations which follows by rational deduction from the definitions will stand in a one-to-one correspondence with the experiences themselves. In this way experience is seen not as a chaos but as an ordered whole. Now in the seventeenth century by far the most highly developed system of abstract reasoning that existed was pure mathematics. There was Aristotelean logic, it is true, but that was quite inadequate to meet the need. The chief purpose it served was to protect the thinker from elementary errors; it gave him no impulse to create a system of thought. Accordingly the experiences which alone at that time could be treated scientifically were those which could be represented mathematically—that is by number, or magnitude: in other words, the only experiences that could be treated scientifically were “measurable” experiences, for measurement is simply an operation, precisely defined in all essential details, which yields a number. It was for that reason, and that reason alone, that Galileo began with the experience of motion. There are other experiences measurable now—*e.g.* temperature and brightness—but in his day motion was almost, if not quite, the only one, and so modern science began with the study of motion.

There was yet a further limitation. All motions could be measured, but the results of the measurements did not always stand in simple relations with one another, and since the ultimate aim was to find such relations there was little incentive to waste time on sterile measurements.

The motions of living creatures, for example, showed no detectable regularity, so they were left alone. The motions of falling bodies offered a better prospect of success, but even they, in their natural form, gave results difficult to relate simply and exactly with one another, so the fall was controlled. Not the experiences that came uncalled for, but those that arose in a carefully prepared situation, were the experiences with which science began, and it began with them because it was not then prepared for dealing with any others, not because others were of essentially different nature.

Both of these features of early science—the restriction to measurement and the “experimental method”, as it is called — have at various times been represented as the essential distinguishing characteristic of science itself. Important as they are, I think their significance is quite misunderstood when they are so represented. They owe their place in science not to their being necessary to its nature but to the fact that they provide the easiest—indeed, human limitations being what they are, the only possible—starting-point for the process of rationalising the whole of experience. The vital essence of science was the realisation that the world of matter, space, and time must be ignored and a fresh start made with bare experience itself. That having once been realised, a perfectly equipped logician provided with an unlimited imagination might have started equally effectively with any kind of experience and covered the whole field in any order he wished (I am assuming, of course, that all experience is susceptible of rationalisation and is not essentially chaotic). Science would then have known no distinction between the measurable and the non-measurable or between experiment and observation. Galileo was, of course, not a perfectly equipped logician, but he was an excellent mathematician according to the standards of his time. He accordingly selected experiences which could be represented by mathematical quantities, and in order that those quantities when determined should be such as his mathematics could relate together by simple formulæ he altered the conditions of his observation until he had attained the desired end. That is how mathematics and experiment gained their present place in science.

The position to-day is very different from that which Galileo knew. During the years that followed the birth of science the metrical process was of course greatly extended, and in biology much rationalisation of experience was carried on in which measurement played no part. In our own time psychology has become scientific; that is to say, it has chosen as its field of study mental phenomena in isolation from those bodily actions which are the property of physics and biology. It has chosen its own concepts—the *unconscious*, the *ego*, the *ensor*, and so on—and in terms of them it has related together experiences which in earlier days would have been rejected as mere illusions, unworthy of a moment's attention. I think this is an event of tremendous significance. It does not matter whether the present conceptions and correlations of psychoanalysis are valid or not—indeed, it is difficult to believe that all of them are—and we might expect them to have the same degree of permanence as the earliest conceptions of electricity or magnetism. What is important is the fact that a science of pure psychology, divorced from physiology on the one hand and from pure reason uncontaminated by experience on the other, has been shown to be possible, for it confirms the belief that the scientific process of starting with experience itself is applicable in even the most difficult fields, and that no experience can be regarded as intrinsically outside the scope of the scientific treatment.

But the greatest hope for the future seems to me to lie in the amazing progress of logic during the last hundred years. Where Galileo had only a rudimentary body of mathematics and a logic of simple syllogisms with which to face experience, we have an enormously developed mathematical octopus and a set of logical calculi in which mathematics itself occupies only a subordinate place. Numbers, which are what measurement always yields, are subsumed within a far greater corpus of systematised concepts, all of which is available for the scientist of imagination to apply to the organisation of those departments of science in which measurement fails. The simple idea that experience must conform to the elementary notions with which mathematics began is a relic of the prescientific days when the units of the world were thought to be material objects. Material objects obey the laws of numbers—no doubt mathematics began with numbers for that reason—and so those laws were thought to comprise the whole machinery of scientific reasoning. But even in physics itself this idea has been transcended. Non-Euclidean geometry and non-commutative algebra are freely called upon to extend the scheme of relations beyond what was possible to Galileo or Newton, and if a two-valued logic



becomes embarrassed there is a three-valued—or indeed an  $n$ -valued—logic ready to take its place. It may or may not be that God made the integers and man made the rest, but it is certain that, if so, man can achieve what would otherwise be impossible, by using his own creations. It seems to me that in this enormous enlargement of the logical machinery—which will doubtless itself be indefinitely extended in the future—lies the possibility of a rationalisation of those psychological and psychical phenomena which have so far seemed to be beyond the possibility of scientific treatment. What is wanted is the genius to choose the right logical system and to correlate its elementary concepts with the appropriate phenomena. When that is done we shall know something.

I would not like to end without saying something about the relation between scientific philosophy and what I have called the practical or commonsense view which sees experience organised into a world of matter in space and time, but before doing so I must interpolate a word about nomenclature, to avoid mere verbal confusion. I have defined the philosopher as one whose ambition it is to see the whole of experience as a single rationally connected system, and I have conjectured that, in the light of present knowledge, the probabilities are overwhelmingly in favour of the belief that the scientific approach—that which begins by grouping together experiences of the same kind and not those associated with the same material object—is best fitted to reach the desired end. That would suggest that science is the right philosophy and all others are wrong; and furthermore, that no one whose aim is less universal can be called a philosopher at all. Words are merely labels, and the acceptance of these conclusions would do no harm if it were not that the words “right” and “wrong” and “philosopher” are so familiar with other meanings that misunderstanding would almost inevitably result. We do not wish, for example, to deny the title of philosopher to Hume because his work made no pretension of illuminating the relation of smelling to hearing, and we do not think it wrong to say that the sun will rise at six o'clock because the hypothesis of a rising sun does not lead to the most comprehensive system of astronomy. Let me, then, in what follows use the words “philosophy” and “philosopher” in the ordinary, rather vague sense, with reference to any attempt to make sense of our experience, whether complete or partial; and let me add that I shall not call any philosophy right or wrong without reference to its purpose. I would then say that if one's purpose is to achieve the full rationalisation of the whole of experience, the scientific philosophy, though still incomplete, is the only one that offers any reasonable prospect of success; but if one has a different object in view, then the choice of a different philosophy may be not only legitimate but the only sensible course to adopt.

Examples of legitimate philosophies are numerous. One thinks, for example, of various systems of theology for which men have been ready to die, yet which, with larger experience, have been discarded. Their holders have found them indispensable for rationalising their own limited religious experience, and it would have been the height of folly for them to have exchanged such theologies for a not yet achieved inclusion of religious experience within the scientific scheme. The art of science is long and life is short. A man must make up his account with the universe in the time allotted to him, and while, if he is a scientist, it is his duty to push forward towards the goal he will not live to reach, it is no less his duty and his privilege as a common man to make such rationalisation of what he feels to be the most constraining elements of his experience as will give him the greatest immediate satisfaction. Similarly, art critics may well find that such a concept as beauty—and especially ugliness, perhaps—as an objective characteristic of a work of art may be necessary for the clear expression of their meaning, notwithstanding that a future inclusion of aesthetics within a wider psychological science may show the inadmissibility of such a concept. Such examples present no difficulty so long as one recognises, on the one hand, that a philosophy must always be evaluated in relation to its purpose and to the knowledge available at the time, and, on the other hand, that the validity, and even necessity, of a philosophy within a limited sphere carries with it no guarantee that that philosophy will survive in a more complete system of correlations.

But the relation of the scientific philosophy to the everyday philosophy based on a world of matter in space and time, though no different in principle, is so much more subtle in detail that it calls for special consideration. Even the most uncompromisingly dedicated scientist must live an ordinary life, and he must recognise that while, for the most exalted scientific purposes, he should see experience organised in the scientific manner, for all other purposes he must see

it organised in the commonsense manner. Indeed, the very practice of science itself is a commonsense undertaking, and in the simplest electric circuit the physicist, often without noticing the incongruity, sees one part as a dance of electrons from atom to atom in an electric field and another part as a galvanometer. The galvanometer also, in the scientific sense, is a dance of electrons from atom to atom in an electric field, but it would not serve his purpose so to consider it; and the conductor which is the centre of interest in his experiment is, in the everyday sense, a piece of gross matter, but the experiment would be meaningless if he regarded it in that way. If the whole circuit is expressed in terms of the scientific philosophy it becomes a subspace in an  $n$ -dimensional manifold represented by certain equations—or by whatever more comprehensive generalisation may succeed such a description. This may be in one sense the truest and purest account we can give of it, but the mixed account of the physicist is nevertheless the one to be preferred for the purpose of making progress in science.

A particularly interesting example of the close interconnection of science and commonsense is afforded by chemistry. Here we have a study which seems to give the lie to the main thesis of this lecture, which is that the essence of scientific philosophy lies in the ignoring of material objects and the adoption of the alternative arrangement of experiences into groups to each of which corresponds a particular science. For it turns out that when we arrange our experiences in this way, all the sensations with which science deals, except those of the movements of living creatures, go to the various branches of physics, and those movements, together with all experiences which are not sensations, are claimed by biology and psychology. There appears to be nothing left for chemistry. A realist might indeed say that chemistry is the science of stinks, and it is true that physics puts in no counterclaim to this not altogether felicitous element of our sensibility, but I am afraid that there is no legitimate escape that way. Unlike G. K. Chesterton's character, who played billiards in the dark entirely by the sense of smell, the chemist does pay some attention to other aspects of his subject-matter, and what he gives us is not an understanding of the olfactory sense but, as it seems, a knowledge of the composition of those very material bodies whose existence we said science did not acknowledge. How are we to account for this?

Well, the truth is that chemistry indeed has no place in the strict scientific scheme, and that this is so can be seen from the fact, already evident even at the present stage of scientific progress, that the ultimate generalisations of chemistry are all derivable, and indeed must inevitably have been reached sooner or later, from the development of physics itself—chiefly the departments of optics and electro-magnetism. The periodic table, originally a product of chemical research, is a product also of spectroscopic research, and with this difference, that instead of showing in each place a chemical symbol and an atomic weight, the spectroscopic table shows a configuration of electric charges which, when fully understood, will undoubtedly prescribe all the varieties of chemical combination that are possible. The whole of chemistry may therefore, so far as final results go, be regarded as a superfluous study.

I need scarcely say, I hope, that I do not draw from this the moral that the pursuit of chemistry has been a grand mistake. Reluctant as I am, and as a loyal physicist should be, to say anything good of chemistry, I cannot deny that, quite apart from its necessity for the amenities of life, it has been indispensable in making possible the rapid progress of physics. Without it the present state of advancement of physics would have been indefinitely postponed. But what does follow, I think, is that the part played by chemistry in the growth of science has been a pragmatism, a heuristic one. It has provided a short cut to knowledge in principle obtainable without it, and so, like all philosophising based on the unanalysed concept of the material object or material substance, it is a means of reaching an end, but does not survive in the end itself. It is important to understand this distinction, because chemistry has played so conspicuous a part in the history of science that otherwise it would seem that an interpretation of science which is not largely based on its peculiar character must necessarily be defective. That, I am convinced, would be a mistake. Chemistry rightly figures prominently in the history of science; in the philosophy of science it should figure not at all. But that very fact shows how intimately the scientific and the commonsense philosophy are interwoven. If I emphasise the necessity for freeing scientific philosophy from the intrusion of commonsense conceptions, it is not in order to depreciate commonsense but because the greater danger to-day lies in their confusion.



This brings me to another, still more topical, problem—the status of the so-called social sciences. Their position in this country is somewhat ambiguous. They are called sciences, yet those who study them are not usually regarded as scientists, they do not come within the purview of the Royal Society or any of the ordinary scientific societies, and they have little, if any, interaction with the studies whose scientific character is unquestioned. In Germany, I believe, they are included with the natural sciences under the general name of *Wissenschaft*; the latter are given the sub-title, *Naturwissenschaft*, and the former that of *Sozialwissenschaft* or *Geisteswissenschaft*. Hence there also the social sciences seem to be regarded as in some sense scientific yet not of the purest blood. Those who feel most strongly that they should be considered sciences seem to base their conviction on the belief that “scientific method” is applicable to their study, though when we reflect that the method of arriving at the principle of least action was to prove that God works in the most economical way, and the method of finding a relation between the wave-lengths in the hydrogen spectrum, from which the whole of modern atomic theory has proceeded, was to hand the figures to a numerical mystic innocent of physical knowledge, we cannot help feeling a little sceptical of a decision on grounds of method. Let us look at the question from the point of view taken in this lecture.

When we do that it becomes immediately obvious that we have a problem very like that of chemistry, and susceptible of a similar solution. Instead, however, of the material substance as the accepted starting-point, we have the *community*—a group of human beings, each of which is a body and mind. Such a highly complex unit is, of course, much further removed from the elementary starting-point of science than is the mere material object of chemistry, but that is only a matter of degree; in principle the situations are just the same. Science as such knows nothing of the community. The group of experiences that underlies such a conception would, in the scientific study, be divided between all the sciences. Physics would take those aspects of the body of every member that are exhibited also by inorganic matter, and biology and psychology would share the remaining bodily behaviour and the activities of the minds. In the course of time, when these sciences have developed sufficiently, we might expect to obtain a knowledge of the community which will transcend that obtainable from existing social studies in the same degree as the physicist's periodic table transcends the chemist's. To obtain the fullest knowledge of sociology we must therefore pursue the studies of physics, biology, and psychology—especially psychology.

Even less wisely, however, than with chemistry could we conclude that therefore the social sciences might be discarded. We have already almost reached the stage when pure science can dispense with chemistry; but the day is not yet in sight when it can do without the independent study of communities. Both in order to make civilised life possible and also in order to accelerate the progress of individual psychology by learning about the behaviour of men in the mass, we must prosecute the study of the social sciences with all the means at our command. This needs no emphasis. What does need emphasis at the present time is that, despite their urgency and indispensability, the social sciences are, from the point of view of pure knowledge, ultimately superfluous and intrinsically incapable of yielding anything like a full understanding of their own subject-matter.

The reason why this conclusion, which follows inevitably enough from a clear understanding of the character and scope of science, needs to be stressed is that there are prominent and influential philosophies in the world to-day which claim special scientific sanction for their own social theories, and even in some instances assert that scientific laws are merely special cases of their own hypotheses of community behaviour. If I select dialectical materialism for special mention it is not because it is unique in this respect, but because it is the most conspicuous example of the philosophies I mean, and the one which probably has the greatest influence on modern thought. Certain principles are stated to be inevitably operative in community life—whether rightly or wrongly is not our present concern—and it is then deduced that those principles must operate also in the realms of physics and biology. The fundamental content of those realms is taken to be matter, and the conclusion is that the laws of matter must be derivable from the laws of human communities, and that all this constitutes science. For instance, the melting of ice when it is heated is an example of the law of internal struggle which is derived from the behaviour of classes of human beings under changing economic conditions.

This is, of course, completely incompatible with the view which I have been trying to present in this lecture. To begin with, the materialistic aspect of the doctrine, exemplified in the assumption that physics is the study of matter, is directly at variance with what I hope to have shown is the historical fact that physics began to advance when it abandoned the study of matter. Again, the method of explaining the simple in terms of the complex is the exact contrary of the scientific practice of building up the complex out of the simple. Whatever may be said for dialectical materialism as a practical guide to politics—and I repeat that I am not here expressing any opinion at all about its merits or demerits in that regard—it is crystal clear that as a universal philosophy it is as unscientific as any system of thought could be. Its advocates would show less muddleheadedness and do better service to whatever of value their philosophy may possess if, instead of claiming that it is what it manifestly is not, they would contrast it with science and try to show its superiority. But I do not expect them to do so.

The achievements of the scientific philosophy, begun so modestly more than 300 years ago, are now among the most amazing elements of our civilisation and culture. In the advancement of pure knowledge—the understanding of the interconnectedness of our experience—scientific philosophy has been without a rival. It has pressed into its service the partial practical philosophies that serve limited ends, and made them contribute to its advancement. In its turn it has enriched them by placing at their disposal knowledge which by themselves they would have been powerless to attain, so that we can manipulate the material world in ways undreamed of so long as matter was treated as a fundamental entity. Few objects could be more worthy of our thought than that of reaching the clearest and fullest understanding of the nature of scientific philosophy itself, and I hope that I have been able to contribute something towards that end.

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XLIII.—On the Gravitational Mass of a System of Particles. By G. L. Clark,  
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SUMMARY

In classical mechanics the mass of a system of gravitating particles can be defined to be the mass of an equivalent particle which gives the same field at great distances, or alternatively the mass can be defined by means of Gauss' Theorem. Reference to the former procedure was made by Eddington and Clark (1938) in a discussion on the problem of  $n$  bodies. The relativistic extension of Gauss' Theorem has been investigated by Whittaker (1935) for a particular form of the line-element and for more general fields by Ruse (1935). The latter, treating the problem from a purely geometrical point of view, expressed the integral of the normal component of the gravitational force as the sum of two volume integrals. The physical significance of one of these integrals was quite obvious but the meaning of the other was far from clear. In this paper the terms in Ruse's result are examined as far as the order  $m^2$  in the case of a fundamental observer at rest and the 1938 discussion modified to bring the two investigations into line. It is concluded that the surface integral of the normal component of the gravitational force taken over an infinite sphere is  $-4\pi \times$  the energy of the system.

1. THE FIELD DUE TO  $n$  BODIES

It is well known that in the case of weak fields it is convenient to introduce in place of the  $g_{\mu\nu}$  quantities  $h_{\mu\nu}$ ,  $\gamma_{\mu\nu}$  defined by the equations

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (1.1)$$

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}h_{\sigma\rho}, \quad (1.2)$$

where the  $\eta_{\mu\nu}$  are the Galilean values of the  $g_{\mu\nu}$ , and to use the convention that Greek indices run over the values 1, 2, 3 and 4 while Latin indices take the values 1, 2 and 3.

Denoting ordinary differentiation by a line followed by a suffix, the linear terms in the expression for the energy-tensor are

$$16\pi T_{\mu\nu} = \gamma_{\mu\nu|s} - \gamma_{\mu s|\nu}, \quad (1.3)$$

provided

$$\gamma_{\mu s|s} = \gamma_{\mu 4|4}. \quad (1.4)$$

The  $g_{\mu\nu}$  of the field having been determined, the track of a body in the field is given by the geodesic

$$\frac{d^2x^\mu}{ds^2} + \{a\beta, \mu\} \frac{dx^a}{ds} \frac{dx^\beta}{ds} = 0. \quad (1.5)$$

Considering only linear terms, (1.5) may be written

$$\frac{d^2x^s}{dt^2} = -\frac{1}{2}h_{44|s} + h_{4s|4} + v^r h_{4s|r} - v^r h_{4r|s} + \frac{1}{2}h_{44|4}v^s + v^r h_{rs|4}, \quad (1.6)$$

where  $v^r$  are the components of the velocity of the test particle. The form (1.6), given originally by de Sitter (1916), was derived by the author as a condition of integrability of the field equations in 1941. If the test particle is instantaneously at rest, (1.6) reduces to

$$\frac{d^2x^s}{dt^2} = -\frac{1}{2}h_{44|s} + h_{4s|4}. \quad (1.7)$$

Two fields will be defined to be equivalent at great distances if a test particle has the same acceleration when placed in each field; it would, indeed, be illogical to say that two fields are equivalent if they produced different accelerations on a test particle. Now, in the case of  $n$  bodies,  $h_{44}$  is of the order  $1/r$  at great distances and the derivative  $h_{44|3}$  is of the order  $1/r^2$ , so for two fields to be equivalent we require the respective  $h_{4s|4}$  to be equal up to order  $1/r^2$ .

In the 1938 paper it was explained that it was desired to determine accelerations correctly to the squares of the potentials, and accordingly, treating the Newtonian potential as of the first order, the order of the terms required to be retained in the  $g_{\mu\nu}$  are 2,  $\frac{3}{2}$ , 1 in  $g_{44}$ ,  $g_{4n}$  and the other  $g_{\mu\nu}$  respectively; and, in particular,  $h_{4s|4} = g_{4s|4}$  is of order 2.

The first approximation to the field due to a system of gravitating particles is given by the expressions (3.1) of the 1938 paper; the field, in fact, is

$$\left. \begin{aligned} \gamma_{44} &= -\sum_i \frac{m_i}{r_i}, \\ \gamma_{4n} &= 4\sum_i \frac{m_i v_{ni}}{r_i}, \end{aligned} \right\} \quad (1.8)$$

where  $v_{1i}$ ,  $v_{2i}$ ,  $v_{3i}$  are the components of the velocity of the  $i$ th particle and  $\gamma_{mn}$  is of order 2. As explained above, we do not require the individual  $\gamma_{mn}$  but only the second-order terms of  $h_{44} = \frac{1}{2}\gamma_{44} + \frac{1}{2}\gamma_{11}$ . After some calculation, it was found that to the required order the field due to  $n$  bodies is

$$\left. \begin{aligned} h_{mn} &= \gamma_0 \delta_{mn}, \\ h_{4n} &= \sum_i \frac{4m_i v_{ni}}{r_i}, \\ h_{44} &= \gamma = \gamma_0 + \frac{1}{2}\gamma_0^2 + \sum_i \left( -\frac{3m_i v_i^2}{r_i} + \frac{2m_i}{r_i} \sum_j \frac{m_j}{\Delta_{ij}} - m_i \frac{\partial^2 r_i}{\partial t^2} \right), \end{aligned} \right\} \quad (1.9)$$

where

$$\gamma_0 = -\sum_i \frac{2m_i}{r_i},$$

and  $\Delta_{ij}$  is the distance between the  $i$ th and  $j$ th bodies (the value  $j=i$  being omitted in the  $j$ -summation).

If we adopt a co-ordinate system in which the centre of mass of the system of  $n$  particles is at rest at the instant considered, so that  $\sum_i m_i v_{ni} = 0$ , the line-element at great distances is of the form

$$ds^2 = (-1 + \gamma_0)(dx^2 + dy^2 + dz^2) + (1 + \gamma)dt^2, \quad (1.10)$$

provided terms of order  $1/r^2$  in  $h_{4n}$  are neglected. By examining the coefficient of  $1/r$  in  $\gamma$  it was shown in the 1938 paper that the mass of the system at an appropriately antedated instant is given by

$$\begin{aligned} M &= \sum_i m_i + \frac{3}{2} \sum_i m_i v_i^2 - \sum_i \sum_j \frac{m_i m_j}{\Delta_{ij}} \\ &= E + \frac{1}{2} \frac{d^2 I}{dt^2}, \end{aligned} \quad (1.11)$$

where

$$E = M_0 + K + \Omega, \quad (1.11a)$$

$M_0$  being the sum of the rest masses,  $K$  the kinetic energy,  $\Omega$  the potential energy and  $I$  the moment of inertia about the centre of gravity. The purpose of the present paper is to criticise the former definition of the mass of a system on the ground that it is only valid in the case when  $h_{4s|4}$  is zero up to order  $-2$  in  $r$ . For, if  $h_{4s|4}$  is not zero, the  $s$ th component of the acceleration of a test particle placed at a great distance from a system of  $n$  bodies would differ from the corresponding component of a test particle placed in the so-called "equivalent field" by an amount  $h_{4s|4}$ . We shall take over the entire 1938 analysis and consider the effect of the additional terms arising from  $h_{4s|4}$ . We conclude, if we retain terms of order  $m^2$ , that a system of particles cannot, in general, be regarded at great distances as equivalent to a

single particle, and we are led to investigate an alternative line of development by means of Gauss' Theorem. We shall find that the inclusion of  $\hbar_{4s|4}$  leads to a term  $-\frac{1}{2}d^2I/dt^2$  which cancels out the second term on the right-hand side of (1.11).

A critic may argue that the discrepancy lies in the fact that the geometrically defined centre of mass does not necessarily have zero acceleration. This objection is not valid since this phenomenon introduces terms of order  $m^3$  and not  $m^2$ . In the first place, we have

$$\begin{aligned}\frac{1}{2} \frac{d^2 I}{dt^2} &= \frac{d^2}{dt^2} \sum_i \frac{1}{2} m_i \{ (x_i - x_0)^2 + (y_i - y_0)^2 + (z_i - z_0)^2 \} \\ &= \frac{d}{dt} \sum_i m_i \{ (x_i - x_0)(\dot{x}_i - \dot{x}_0) + \dots \} \\ &= \sum_i m_i \{ (\ddot{x}_i x_i + \dot{x}_i^2 - \ddot{x}_0 x_i + \dot{x}_0^2) + \dots \} \\ &= \frac{1}{2} \sum_i m_i \frac{d^2}{dt^2} (x_i^2 + y_i^2 + z_i^2) - (\ddot{x}_0 \sum_i m_i x_i + \dots) + \sum_i m_i (\dot{x}_0^2 + \dots),\end{aligned}\quad (1.12)$$

on putting the co-ordinates  $(x_0, y_0, z_0)$  of the centre of mass zero after differentiation. Now the accelerations  $\ddot{x}_0, \ddot{y}_0, \ddot{z}_0$  and the squares of the velocities  $\dot{x}_0^2, \dot{y}_0^2, \dot{z}_0^2$  are at least of order  $m^2$ , and the terms involving  $m_i \ddot{x}_0, m_i \dot{x}_0^2$  in (1.12) are of the order  $m^3$  at least. Again, in the second place, we note that if it were possible to replace the system of  $n$  bodies by an equivalent particle having the same acceleration as well as the same velocity as the centre of mass of the system, the line-element would be of the form

$$ds^2 = \left( -1 - \frac{2M}{r_0} \right) (dx^2 + dy^2 + dz^2) + \frac{4Mv^s dx^s dt}{r_0} + \left( 1 - \frac{2M}{r_0} \right) dt^2, \quad (1.13)$$

and the acceleration of a test particle would be

$$\ddot{x}^s = -\frac{M(x^s - x_0^s)}{r_0^3} + \frac{4M\dot{v}^s}{r_0} + \frac{4M(x^r - x_0^r)v^r v^s}{r_0^3}, \quad (1.14)$$

where  $(x_0, y_0, z_0)$  are the co-ordinates of the centre of mass of the system and

$$v^r = \frac{dx_0^r}{dt}, \quad \dot{v}^r = \frac{dv^r}{dt}, \quad r_0^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2. \quad (1.15)$$

Since we have chosen a co-ordinate system in which the centre of mass is at rest at the Newtonian level, the second and third terms on the right of (1.14) are of the order  $m^3$  and consequently outside the scope of the present discussion. However, we may note that there may be terms of order  $-1$  in  $r$  in the equations of motion, when we retain terms of order  $m^3$ . It is clear that to the order considered in this discussion we may neglect any acceleration of the geometrically defined centre of mass.

We now proceed to expand  $\gamma_{4n}$  in powers of  $1/r$ . We have

$$\begin{aligned}r_i^2 &= (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 \\ &= r^2 - 2(xx_i + yy_i + zz_i) + r_{0i}^2,\end{aligned}\quad (1.16)$$

where

$$\left. \begin{aligned}r^2 &= x^2 + y^2 + z^2 \\ r_{0i}^2 &= x_i^2 + y_i^2 + z_i^2.\end{aligned} \right\} \quad (1.17)$$

Expanding  $1/r_i$  in powers of  $1/r$  we obtain

$$\frac{1}{r_i} = \frac{1}{r} + \frac{xx_i + yy_i + zz_i}{r^3} + \frac{3(xx_i + yy_i + zz_i)^2}{2r^5} - \frac{r_{0i}^2}{2r^3} + \dots \quad (1.18)$$

Substituting (1.18) in (1.8) and remembering that at the Newtonian level  $\sum_i m_i v_{ni} = 0$ , we find, considering terms up to order  $-2$  in  $r$ ,

$$\gamma_{4n} = \frac{4}{r^3} \{ x \sum_i m_i v_{ni} x_i + y \sum_i m_i v_{ni} y_i + z \sum_i m_i v_{ni} z_i \}, \quad (1.19)$$

that is,

$$\gamma_{41} = \frac{4}{r^3} \left\{ x \sum_i m_i x_i \frac{dx_i}{dt} + y \sum_i m_i y_i \frac{dy_i}{dt} + z \sum_i m_i z_i \frac{dz_i}{dt} \right\}, \quad (1.20)$$

and similar expressions for  $\gamma_{42}$  and  $\gamma_{43}$ .

At this point it is instructive to consider a double star having components of equal mass. Putting  $m_1 = m_2$  in equation (8.2) of the 1938 paper, it is seen that the centre of mass has no acceleration of the order  $m^2$ . On the other hand, we may choose co-ordinates so that to a first approximation,

$$\left. \begin{aligned} x_1 &= \frac{1}{2}a \cos \omega t, & y_1 &= \frac{1}{2}a \sin \omega t, & z_1 &= 0, \\ x_2 &= -\frac{1}{2}a \cos \omega t, & y_2 &= -\frac{1}{2}a \sin \omega t, & z_2 &= 0, \\ \dot{x}_1 &= -\frac{1}{2}a\omega \sin \omega t, & \dot{y}_1 &= \frac{1}{2}a\omega \cos \omega t, & \dot{z}_1 &= 0, \\ \dot{x}_2 &= \frac{1}{2}a\omega \sin \omega t, & \dot{y}_2 &= -\frac{1}{2}a\omega \cos \omega t, & \dot{z}_2 &= 0, \end{aligned} \right\} \quad (1.21)$$

where  $\omega$  is constant,  $a$  is the distance between the bodies, and  $a\omega$  is related to the mass  $m$  of one of the components of the double star by means of

$$\frac{1}{2}ma^2\omega^2 = \frac{m^2}{a}. \quad (1.22)$$

Substituting (1.21) in (1.19) we obtain

$$\left. \begin{aligned} \gamma_{41} &= \frac{ma^2\omega}{r^3} \{-x \sin 2\omega t + (\cos 2\omega t - 1)y\}, \\ \gamma_{42} &= \frac{ma^2\omega}{r^3} \{x(1 + \cos 2\omega t) + y \sin 2\omega t\}, \\ \gamma_{43} &= 0. \end{aligned} \right\} \quad (1.23)$$

According to the 1938 discussion the "equivalent mass" of the double star is given by (1.11), namely

$$M = 2m - \frac{ma^2\omega^2}{4} = 2m - \frac{m^2}{2a}, \quad (1.24)$$

since

$$K = \frac{1}{4}ma^2\omega^2, \quad \Omega = -\frac{m^2}{a}.$$

The motion of a particle at great distances is, by (1.7) and (1.23),

$$\left. \begin{aligned} \frac{d^2x}{dt^2} &= -\frac{Mx}{r^3} + \frac{2ma^2\omega^2}{r^3} (-x \cos 2\omega t - y \sin 2\omega t), \\ \frac{d^2y}{dt^2} &= -\frac{My}{r^3} + \frac{2ma^2\omega^2}{r^3} (-x \sin 2\omega t + y \cos 2\omega t), \\ \frac{d^2z}{dt^2} &= -\frac{Mz}{r^3}, \end{aligned} \right\} \quad (1.25)$$

where  $M$  is given by (1.24).

Since the centre of mass of the system has no acceleration up to order  $m^2$ , it is obvious that the periodic terms in (1.25) are not due to any motion of the centre of gravity. Moreover, if we use present instead of retarded values and retain the term  $\sum m\ddot{r}$  in (1.9), we find that entirely new terms appear in (1.25). The periodic terms in (1.25) are not therefore connected in any way with the fact that retarded potentials are used in evaluating  $M$ . As a consequence of these facts we are led to the conviction that the field due to two bodies of equal mass cannot be regarded as equivalent to the field due to a single particle at great distances if terms of order  $m^2$  are retained. As a matter of fact the potentials (1.23) are of precisely the same form as potentials due to a rotating cohesive system such as a rotating rod, but the equations (1.25) are not valid for the latter system owing to the presence of additional periodic terms of order  $1/r$  in  $h_{44}$ .

In general it is not possible to express the potentials in forms which are valid for all time, but we can usually obtain the instantaneous values of the potentials. For example, in the case of a rigid continuous system rotating with angular velocities  $(\omega_1, \omega_2, \omega_3)$  about the axes, we may write

$$\left. \begin{aligned} v_{1i} &= \omega_2 z_i - \omega_3 y_i, \\ v_{2i} &= \omega_3 x_i - \omega_1 z_i, \\ v_{3i} &= \omega_1 y_i - \omega_2 x_i, \end{aligned} \right\} \quad (1.26)$$

and, if  $A, B, C, F, G, H$  denote the moments and products of inertia, (1.19) becomes

$$\gamma_{41} = \frac{4}{r_3} \left\{ x(G\omega_2 - H\omega_3) + y \left( F\omega_2 - \frac{C+A-B}{2}\omega_3 \right) + z \left( \frac{A+B-C}{2}\omega_2 - F\omega_3 \right) \right\}, \quad (1.27)$$

and similar expressions for  $\gamma_{42}$  and  $\gamma_{43}$ .

If the moments and products of inertia are expressed in terms of the principal moments of inertia referred to axes moving with the body and instantaneously coinciding with the coordinate axes, we have [Clark, 1941, equation (7.4)]

$$\left. \begin{aligned} G\omega_2 &= (A-C)\omega_2^2 t, \\ H\omega_3 &= (B-A)\omega_3^2 t, \\ \omega_3 &= \omega_3 + \dot{\omega}_3 t, \end{aligned} \right\} \quad (1.28)$$

where  $\omega_s, \dot{\omega}_s$  are the instantaneous values of the components of angular velocity and acceleration, and cubes and higher powers of  $\omega_s$  are neglected. The angular acceleration is given by Euler's equations and is consequently of the order  $\omega^2$ . On substituting (1.28) in (1.27) we obtain

$$\gamma_{41} = \frac{4}{r_3} \left\{ x[(A-C)\omega_2^2 t + (A-B)\omega_3^2 t] + y \left[ (C-B)\omega_1 \omega_2 t - \frac{C+A-B}{2}(\dot{\omega}_3 + \omega_3 t) \right] + z \left[ \frac{A+B-C}{2}(\omega_2 + \dot{\omega}_2 t) - (C-B)\omega_1 \omega_2 t \right] \right\}. \quad (1.29)$$

The component of the acceleration in the direction of the  $x$ -axis of a test particle placed at a great distance from the system is accordingly

$$\frac{d^2 x}{dt^2} = -\frac{Mx}{r^3} + \frac{4x}{r^3} \left\{ (A-C)\omega_2^2 + (A-B)\omega_3^2 \right\} + \frac{4y}{r^3} \left\{ (C-B)\omega_1 \omega_2 - \frac{C+A-B}{2}\dot{\omega}_3 \right\} + \frac{4z}{r^3} \left\{ \frac{A+B-C}{2}\dot{\omega}_2 - (C-B)\omega_1 \omega_2 \right\}, \quad (1.30)$$

where, as in (1.25),  $M$  denotes the "equivalent mass" in the 1938 sense.

We conclude this part of the discussion by remarking that it is evident from (1.20) that the conditions that (1.10) should represent the system are:

$$\left. \begin{aligned} \Sigma_i m_i x_i \frac{dx_i}{dt} &= \Sigma_i m_i y_i \frac{dx_i}{dt} = \Sigma_i m_i z_i \frac{dx_i}{dt} = 0, \\ \Sigma_i m_i x_i \frac{dy_i}{dt} &= \Sigma_i m_i y_i \frac{dy_i}{dt} = \Sigma_i m_i z_i \frac{dy_i}{dt} = 0, \\ \Sigma_i m_i x_i \frac{dz_i}{dt} &= \Sigma_i m_i y_i \frac{dz_i}{dt} = \Sigma_i m_i z_i \frac{dz_i}{dt} = 0. \end{aligned} \right\} \quad (1.31)$$

Hence

$$\frac{1}{2} \frac{d^2 I}{dt^2} = \frac{1}{2} \frac{d^2}{dt^2} \Sigma_i m_i r_{0i}^2 = \frac{d}{dt} \Sigma_i m_i \left( x_i \frac{dx_i}{dt} + y_i \frac{dy_i}{dt} + z_i \frac{dz_i}{dt} \right), \quad (1.32)$$

and consequently (1.11) reduces to

$$M = E. \quad (1.33)$$

The result (1.32) also holds in the case of a rigid system rotating about an axis. That is, it applies when terms of the type (1.29) are added to the line-element (1.10). To investigate the effect of the terms (1.19) in the general case, it is necessary to consider the problem from the point of view of Gauss' Theorem.

## 2. GAUSS' THEOREM

Taking a line-element of the form

$$ds^2 = g_{44}dt^2 + g_{mn}dx^m dx^n, \quad (2.1)$$

where  $g_{44}$  may be a function of all four co-ordinates and  $g_{mn}$  functions of  $x^s$  only, Whittaker (1935) showed that the quantity

$$4\pi(T_4^4 - T_1^1 - T_2^2 - T_3^3)\sqrt{(-g)} \quad (2.2)$$

could be expressed as a divergence

$$(\sqrt{(-g)}g^{44}\{44, s\})_{|s}, \quad (2.3)$$

where the Christoffel bracket  $\{44, s\}$  reduces to  $-\frac{1}{2}g^{sq}g_{44|q}$  for the metric (2.1). Integrating throughout the volume considered he obtained Gauss' Theorem in the form

$$\begin{aligned} \iiint (\sqrt{(-g)}g^{44}\{44, s\})_{|s} dx dy dz &= 4\pi \iiint (T_4^4 - T_1^1 - T_2^2 - T_3^3)\sqrt{(-g)} dx dy dz \\ &= 4\pi \iiint (T - 2T_1^1 - 2T_2^2 - 2T_3^3)\sqrt{(-g)} dx dy dz, \end{aligned} \quad (2.4)$$

which, in the case of weak fields, may be written

$$\iiint (-\frac{1}{2}h_{44|s})_{|s} dx dy dz = -4\pi \iiint (T + 2T_{11})\sqrt{(-g)} dx dy dz. \quad (2.5)$$

Now the invariant mass,  $m$ , of a single particle is given by

$$m = \iiint T\sqrt{(-g)}\frac{dt}{ds} dx dy dz, \quad (2.6)$$

and so

$$\iiint T\sqrt{(-g)} dx dy dz = m\frac{ds}{dt} = m - \frac{1}{2}mv^2 + \frac{1}{2}mh_{44}.$$

Summing over all particles, we get

$$\iiint T\sqrt{(-g)} dx dy dz = M_0 - K + 2\Omega, \quad (2.7)$$

where, as in (1.11a),  $M_0$  is the sum of the rest masses,  $K$  the kinetic energy and  $\Omega$  the potential energy.

We also may write

$$T_{mn} = \rho v^m v^n + p_{mn}, \quad (2.8)$$

where  $p_{mn}$  represent the stresses. In a subsequent paper we shall derive the equation

$$\int p_{11} dV = 0. \quad (2.9)$$

Using the result (2.9), we have to the order considered

$$2 \iiint T_{11}\sqrt{(-g)} dx dy dz = 4K. \quad (2.10)$$

Substituting (2.7) and (2.10) in (2.5), we see that the volume integral on the right-hand side of the equation has the value

$$-4\pi(\sum_i m_i + 3K + 2\Omega) = -4\pi\left(E + \frac{1}{2}\frac{d^2 I}{dt^2}\right), \quad (2.11)$$

\* See p. 432, equ. (5.8).



on using the relation [Eddington, 1916, equation (4)]

$$2K + \Omega = \frac{1}{2} \frac{d^2 I}{dt^2} \quad (2.12)$$

for a system of particles. The result (2.11) is identical with (1.11) and the two methods of analysis are equivalent. However, as already mentioned, the term  $d^2 I/dt^2$  is zero for rigid continuous systems.

We now proceed to consider the additional terms appearing in (2.4) when the line-element is of the general form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (2.13)$$

Throughout the discussion we have restricted ourselves to the consideration of terms up to order  $m^2$ , and consequently the only contribution to (2.11) from the squares of the potentials  $h_{\mu\nu}$  is

$$-8\pi\Omega = -2\pi \int T h_{44} dV.$$

In passing over to the general form (2.13) we naturally retain this term, but the additional terms all involve  $h_{\mu\nu}$  linearly and the investigation is therefore quite straightforward.

### 3. THE GRAVITATIONAL FORCE

The linear terms in the potentials  $h_{\mu\nu}$  in the equations of motion of a test particle are given in (1.6). Taking the test particle to be of unit mass, we see that the expression on the right-hand side of this equation is the gravitational force  $F^s$ , and if the test particle is at rest, at the instant considered the gravitational force is simply

$$F^s = -\frac{1}{2} h_{44|s} + h_{4s|4}, \quad (3.1)$$

and the divergence of  $F^s$  is

$$F^s|_s = -\frac{1}{2} h_{44|ss} + h_{4s|s4}. \quad (3.2)$$

Introducing the  $\gamma_{\mu\nu}$  by means of (1.2), this divergence may be written

$$F^s|_s = -\frac{1}{2} (\gamma_{44|ss} - \gamma_{44|44}) - \frac{1}{2} (\gamma_{4s|ss} - \gamma_{4s|44}) + h_{4s|s4} - \frac{1}{2} h_{44|44}. \quad (3.3)$$

If the condition (1.4) is satisfied, (3.3) becomes

$$F^s|_s = -4\pi(T_{44} + T_{44}) + h_{4s|s4} - \frac{1}{2} h_{44|44} \quad (3.4)$$

on using (1.3). As we have seen,  $\gamma_{mn}$  is of the second order and its derivative  $\gamma_{mn|44}$  is of the third order; we may therefore express (3.4), with the aid of (1.4), in the form

$$F^s|_s = -4\pi(T_{44} + T_{44}) + \gamma_{4s|s4} - \frac{1}{2} \gamma_{44|44} \quad (3.5)$$

$$= -4\pi(T + 2T_{44}) + \frac{3}{2} \gamma_{4s|s4}, \quad (3.6)$$

since to the order considered

$$T = T_{44} - T_{44}. \quad (3.7)$$

As far as the linear terms are concerned, Gauss' Theorem accordingly takes the form

$$\begin{aligned} \int (n_s F^s) dS &= \int (F^s|_s) dV \\ &= -4\pi \int (T + 2T_{44}) dV + \frac{3}{2} \int \gamma_{4s|s4} dV, \end{aligned} \quad (3.8)$$

where  $n_s$  is the outward normal and  $F^s$  is given by (3.1).

Comparing (3.8) with (2.6) we see that the effect of the additional potential  $\gamma_{4n}$  in the line-element is to increase the gravitational force by  $h_{4s|4}$  and to introduce a further volume integral  $\frac{3}{2} \int \gamma_{4s|s4} dV$ . Applying the relations (2.8) and (2.10) to the first integral on the right-hand side of (3.8), we find that its value is

$$-4\pi(\Sigma_i m_i + 3K). \quad (3.9)$$

We mentioned at the end of the previous section that the contribution  $-8\pi\Omega$  arising from the squares of the potentials  $h_{\mu\nu}$  must be added to (3.9); the sum

$$-4\pi(\sum_i m_i + 3K + 2\Omega) \quad (3.10)$$

being in agreement with (2.10) and (1.11). Moreover, the form of the equation (1.3) indicates that retarded values must be used in calculating the potentials.

In a rigorous discussion, covering the case of strong fields, Ruse (1935, p. 151) expresses Gauss' Theorem in a form similar to (3.8). One integral gives immediately the expression (3.10) for the system considered in this paper. The integral corresponding to the second integral on the right-hand side of (3.8) is

$$-\int\{(\lambda^\alpha)_\beta(\lambda^\beta)_\alpha + \lambda^\alpha(\lambda^\beta)_{\beta\alpha}\}dV, \quad (3.11)$$

where  $\lambda^\alpha$  is the unit 4 vector of the fundamental observer and the brackets ( ) denote covariant differentiation. To the order of magnitude considered in this paper we may take

$$\left. \begin{aligned} \lambda_i &= 0, & \lambda_4 &= 1 + \frac{1}{2}h_{44}, \\ \lambda^i &= 0, & \lambda^4 &= 1 - \frac{1}{2}h_{44}. \end{aligned} \right\} \quad (3.12)$$

The only terms of order  $m^2$  in (3.11) come from  $-\lambda^\alpha(\lambda^\beta)_{\beta\alpha}$ , namely

$$\begin{aligned} -\lambda^4 \frac{\partial^2}{\partial t^2} \{\lambda^4 \sqrt{(-g)}\} &= -\frac{\partial^2}{\partial t^2} [(1 - \frac{1}{2}h_{44})(1 - \frac{1}{2}h_{11} + \frac{1}{2}h_{44})] \\ &= \frac{1}{2}h_{11|44} \\ &= \frac{3}{2}\gamma_{44|44} \\ &= \frac{3}{2}\gamma_{4s|s4}, \end{aligned} \quad (3.13)$$

in agreement with (3.8).

Since  $\gamma_{4s|s4}$  is in the form of a divergence, the volume integral  $\int \gamma_{4s|s4} dV$  can be evaluated by means of a surface integral; that is, (3.8) may be written in the form

$$\int (n_s F^s) dS = -4\pi \left\{ \int (T + 2T_{11}) dV + \int \frac{1}{2} T h_{44} dV \right\} + \frac{3}{2} \int (n_s \gamma_{4s|s4}) dS \quad (3.14)$$

on including the quantity

$$-8\pi\Omega = -2\pi \int T h_{44} dV.$$

Now if we integrate first through regions containing matter only and then throughout empty space, the surface integrals, evaluated at the boundaries, will cancel out in pairs since the potentials and their first derivatives are continuous. Consequently, when integrating throughout all space we need only evaluate the surface integrals over an infinite sphere and, in the case of  $n$  bodies, we may use the approximate value of  $\gamma_{4n}$  given by (1.19). We then have

$$\gamma_{4n|4} = \frac{4}{r^3} \frac{d}{dt} \{x \sum_i m_i x_i v_{ni} + y \sum_i m_i y_i v_{ni} + z \sum_i m_i z_i v_{ni}\}. \quad (3.15)$$

Also the direction cosines of the outward normal for a sphere are  $x^s/r$ , and so

$$\int (n_s \gamma_{4s|4}) dS = \int \frac{x^s}{r} \gamma_{4s|4} dS. \quad (3.16)$$

From (3.15) and (3.16) we see that the integral consists of partial integrals of the type

$$\int \frac{x^s x^r}{r^4} dS.$$

This is zero unless  $s=r$ , in which case we have

$$\frac{1}{r^4} \int x^2 dS = \frac{1}{r^4} \int y^2 dS = \frac{1}{r^4} \int z^2 dS = \frac{4\pi}{3}. \quad (3.17)$$

With the help of (3.17) we then obtain from (3.16) the result

$$\begin{aligned} \frac{3}{4} \int (\gamma_{4s|4}) dS &= \pi \frac{d}{dt} \{4 \sum_i m_i (x_i v_{1i} + y_i v_{2i} + z_i v_{3i})\} \\ &= 2\pi \sum_i m_i \frac{d^2}{dt^2} r_{0i}^2 \\ &= 2\pi \frac{d^2 I}{dt^2}, \end{aligned} \quad (3.18)$$

where  $r_{0i}$  is defined by (1.17).

Combining (3.18) and (2.11), we finally obtain Gauss' Theorem in the form

$$\int (\gamma_s F^s) dS = -4\pi E, \quad (3.19)$$

as the terms in  $d^2 I/dt^2$  cancel. The result (3.19) may be stated thus: "The outward flux of the gravitational force due to several moving particles taken over an infinite sphere is equal to  $-4\pi \times$  the energy of the system."

In the above analysis we expressed the volume integral  $\int \gamma_{4s|s4} dV$  as a surface integral which we evaluated over an infinite sphere. It may be thought that a more detailed investigation would show that the only contributions to the volume integral come from regions containing matter. This does not appear to be the case; in fact, it seems that

$$\frac{3}{4} \int \gamma_{4s|s4} dV = 4\pi \sum_i m_i x_i^s \frac{d^2 x_i^s}{dt^2} \quad (3.20)$$

when integration is over the part of space containing no matter, and

$$\frac{3}{4} \int \gamma_{4s|s4} dV = 4\pi \sum_i m_i v_i^2 \quad (3.21)$$

when integration is over the part of space containing matter. The sum of (3.20) and (3.21) gives

$$\frac{3}{4} \int \gamma_{4s|s4} dV = 2\pi \frac{d^2}{dt^2} \sum_i m_i r_{0i}^2, \quad (3.22)$$

as in (3.18), integration now being over all space.

For the sake of illustration let us suppose the  $i$ th body to be spherical, of constant density and of small radius  $a_i$ , which can be made to vanish in the limit. The part of  $\gamma_{4s|s4}$  due to the other particles will accordingly not contribute to

$$\frac{3}{4} \int \gamma_{4s|s4} dV. \quad (3.23)$$

$i$ th particle

The internal potential  $\gamma_{4s}$  for the  $i$ th particle is simply

$$(\gamma_{4s})_{\text{int}} = \frac{2m_i v_{si}}{3} (3a_i^2 - r_i^2), \quad (3.24)$$

for the expression (3.24) and its derivatives is continuous at  $r_i = a_i$  with

$$(\gamma_{4s})_{\text{ext}} = \frac{4m_i v_{si}}{r_i}. \quad (3.25)$$

Substituting (3.24) in (3.23) gives

$$\frac{3}{4} \int \gamma_{4s|s4} dV = 4m_i v_i^2, \quad (3.26)$$

$i$ th particle

in agreement with (3.21). The result (3.26) can be verified by considering the integral

$$\frac{3}{4} \int (n_s \gamma_{4s|4}) dS$$

extended over the surface of the  $i$ th particle and using either the internal or external form of the potential  $\gamma_{4s}$ . Further insight into the theory is obtained by writing (3.20) and (3.21) in the forms

$$\frac{3}{4} \int_{\text{empty space}} \gamma_{4s|s4} dV = 4\pi \left( \frac{1}{2} \frac{d^2 I}{dt^2} - 2K \right) = 4\pi \Omega \quad (3.27)$$

and

$$\frac{3}{4} \int_{\text{matter}} \gamma_{4s|s4} dV = 4\pi (2K). \quad (3.28)$$

The formula (3.19) may then be written

$$\int (n_s F^s) dS = -4\pi (E_i + E_0), \quad (3.29)$$

where  $E_i$ ,  $E_0$  are respectively the contributions from regions containing matter and empty space. The actual values of  $E_i$  and  $E_0$  are found from (3.10), (3.27) and (3.28); they are:

$$\begin{aligned} E_i &= M_0 + 3K + 2\Omega - 2K \\ &= M_0 + K + 2\Omega \end{aligned} \quad (3.30)$$

$$= M_0 - 3K + \frac{d^2 I}{dt^2}, \quad (3.30a)$$

and

$$\begin{aligned} E_0 &= -\Omega \\ &= 2K - \frac{1}{2} \frac{d^2 I}{dt^2}, \end{aligned} \quad (3.31)$$

where  $M_0$  is the sum of the rest masses, and the total energy is

$$\begin{aligned} E &= E_i + E_0 \\ &= M_0 + K + \Omega \\ &= M_0 - K + \frac{1}{2} \frac{d^2 I}{dt^2}. \end{aligned} \quad (3.32)$$

For steady systems, for which  $d^2 I/dt^2$  is zero,

$$E = M_0 - K. \quad (3.33)$$

#### 4. ISOLATED SYSTEMS

Throughout the preceding analysis we have restricted the investigation to linear motion, and the results we have obtained depend on equation (2.9). In the case of a rotating cohesive system, the analysis from which this equation was deduced no longer applies and the result (2.9) no longer holds. I have elsewhere (1946) calculated the gravitational field of a slowly rotating nearly spherical ellipsoid. For this system, I have found that

$$\int T_{ii} dV = 0. \quad (4.1)$$

That is,

$$\begin{aligned} \int p_{ii} dV &= - \int \rho v^s v^s dV \\ &= -2K_R. \end{aligned} \quad (4.2)$$

We may regard the integral on the left-hand side of (4.2) as being a rotational potential energy  $\Omega_R$ ; the equation (4.2) then expresses the fact that

$$\Omega_R = -2K_R \quad (4.3)$$

and is analogous to equation (2.12) for systems (as in this case) for which  $d^2I/dt^2$  is zero.

The equations (3.20) and (3.21) and therefore (3.30) and (3.31) do not apply to this system. I have substituted the potential  $\gamma_{4s}$ , for the internal and external fields due to the rotating system, in the volume integral  $\int \gamma_{4s|s4} dV$ . It is found that the integral vanishes when integrated in the region containing matter and also when integrated throughout empty space. That is,

$$E_0 = 0, \quad (4.4)$$

and

$$\begin{aligned} E_i &= M_0 + 3K_R + 2\Omega_R \\ &= M_0 - K_R, \end{aligned} \quad (4.5)$$

on using (4.3).

A general system will consist of several bodies having both linear and angular velocities. The total kinetic and potential energies will be given by

$$K = K_L + K_R, \quad (4.6)$$

$$\Omega = \Omega_L + \Omega_R, \quad (4.7)$$

where  $K_L$  is the sum of the kinetic energies of the linear motion of each body,  $K_R$  is the sum of the kinetic energies of the rotational motion of each body relative to its centre of gravity,  $\Omega_L$  is the ordinary potential energy and  $\Omega_R$  is the rotational energy defined by

$$\Omega_R = \int p_{11} dV. \quad (4.8)$$

From (2.12) and (4.3) we have

$$\Omega_L = -2K_L + \frac{1}{2} \frac{d^2I}{dt^2} \quad (4.9)$$

and

$$\Omega_R = -2K_R, \quad (4.10)$$

so that

$$\Omega = -2K + \frac{1}{2} \frac{d^2I}{dt^2}. \quad (4.11)$$

Also from (3.30), (3.31), (4.4) and (4.5) we have, on using (4.9)–(4.11),

$$E_0 = -\Omega_L, \quad (4.12)$$

$$\begin{aligned} E_i &= M_0 + K_L + 2\Omega_L + 3K_R + 2\Omega_R \\ &= M_0 + K_L + 2\Omega_L + K_R + \Omega_R \end{aligned} \quad (4.13)$$

$$= M_0 - 3K_L - K_R + \frac{d^2I}{dt^2}, \quad (4.14)$$

and

$$\begin{aligned} E &= M_0 + K_L + \Omega_L + K_R + \Omega_R \\ &= M_0 + K + \Omega \end{aligned} \quad (4.15)$$

$$= M_0 - K + \frac{1}{2} \frac{d^2I}{dt^2}. \quad (4.16)$$

In the case of steady systems the expressions (4.15) and (4.16) link up with an alternative line of development given by Eddington in his Dublin Lectures (1943, sections 11 and 13). In these lectures Eddington partitions the total energy-tensor into a particle energy-tensor  $K_{\mu\nu}$  and a field energy-tensor  $\Omega_{\mu\nu}$  by means of

$$T_{\mu\nu} = \Omega_{\mu\nu} + K_{\mu\nu}, \quad (4.17)$$

and

$$\Omega_{\mu\nu} = -(\dot{k} + 1)K_{\mu\nu}, \quad (4.18)$$

$$T_{\mu\nu} = -\dot{k}K_{\mu\nu}, \quad (4.19)$$

where the factor  $k$  represents the number of independent components of the energy-tensor and  $k=1$  for particles considered in molar relativity theory. Putting  $k=1$  in (4.18) and (4.19) we obtain the results (4.11) and (4.16) with  $d^2I/dt^2$  zero. The expressions (4.12) and (4.13) assert, moreover, that in empty space the field energy is  $-\Omega_L$  and in regions containing matter it is  $2\Omega_L + \Omega_R$ , the total field energy being  $\Omega = \Omega_L + \Omega_R$ .

When the system consists of a single body  $\Omega_L = 0$ . In the absence of rotation, we then have

$$E_0 = 0, \quad (4.20)$$

$$E = E_i = M_0 + K_L, \quad (4.21)$$

and there is no inversion of energy. On the other hand when the isolated particle has rotational as well as linear motion,

$$\begin{aligned} E &= M_0 + K_L + K_R + \Omega_R \\ &= M_0 + K_L - K_R. \end{aligned} \quad (4.22)$$

That is, there is inversion of rotational energy but not of the energy of the linear motion. In conclusion, we remark that the result (4.20) follows directly from (3.20), since the acceleration of an isolated particle is zero.

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XLIV.—The Equivalence of the Gravitational and Invariant Mass of an Isolated Body at Rest.\* By G. L. Clark, Trinity College, Cambridge. *Communicated by Sir EDMUND WHITTAKER, F.R.S.*

(MS. received May 22, 1946)

THROUGHOUT the preceding discussion we have considered only those contributions to the stresses which are due to either the motion of the body or the presence of other bodies. That is, we have not considered the stresses which occur in the systems discussed by Whittaker, and we have, in effect, assumed that the gravitational mass of an isolated body at rest is the same as its invariant mass at least as terms of order  $m^2$  are concerned. In this appendix we complete the investigation by demonstrating the validity of this assumption.

Retaining terms up to order  $m^2$  only, Whittaker's analysis gives

$$M = \int T\sqrt{(-g)}dV - 2 \int T_i^i dV, \quad (1)$$

since  $T_i^i$  is of order  $m^2$ .

Now the invariant mass  $M_0$  is given by

$$\begin{aligned} M_0 &= \int T \frac{dt}{ds} \sqrt{(-g)} dV \\ &= \int T\sqrt{(-g)}dV - \frac{1}{2} \int T h_{44} dV. \end{aligned} \quad (2)$$

That is,

$$\int T\sqrt{(-g)}dV = M_0 - \int \phi dm, \quad (3)$$

where  $\phi = -\frac{1}{2}h_{44}$  is the Newtonian potential and  $dm$  is an element of mass of the body. The gravitational energy  $\Omega_s$  of a body is given by (see, for example, Ramsey, 1940, p. 57)

$$\Omega_s = -\frac{1}{2} \int \phi dm. \quad (4)$$

The equation (3) may therefore be written

$$\int T\sqrt{(-g)}dV = M_0 + 2\Omega_s. \quad (5)$$

We shall also prove that

$$\int T_i^i dV = - \int T_{ii} dV = \Omega_s. \quad (6)$$

Substituting (5) and (6) in (1) then gives

$$M = M_0. \quad (7)$$

From equations (12.8), (12.9) and (12.10) of Clark (1941, p. 249) we have

$$16\pi T_{ii} = \gamma_{ii|ss} - \frac{7}{8}\gamma_{44|s}\gamma_{44|s} - \gamma_{44}\gamma_{44|ss} \quad (8)$$

$$= (\gamma_{ii} - \frac{7}{16}\gamma_{44}^2)_{|ss} - \frac{1}{8}\gamma_{44}\gamma_{44|ss}, \quad (9)$$

or

$$T_{ii} = \frac{1}{16\pi} (\gamma_{ii} - \frac{7}{16}\gamma_{44}^2)_{|ss} + \frac{1}{2}\phi\rho, \quad (10)$$

\* This paper is to be regarded as an appendix to the paper, "On the Gravitational Mass of a System of Particles", p. 412 *supra*.

since

$$\phi = -\frac{1}{2}h_{44} = -\frac{1}{2}\gamma_{44} \quad (11)$$

and

$$16\pi\rho = \gamma_{44}|_{ss} \quad (12)$$

correct to order  $m$ .

From (10), we have

$$\int T_{44} dV = \frac{1}{16\pi} \int (\gamma_{44} - \frac{1}{2}\gamma_{44}^2)|_{ss} dV - \Omega_s \quad (13)$$

on using (4).

The first volume integral on the right-hand side of (13) may be expressed as a surface integral. Also since the energy-tensor is zero in empty space and the  $\gamma_{\mu\nu}$  and  $\gamma_{\mu\nu}|_s$  are continuous at the boundary of the body, we may integrate the volume integrals throughout all space and the surface integral over an infinite sphere. Now since  $\gamma_{44}^2$  and  $\gamma_{44}$  are both of order  $-2$  in  $r$ , the surface integral is of the order  $-1$  in  $r$  and accordingly vanishes. In any case, at sufficiently great distances, the field will be the same as that due to an equivalent spherical particle, and the actual expressions for  $\gamma_{44}^2$  and  $\gamma_{44}$  will be

$$\gamma_{44}^2 = \frac{16m^2}{r^2}, \quad (14)$$

$$\gamma_{mn} = \frac{7m^2 x^m x^n}{r^4}, \quad \gamma_{44} = \frac{7m^2}{r^2}. \quad (15)$$

The result (15) is taken from the expression (unnumbered) for  $\gamma'_{mn}$  on p. 242 of Clark (1941), and from equation (13.3), p. 97, of the Einstein, Infeld and Hoffmann paper (1938).

For a sphere we have therefore

$$\gamma_{44} - \frac{1}{2}\gamma_{44}^2 = 0 \quad (16)$$

for all values of  $r$ . Consequently

$$(\gamma_{44} - \frac{1}{2}\gamma_{44}^2)|_s = 0, \quad (17)$$

and the surface integral vanishes when evaluated over a sphere of finite radius. On using (17), the equation (13) takes the form

$$\int T_{44} dV = -\Omega_s,$$

which is equation (6).

It is interesting to verify the result (6) in the particular case of a sphere of constant density and radius  $a$ . The Newtonian potential is given by

$$\phi = \frac{m}{2a^3}(3a^2 - r^2), \quad (18)$$

and consequently

$$\Omega_s = -\frac{1}{2} \int \phi dm = -\frac{3m^2}{5a}. \quad (19)$$

The line-element for this system has been given by Schwarzschild (see, for example, Eddington, 1924, para. 72). This solution gives isotropic hydrostatic pressure at every point. To the required order of approximation, the pressure is

$$P = -T_1^1 = -T_2^2 = -T_3^3 = \frac{3m^2}{8\pi a^6}(a^2 - r^2). \quad (20)$$



The expression (20) is obtained from Eddington's equation (72.4) on writing  $\alpha = 2m/a^3$ . Now from (20) we have

$$-\int T_1^1 dV = \int T_{nn} dV = \frac{3m^2}{5a}. \quad (21)$$

Combining (19) and (21) we then have

$$\Omega_s = -\int T_{uu} dV = -\frac{3m^2}{5a}, \quad (22)$$

in agreement with (6).

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# XLV.—The Internal and External Fields of a Particle in a Gravitational Field.

By **G. L. Clark**, Trinity College, Cambridge. *Communicated by Sir EDMUND WHITTAKER, F.R.S.*

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## SUMMARY

THE gravitational field of a system of particles was investigated by de Sitter as far back as 1916. A minor alteration to the analysis was made by Eddington and Clark in 1938. The amended value of the potential  $g_{44}$  is the same as that derived by Einstein, Infeld and Hoffmann without making use of the energy-tensor; this agreement suggests that the revised de Sitter argument is correct. In this paper we show that this is not the case, for the de Sitter analysis completely overlooked any possible interaction terms in the stress components of the energy-tensor. We find the value of these terms,  $p_{mn}$ , and show that the agreement mentioned above is due to the fact that the volume integral of  $p_{ll}$  vanishes.

## 1. INTRODUCTION

In this paper we adopt the notation and conventions used in the previous paper. We also restrict ourselves to fields satisfying the co-ordinate conditions

$$\gamma_{\mu s|s} - \gamma_{\mu 4|4} = 0. \quad (1.1)$$

For gravitational systems the terms of order  $m$  are of the form

$$\gamma_{44} = 2\phi, \quad (1.2)$$

where  $-\phi$  is the Newtonian potential and

$$\gamma_{4n} = 0, \quad \gamma_{mn} = 0. \quad (1.3)$$

A tedious calculation gives the values of the components  $T_{44}$ ,  $T_{mn}$  of the energy-tensor correct to  $m^2$ ,

$$16\pi T_{44} = \gamma_{44|ss} + 6\phi\phi_{|ss} + \frac{3}{2}\phi_{|s}\phi_{|s}, \quad (1.4)$$

and

$$16\pi T_{mn} = \gamma_{mn|ss} + 2\phi\phi_{|mn} - 2\phi_{|s}\phi_{|s}\delta_{mn} + \phi_{|m}\phi_{|n} - \frac{3}{2}\phi_{|s}\phi_{|s}\delta_{mn}. \quad (1.5)$$

The expression (1.5) is essentially (12.8) of Clark (1941). From (1.5) we have

$$16\pi T_{ll} = \gamma_{ll|ss} - 4\phi\phi_{|ss} - \frac{7}{2}\phi_{|s}\phi_{|s}. \quad (1.6)$$

We now put

$$\phi = U + \frac{1}{2}\gamma'_{44}, \quad (1.7)$$

where  $-\frac{1}{2}\gamma'_{44}$  is the Newtonian potential due to the particle, which we shall assume is spherical, and  $U$  is the potential due to the other bodies. We take

$$\gamma'_{44} = -\frac{4m}{r} \quad (1.8)$$

and

$$\gamma'_{44} = \frac{2m}{a^3}(-3a^2 + r^2), \quad (1.9)$$

where  $m$  is a constant, for the external and internal fields respectively. Moreover, in the

neighbourhood of the particle, we may write

$$U = \bar{U} + x^s \bar{U}_{|s} + \dots, \quad (1.10)$$

where  $\bar{U}, \bar{U}_{|s}, \dots$  are the values of  $U, U_{|s}, \dots$  at the centre of mass of the particle which, for convenience, we have taken to be at the origin. In the Einstein paper and in a similar discussion given by Clark (1941, paras. 8-10) only those terms involving  $\bar{U}_{|s}$ , which were necessary to obtain the equations of motion, were considered. The present analysis requires us to consider all terms containing  $\bar{U}_{|s}$  but not  $\bar{U}_{|mn}, \bar{U}_{|mnp}, \dots$ . We may note also that another minor difference between this and the previous investigations is due to the use of the co-ordinate condition

$$\gamma_{ms|s} - \gamma_{4m|4} = 0, \quad (1.11)$$

instead of

$$\gamma_{ms|s} = 0. \quad (1.12)$$

By making use of (1.7) and (1.10), the expressions (1.4), (1.5) and (1.6) may be written, in the neighbourhood of the particle,

$$16\pi T_{44} = \gamma_{44|ss} + 3\bar{U}\gamma'_{44|ss} + 3\bar{U}_{|s}x^s\gamma'_{44|rr} + \frac{3}{2}\bar{U}_{|s}\gamma'_{44|ss}, \quad (1.13)$$

$$16\pi T_{mn} = \gamma_{mn|ss} + \bar{U}\gamma'_{44|mn} - \bar{U}\gamma'_{44|ss}\delta_{mn} + \bar{U}_{|s}x^s\gamma'_{44|mn} - \bar{U}_{|s}x^s\gamma'_{44|rr}\delta_{mn} \\ + \frac{1}{2}\bar{U}_{|m}\gamma'_{44|n} + \frac{1}{2}\bar{U}_{|n}\gamma'_{44|m} - \frac{3}{2}\bar{U}_{|s}\gamma'_{44|s}\delta_{mn}, \quad (1.14)$$

and

$$16\pi T_{4l} = \gamma_{4l|ss} - 2\bar{U}\gamma'_{44|ss} - 2\bar{U}_{|s}x^s\gamma'_{44|rr} - \frac{1}{2}\bar{U}_{|s}\gamma'_{44|s}. \quad (1.15)$$

## 2. THE EXTERNAL FIELD IN THE NEIGHBOURHOOD OF THE PARTICLE

Substituting (1.8) in (1.13)-(1.15) gives

$$16\pi T_{44} = \gamma_{44|ss} + 6\bar{U}_{|s}\frac{mx^s}{r^3}, \quad (2.1)$$

$$16\pi T_{mn} = \gamma_{mn|ss} + 4\bar{U}m\left(\frac{1}{r^3}\delta_{mn} - \frac{3x^m x^n}{r^5}\right) + \bar{U}_{|s}mx^s\left(-\frac{2}{r^3}\delta_{mn} - \frac{12x^m x^n}{r^5}\right) \\ + 2\bar{U}_{|n}\frac{mx^m}{r^3} + 2\bar{U}_{|m}\frac{mx^n}{r^3}, \quad (2.2)$$

and

$$16\pi T_{4l} = \gamma_{4l|ss} - 14\bar{U}_{|s}\frac{mx^s}{r^3}. \quad (2.3)$$

The condition for free space requires us to take solutions

$$\gamma_{44} = \frac{\alpha\bar{U}m}{r} + \beta\bar{U}_{|s}\frac{mx^s}{r^3} - 3\bar{U}_{|s}\frac{mx^s}{r}, \quad (2.4)$$

$$\gamma_{mn} = -2\bar{U}m\left(\frac{1}{r}\delta_{mn} + \frac{x^m x^n}{r^3}\right) - \bar{U}_{|s}\frac{mx^s x^m x^n}{r^3} - 2\bar{U}_{|s}\frac{mx^s}{r}\delta_{mn} \\ + \frac{ma^2}{5}\left(-\frac{x^s}{r^3}\bar{U}_{|s}\delta_{mn} + \frac{x^n}{r^3}\bar{U}_{|m} + \frac{x^m}{r^3}\bar{U}_{|n}\right), \quad (2.5)$$

where  $\alpha$  and  $\beta$  are constants.

From (2.5) we have

$$\gamma_{ms|s} = -\frac{2m}{r}\bar{U}_{|m}, \quad (2.6)$$

and

$$\gamma_{4l} = -\frac{8\bar{U}m}{r} - 7m\bar{U}_{|s}\frac{x^s}{r} - ma^2\bar{U}_{|s}\frac{x^s}{5r^3}. \quad (2.7)$$

The equations (1.4) and (1.6) can be integrated in forms which are valid for all space. The solutions, apart from additive harmonic functions, are

$$\gamma_{44} = -\frac{3}{4}\phi^2 \quad (2.8)$$

and

$$\gamma_{11} = \frac{7}{4}\phi^2. \quad (2.9)$$

The condition that the expressions which are valid for all space should reduce to the values (2.4) and (2.7) in the neighbourhood of the particle fixes the value of the additive harmonic functions which appear in (2.8) and (2.9). The result is

$$\gamma_{44} = -\frac{3}{4}\phi^2 + (\alpha - 3)\frac{\bar{U}m}{r} + \beta m \bar{U}_{|s} \frac{x^s}{r^3} \quad (2.8a)$$

and

$$\gamma_{11} = \frac{7}{4}\phi^2 - \frac{\bar{U}m}{r} - \frac{1}{2}m \bar{U}_{|s} \frac{x^s}{r^3}. \quad (2.9a)$$

The values of  $\alpha$  and  $\beta$  in (2.4) are fixed by the condition

$$\gamma_{4s|s} - \gamma_{44|4} = 0. \quad (2.10)$$

It is found [Clark, 1941, equations (9.4) and (9.9)] that  $\alpha = 4$  and  $\beta = 0$ . We have therefore, in free space,

$$\begin{aligned} h_{44} &= \frac{1}{2}\gamma_{44} + \frac{1}{2}\gamma_{11} \\ &= \phi^2 - \frac{\bar{U}m}{r} - m \bar{U}_{|s} \frac{x^s}{r^3}. \end{aligned} \quad (2.11)$$

Putting  $m = m_i$ ,  $r = r_i$  and  $\bar{U} = -2 \sum_j \frac{m_j}{\Delta_{ij}}$ , the second term on the right-hand side of (2.11) may be written

$$\frac{2m_i \sum_j \frac{m_j}{\Delta_{ij}}}{r_i},$$

in agreement with previous calculations. The third term, which is of the order  $-2$  in  $r$ , is new; its appearance is due to the fact that we have considered terms of order  $-3$  in  $r$  in  $\gamma_{ms|s}$ . Terms of this order were not included in the 1941 and earlier discussions.

### 3. THE INTERNAL FIELD

We now have to find potentials which are finite at the origin and which, with their derivatives, are continuous at the boundary  $r = a$ . We impose the further restriction that the co-ordinate conditions (1.1) are satisfied. After some calculation we find that the solutions are

$$\gamma_{44} = \frac{\bar{U}m}{a^3} (3a^2 - r^2) + \bar{U}_{|s} \frac{mx^s}{2a^3} (9a^2 - 3r^2), \quad (3.1)$$

$$\begin{aligned} \gamma_{mn} &= \frac{\bar{U}m}{a^5} (-5a^2 + 3r^2) x^m x^n + \frac{\bar{U}m}{a^5} \left( -\frac{15a^4}{2} + 10a^2 r^2 - \frac{9}{2} r^4 \right) \delta_{mn} \\ &\quad + \bar{U}_{|s} \frac{mx^s x^m x^n}{2a^5} (-5a^2 + 3r^2) + \bar{U}_{|s} \frac{mx^s}{a^5} \left( -\frac{1}{2} a^4 + \frac{7}{10} a^2 r^2 - 3r^4 \right) \delta_{mn} \\ &\quad + \frac{m}{a^5} (\bar{U}_{|m} x^n + \bar{U}_{|n} x^m) \left( \frac{7}{8} a^4 - \frac{3}{2} a^2 r^2 + \frac{3}{8} r^4 \right). \end{aligned} \quad (3.2)$$

From (3.2) we have

$$\gamma_{ms|s} = \frac{m\bar{U}}{a^3} |m| (-3a^2 + r^2). \quad (3.3)$$

Also

$$\gamma_{44|ss} = -\frac{6m\bar{U}}{a^3} - \frac{15m}{a^3} \bar{U}_{|s} x^s, \quad (3.4)$$

and

$$\begin{aligned}\gamma_{mn|ss} = & \frac{\bar{U}m}{a^5} (50a^2\delta_{mn} - 84r^2\delta_{mn} + 42x^mx^n) \\ & + \frac{m}{a^5} \bar{U}_{|s} x^s \{ (68a^2 - 84r^2)\delta_{mn} + 27x^mx^n \} \\ & + \frac{m}{2a^5} (x^m \bar{U}_{|n} + x^n \bar{U}_{|m}) (-31a^2 + 27r^2). \end{aligned} \quad (3.5)$$

The contributions from the quadratic terms in (1.13) and (1.14) are respectively

$$\frac{36m\bar{U}}{a^3} + \frac{42m}{a^3} \bar{U}_{|s} x^s \quad (3.6)$$

and

$$-\frac{8m}{a^3} \bar{U}\delta_{mn} - \frac{14m}{a^3} \bar{U}_{|s} x^s \delta_{mn} + \frac{2m}{a^3} (\bar{U}_{|n} x^m + \bar{U}_{|m} x^n). \quad (3.7)$$

From (3.4)–(3.7) we obtain

$$T_{44} = \frac{3m}{4\pi a^3} \left( \frac{5}{2} \bar{U} + \frac{9}{4} x^s \bar{U}_{|s} \right) \quad (3.8)$$

and

$$\begin{aligned}T_{mn} = & \frac{\bar{U}m}{8\pi a^5} \{ (21a^2 - 42r^2)\delta_{mn} + 21x^mx^n \} \\ & + \frac{27m}{16\pi a^5} \bar{U}_{|s} x^s \{ (2a^2 - 3r^2)\delta_{mn} + x^mx^n \} \\ & + \frac{27m}{32\pi a^5} (x^m \bar{U}_{|n} + x^n \bar{U}_{|m}) (r^2 - a^2). \end{aligned} \quad (3.9)$$

#### 4. TERMS INVOLVING VELOCITY AND ACCELERATION

We next turn our attention to the linear terms involving the velocity and acceleration. The external field is given by

$$\begin{aligned}\gamma_{44} = & -\frac{4m}{r} - 2m\ddot{r} - \frac{2mv^sv^s}{r} \\ = & -\frac{4m}{r} + \frac{2mx^s\dot{v}^s}{r} - \frac{4mv^sv^s}{r} + \frac{2mx^sx^r v^s v^r}{r^3}, \end{aligned} \quad (4.1)$$

$$\gamma_{4n} = \frac{4mv^n}{r}, \quad (4.2)$$

$$\gamma_{mn} = -\frac{4mv^m v^n}{r}, \quad (4.3)$$

where a dot denotes differentiation with respect to the time. The potentials (4.1)–(4.3) satisfy

$$\gamma_{4s|s} - \gamma_{44|4} = 0 \quad (4.4)$$

and

$$\gamma_{4m|4} - \gamma_{ms|s} = \frac{4m\dot{v}^n}{r} \quad (4.5)$$

if terms of order greater than 2 in  $m$  are neglected. As in the previous investigations we add (2.6) to (4.5) and obtain the equation of motion

$$\ddot{v}^m + \frac{1}{2} \bar{U}_{|m} = 0. \quad (4.6)$$

The corresponding internal field, satisfying the boundary conditions at  $r=a$ , is

$$\gamma_{44} = -\frac{2m}{a^3}(3a^2 - r^2) + \frac{m\dot{v}^s x^s}{a^3}(3a^2 - r^2) + \frac{2mv^2}{a^3}(-3a^2 + r^2) + \frac{mx^s x^r}{a^5}(5a^2 - 3r^2)v^s v^r, \quad (4.7)$$

$$\gamma_{4n} = \frac{2m}{a^3}(3a^2 - r^2)v^n, \quad (4.8)$$

$$\gamma_{mn} = -\frac{2m}{a^3}(3a^2 - r^2)v^m v^n. \quad (4.9)$$

The expressions (4.7)–(4.9) satisfy (4.4) and

$$\gamma_{4m|4} - \gamma_{ms|s} = \frac{2m}{a^3}(3a^2 - r^2)\dot{v}^m. \quad (4.10)$$

Hence, by (3.3), we have on including the interaction terms,

$$\gamma_{4m|4} - \gamma_{ms|s} = \frac{2m}{a^3}(3a^2 - r^2)(\dot{v}^m + \frac{1}{2}\ddot{U}_{|m}). \quad (4.11)$$

The expression on the right-hand side of (4.11) vanishes if (4.6) is satisfied.

To the order considered, the energy-tensor is given by

$$16\pi T_{\mu\nu} = \gamma_{\mu\nu|ss} - \gamma_{\mu\nu|44}. \quad (4.12)$$

The external field given by (4.1)–(4.3) gives a zero energy-tensor. The internal solution gives

$$T_{44} = \frac{3m}{4\pi a^3} \left( 1 - \frac{1}{2}\dot{v}^s x^s + \frac{3}{2}v^2 - \frac{1}{2}\frac{r^2 v^2}{a^2} - \frac{7x^s x^r v^s v^r}{2a^2} \right) \quad (4.13)$$

$$= \frac{3m}{4\pi a^3} \left( 1 + \frac{1}{4}\ddot{U}_{|s} x^s + \frac{3}{2}v^2 - \frac{1}{2}\frac{r^2 v^2}{a^2} - \frac{7}{2}\frac{x^s x^r v^s v^r}{a^2} \right), \quad (4.13a)$$

$$T_{4n} = -\frac{3m}{4\pi a^3}v^n, \quad (4.14)$$

$$T_{mn} = \frac{3m}{4\pi a^3}v^m v^n, \quad (4.15)$$

where  $v^2 = v^s v^s$  and (4.13a) is derived from (4.13) by means of (4.6).

## 5. THE COMPLETE ENERGY-TENSOR

The complete energy-tensor is obtained by adding (3.8) and (3.9) to (4.13a) and (4.15) respectively. The result is

$$T_{44} = \rho \left( 1 + \frac{3}{2}v^2 - \frac{1}{2}\frac{r^2 v^2}{a^2} - \frac{7x^s x^r v^s v^r}{2a^2} + \frac{5}{2}\ddot{U} + \frac{5}{2}x^s \ddot{U}_{|s} \right), \quad (5.1)$$

$$T_{4n} = -\rho v^n, \quad (5.2)$$

$$T_{mn} = \rho v^m v^n + p_{mn}, \quad (5.3)$$

where

$$\rho = \frac{3m}{4\pi a^3} \quad (5.4)$$

and  $p_{mn}$  is given by the expression on the right-hand side of (3.9). Now, at the boundary,

$$p_{mn} = \frac{21\ddot{U}m}{8\pi a^5}(-a^2\delta_{mn} + x^m x^n) + \frac{27m}{16\pi a^5}(-a^2\delta_{mn} + x^m x^n)x^s \ddot{U}_{|s}, \quad (5.5)$$

so that

$$\begin{aligned} p_{ms}x^s &= \frac{21\bar{U}mx^m}{8\pi a^3}(-1+1) + \frac{27m\bar{U}}{16\pi a^3}(-1+1)x^s\bar{U}_{|s} \\ &= 0. \end{aligned} \quad (5.6)$$

Since the direction cosines of the normal to the surface of the body are  $x^s/a$ , the equation (5.6) expresses the condition that the normal components of the stresses vanish at the boundary.

Again, we have

$$p_{11} = \frac{21\bar{U}m}{8\pi a^5}(3a^2 - 5r^2) + \frac{27m\bar{U}_{|s}x^s}{16\pi a^5}(5a^2 - 7r^2). \quad (5.7)$$

Integrating through the body we obtain the result

$$\int p_{11}dV = 0. \quad (5.8)$$

In the 1938 investigation the equation (3.3), from which the second-order terms in  $h_{44}$  were derived, was assumed (using the present notation) to be of the form

$$h_{44|ss} = 8\pi(\rho_0 + 2\rho_0v^2) + \frac{1}{2}\gamma'_{44|44} + \frac{1}{4}\gamma'_{44|s}\gamma'_{44|s}, \quad (5.9)$$

where  $\rho_0$  is the invariant density. The present investigation shows that the stress terms  $p_{mn}$  should have been considered, and (5.9) should, in fact, have been written

$$h_{44|ss} = 8\pi(\rho_0 + 2\rho_0v^2 + 2p_{11}) + \frac{1}{2}\gamma'_{44|44} + \frac{1}{4}\gamma'_{44|s}\gamma'_{44|s}. \quad (5.10)$$

The correctness of the amended de Sitter result in the 1938 paper is entirely due to the equation (5.8).

The proper density is given by

$$T = \rho \left( 1 + \frac{1}{2}v^2 - \frac{r^2v^2}{2a^2} - \frac{7x^sx^rv^sv^r}{2a^2} \right) + \frac{3}{2}\rho(\bar{U} + x^s\bar{U}_{|s}) - p_{11} \quad (5.11)$$

$$= \rho \left( 1 + \frac{1}{2}v^2 - \frac{r^2v^2}{2a^2} - \frac{7x^sx^rv^sv^r}{2a^2} \right) + \frac{3}{2}\rho U - p_{11}, \quad (5.11a)$$

on using (1.10) and neglecting  $\bar{U}_{|sr}$  and higher derivatives. Also, to the order considered,

$$\frac{dt}{ds}\sqrt{(-g)} = (1 - \frac{3}{2}U + \frac{1}{2}v^2). \quad (5.12)$$

Accordingly, we have

$$T \frac{dt}{ds}\sqrt{(-g)} = \rho \left( 1 + v^2 - \frac{r^2v^2}{2a^2} - \frac{7x^sx^rv^sv^r}{2a^2} \right) - p_{11}. \quad (5.13)$$

The invariance of the quantity

$$\iiint T \frac{dt}{ds}\sqrt{(-g)} dx dy dz$$

is known from general theory, but it is instructive to deduce this result directly from (5.13); we have

$$\begin{aligned} \iiint T \frac{dt}{ds}\sqrt{(-g)} dx dy dz &= m \{ 1 + v^2(1 - \frac{3}{10} - \frac{7}{10}) \} \\ &= m \end{aligned} \quad (5.14)$$

on using (5.4) and (5.8).

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XLVI.—The Mechanics of Continuous Matter in the Relativity Theory. By  
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## I. INTRODUCTION

THE extension of the classical theory of elasticity to relativity mechanics presents many problems of great complexity. The chief difficulty arises from the fact that whereas in the classical theory the stress-strain relations are valid only in the case of small strain, in a complete relativistic treatment the squares and cubes of the components of strain must necessarily be retained. Although the general formulation of the laws of elasticity is still unsolved, as far back as 1917 Lorentz (1) published a theory applicable to the case of small strain, and as an illustration considered the problem of an incompressible homogeneous disc rotating with a small angular velocity about its axis. He claimed that the radius measured by an observer at rest on the disc is reduced from its original value  $a$  by an amount  $\frac{1}{2}\omega^2 a^3$ . This result has never been universally accepted but, on the contrary, many irrelevant and unjustified criticisms of Lorentz' argument have been expressed. Some of these have been collected together and reproduced, with references, in a *Memoir* by Seyuan Shu (2). The author of the present paper takes the view that Lorentz' theory is in the main correct, but points out that both Lorentz and his critics have overlooked the postulate that the velocity of propagation of the dilatation cannot exceed the velocity of light. On working out the consequence of this postulate it is found that there is no contraction of the radius. For material in which the waves of dilatation travel with the fundamental velocity there is no alteration in the radius of the disc.

In the second half of the paper (§ 5) the equation of equilibrium of a continuous static distribution of matter is discussed in the case in which terms involving the cube but not the fourth power of the density are retained.

## 2. THE PROBLEM OF A ROTATING DISC OR CYLINDER: THE CLASSICAL THEORY

In textbooks on elasticity the following equations and expressions are derived and explained in some detail for isotropic bodies. When the body forces may be neglected, the equations of equilibrium are, using rectangular cartesian co-ordinates,\*

$$\frac{\partial p_{ij}}{\partial x_j} = \rho f_i, \quad (2.1)$$

where  $\rho$  is the density,  $f_i$  are the components of the acceleration, and the components of stress  $p_{ij}$  are given by

$$p_{ij} = \lambda \Delta \delta_{ij} + 2\mu e_{ij}, \quad (2.2)$$

where the strains  $e_{ij}$  are related to the components  $u_i$  of the displacement by the expressions

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \Delta = e_{ii}, \quad (2.3)$$

and  $\lambda$ ,  $\mu$  are constants for the body. Further, it is shown that when the material is homogeneous the dilatation  $\Delta$  is propagated according to the equation

$$(\lambda + 2\mu) \nabla^2 \Delta = \rho \frac{\partial^2 \Delta}{\partial t^2}.$$

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\* Throughout the paper italic indices run over the values 1, 2, 3 only.

The velocity ( $c_0$ ) of propagation of  $\Delta$  is accordingly

$$c_0 = \sqrt{\left\{ \frac{\lambda + 2\mu}{\rho} \right\}}. \quad (2.4)$$

For incompressible material  $\lambda \gg \mu$ , and in this case (2.4) reduces to

$$c_0 = \sqrt{(\lambda/\rho)}. \quad (2.5)$$

The equations (2.1)–(2.3) can readily be expressed in alternative forms when other co-ordinate systems are used. It is, in particular, convenient to use cylindrical polars ( $r, \theta, z$ ) when the displacement is symmetrical about the  $x_3 = z$ -axis. If, in addition, there is no displacement in the direction of the  $z$ -axis, the equations (2.1)–(2.3) take the forms

$$(\lambda + 2\mu) \frac{d}{dr} \left( \frac{dU}{dr} + \frac{U}{r} \right) = \rho f_r, \quad (2.6)$$

where  $f_r$  is the radial acceleration, and the only non-zero components of strain are

$$e_{rr} = \frac{dU}{dr}, \quad e_{\theta\theta} = \frac{U}{r}, \quad (2.7)$$

where  $U$  is a function of  $r$  only. The only non-vanishing components of stress are given by

$$\begin{aligned} p_{rr} &= \lambda \left( \frac{dU}{dr} + \frac{U}{r} \right) + 2\mu \frac{dU}{dr}, \\ p_{\theta\theta} &= \lambda \left( \frac{dU}{dr} + \frac{U}{r} \right) + 2\mu \frac{U}{r}, \\ p_{zz} &= \lambda \left( \frac{dU}{dr} + \frac{U}{r} \right). \end{aligned}$$

When the system under consideration consists of a homogeneous cylinder of radius  $a$  rotating with angular velocity  $\omega$  about its axis, the solution is found to be [*e.g.* Love (3)]

$$U = \frac{\rho \omega^2 r}{8(\lambda + 2\mu)} \left( \frac{2\lambda + 3\mu}{\lambda + \mu} a^2 - r^2 \right).$$

For an incompressible body we take  $\lambda \gg \mu$ , and so the displacement and stresses are given respectively by

$$U = \frac{\rho \omega^2 r}{8\lambda} (2a^2 - r^2) = \frac{\omega^2 r}{8c_0^2} (2a^2 - r^2), \quad (2.8)$$

and

$$p_{rr} = p_{\theta\theta} = p_{zz} = \frac{1}{2} \rho \omega^2 (a^2 - r^2)$$

on using (2.5). The solution is applicable to the greater part of the cylinder but is defective near its ends.

### 3. THE RELATIVITY MODIFICATION OF LORENTZ

We now give a brief account of the argument given by Lorentz. Denoting the velocity of light by  $c$  and the time co-ordinate  $x_4$  by  $t$ , the metric for an observer  $S$  on the cylinder when it is stationary is

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + c^2 dt^2. \quad (3.1)$$

Applying the transformation

$$\left. \begin{aligned} x_1 &= x_1' \cos \omega t' - x_2' \sin \omega t', & x_3 &= x_3', \\ x_2 &= x_1' \sin \omega t' + x_2' \cos \omega t', & t &= t', \end{aligned} \right\} \quad (3.2)$$

the metric for an observer  $S'$  at rest on the rotating system is

$$ds^2 = -dx_1'^2 - dx_2'^2 - dx_3'^2 + 2\omega x_2' dx_1' dt' - 2\omega x_1' dx_2' dt' + \{c^2 - \omega^2(x_1'^2 + x_2'^2)\} dt'^2. \quad (3.3)$$

Now for the metric

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu, \quad (\mu, \nu = 1, 2, 3, 4)$$

the invariant spatial interval  $dl$  is given by

$$-dl^2 = \left( g_{mn} - \frac{g_{4m}g_{4n}}{g_{44}} \right) dx_m dx_n, \quad (m, n = 1, 2, 3). \quad (3.4)$$

Hence the spatial intervals for  $S$  and  $S'$  are respectively

$$\begin{aligned} dl^2 &= dx_1^2 + dx_2^2 + dx_3^2, \\ dl'^2 &= \left( 1 + \frac{\omega^2 x_2'^2}{c^2 - r^2 \omega^2} \right) dx_1'^2 - \frac{2\omega^2 x_1' x_2'}{c^2 - r^2 \omega^2} dx_1' dx_2' + \left( 1 + \frac{\omega^2 x_1'^2}{c^2 - r^2 \omega^2} \right) dx_2'^2, \end{aligned} \quad (3.5)$$

where

$$r^2 = x_1^2 + x_2^2 = x_1'^2 + x_2'^2.$$

In cylindrical polar co-ordinates (3.5) take the forms

$$\left. \begin{aligned} dl^2 &= dr^2 + r^2 d\theta^2 + dz^2, \\ dl'^2 &= dr^2 + \frac{r^2 d\theta^2}{1 - r^2 \omega^2 / c^2} + dz^2. \end{aligned} \right\} \quad (3.6)$$

We accordingly deduce that the length of a standard measuring rod pointing in the radial direction is unaltered by the rotation, but a factor  $1/\sqrt{(1 - r^2 \omega^2 / c^2)}$  has to be applied when the rod is at right angles to the radius. When  $r\omega/c$  is small, this results in  $S'$  observing a strain  $\epsilon'_{\theta\theta} = r^2 \omega^2 / 2c^2$  which has to be inserted in (2.7). Taking  $\lambda$  to be infinite, the dilatation  $\Delta$  is unaltered if we have an additional displacement  $U'$ , corresponding to  $\epsilon'_{\theta\theta}$ , where

$$\frac{dU'}{dr} + \frac{U'}{r} + \frac{r^2 \omega^2}{2c^2} = 0.$$

The solution of this equation is

$$U' = -r^3 \omega^2 / 8c^2. \quad (3.7)$$

Adding (2.8) and (3.7), we find that the displacement has the value

$$U = \frac{\omega^2 r}{8} \left\{ \frac{1}{c_0^2} (2a^2 - r^2) - \frac{r^2}{c^2} \right\}. \quad (3.8)$$

The density in the strained state can be calculated, since the number of particles in a ring of radius  $r$  and width  $dr$  in the unstrained state is the same as the number in a ring of radius  $r + U$  and width  $dr(1 + dU/dr)$  in the strained state. Remembering to insert the Lorentz factor  $1/\sqrt{(1 - r^2 \omega^2 / c^2)}$ , the density  $\rho'$  of the rotating cylinder is found to be

$$\begin{aligned} \rho' &= \rho \left\{ 1 + \frac{\omega^2}{2c_0^2} (r^2 - a^2) \right\} \\ &= \rho - \rho / c_0^2. \end{aligned} \quad (3.9)$$

The change in the radius of the cylinder is determined by putting  $r = a$  in (3.8); this gives

$$U = \frac{\omega^2 a^3}{8} \left( \frac{1}{c_0^2} - \frac{1}{c^2} \right). \quad (3.10)$$

By taking  $c_0$  to be infinite, Lorentz obtains the result

$$U = -\frac{\omega^2 a^3}{8c^2}. \quad (3.11)$$

The contraction (3.11) was accepted by Eddington (4), who gave an alternative method of attacking the problem. According to him, the particle density (referred to proper measure) is unaltered by rotation in the case of an *incompressible* disc or cylinder. That is, Eddington

asserts that  $\rho' = \rho$ ; and consequently he is, in effect, assuming like Lorentz that  $c_0$  is infinite [see equ. (3.9)].

Now, from (3.6), the proper element of volume for the rotating system is

$$dV = (1 - r^2\omega^2/c^2)^{-\frac{1}{2}} r dr d\theta dz, \quad (3.12)$$

and the total number of particles in a disc of thickness  $b$  is accordingly

$$\int \rho' dV = 2\pi b \int_0^{a'} \rho' (1 - r^2\omega^2/c^2)^{-\frac{1}{2}} r dr,$$

where  $a'$  is the radius of the rotating disc and  $\rho'$  is given by (3.9). Since this number must be unaltered by the rotation,  $a'$  must be a function of  $\omega$  such that

$$\int_0^{a'} \rho' (1 - r^2\omega^2/c^2)^{-\frac{1}{2}} r dr = \text{constant}.$$

Expanding the square root and neglecting  $a^4\omega^4/c^4$ , we find

$$a' = a \left\{ 1 + \frac{a^2\omega^2}{8} \left( \frac{1}{c_0^2} - \frac{1}{c^2} \right) \right\},$$

in agreement with (3.10).

By assuming that  $c_0$  is infinite, Eddington naturally obtains the Lorentz result

$$a' = a \left( 1 - \frac{a^2\omega^2}{8c^2} \right).$$

It has already been pointed out, however, that the greatest possible value of  $c_0$  is  $c$ . In this case (3.10) gives  $U=0$  at the boundary and there is no alteration in the radius. If  $c_0 < c$ , there is necessarily an expansion.

#### 4. FURTHER REMARKS ON THE LORENTZ-EDDINGTON THEORY

In the previous section we have made no attempt to deal with the criticisms which have been brought against the arguments of Lorentz and Eddington. As we mentioned in the Introduction, many of these criticisms are irrelevant and need not be discussed. In this section, therefore, we shall only comment briefly on the two most interesting points raised by the critics.

In his review of the problem Seyuan-Shu (ref. (2), p. 68), without giving any reason, rejects the use of the transformation (3.2). In my view, the justification for this transformation is that it does transform the mass motion part of the energy-tensor in the required manner. Retaining only the first power of the density, the non-vanishing contravariant components of the energy-tensor representing a system under isotropic stress rotating with constant angular velocity  $\omega$  about the  $x_3$ -axis are

$$\begin{aligned} T^{11} &= T^{22} = \beta^2 \rho_{000} \omega^2 \frac{x_2^2}{c^2} - p, & T^{12} &= -\beta^2 \rho_{000} \omega^2 \frac{x_1 x_2}{c^2}, \\ T^{33} &= -p, & T^{41} &= -\beta^2 \rho_{000} \omega \frac{x_2}{c}, \\ T^{44} &= \beta^2 \rho_{000} - \frac{p^2}{c^2}, & T^{42} &= \beta^2 \rho_{000} \omega \frac{x_1}{c}, \end{aligned}$$

where  $1/\beta^2 = 1 - r^2\omega^2/c^2$ ,  $r^2 = x_1^2 + x_2^2$ ,  $\rho_{000}$  and  $p$  are invariants, and the leading terms in the metric are given by (3.1).

On applying the transformation (3.2), the new values of the contravariant components are

$$T'^{mn} = g'^{mn} p, \quad T'^{4n} = g'^{4n} p, \quad T'^{44} = \beta^2 \rho_{000} + g'^{44} p.$$

That is, the energy-tensor is calculated with respect to an observer at rest on the rotating system.

A further reason for retaining the transformation (3.2) is because it explains Michelson's 1925 experiment (5). In cylindrical polar co-ordinates (3.3) takes the form

$$ds^2 = -dr^2 - r^2 d\theta^2 - dz^2 - 2\omega r^2 d\theta dt + (c^2 - r^2 \omega^2) dt^2,$$

and so the velocity of light in a direction perpendicular to the radius is

$$V = c \mp r\omega = c \mp v, \quad (4.1)$$

where  $v$  is the velocity of an observer at rest on the rotating system, and the negative sign is to be taken if the ray is travelling in the same sense as the rotation. In consequence of (4.1), if it were possible to transmit two rays of light in opposite directions round the earth, parallel to the equator, they would return to the starting-point at different times. In 1904 Michelson (6) showed that it is not necessary that the track should completely encircle the earth and calculated the difference in path between two light rays, one of which travels in a clockwise direction and the other in an anticlockwise direction. The experiment which he carried out twenty years later verified the validity of (4.1).

In his discussion on the problem, Eddington obtains the proper element,  $dV$ , of volume by a different method from the one used by Lorentz and reproduced in § 3. He shows that  $dV$  can be expressed in the form

$$dV = \sqrt{(-g)} \frac{dt}{ds} dx_1 dx_2 dx_3, \quad (4.2)$$

and this reduces to

$$dV = (1 - r^2 \omega^2 / c^2)^{-\frac{1}{2}} r dr d\theta dz$$

for the metric (3.3). Although this expression is entirely in agreement with (3.6), Berenda (7) has, for some inexplicable reason, criticised Eddington's treatment and has merely reproduced the Lorentz formulæ (3.4) and (3.6). To remove any misunderstanding on this point, we observe that it is easy to see that (4.2) is only an alternative way of expressing (3.4). We find, by using the elementary theory of determinants, that the determinant  $g$  is equal to

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{12} & g_{22} & g_{23} & g_{24} \\ g_{13} & g_{23} & g_{33} & g_{34} \\ g_{14} & g_{24} & g_{34} & g_{44} \end{vmatrix} = g_{44} K, \quad (4.3)$$

where

$$K = ||k_{mn}||, \quad k_{mn} = g_{mn} - g_{4m} g_{4n} / g_{44}.$$

On account of (4.3), (4.2) may be written

$$dV = \sqrt{g_{44}} \sqrt{(-K)} \frac{dt}{ds} dx_1 dx_2 dx_3. \quad (4.4)$$

Now for an observer at rest

$$\left( \frac{ds}{dt} \right)^2 = g_{44},$$

and so (4.4) reduces to

$$dV = \sqrt{(-K)} dx_1 dx_2 dx_3;$$

this is the volume element for the spatial interval (3.4).

This spatial interval can also be calculated by referring an element of volume at a given point to "local" co-ordinates; that is, by considering the two-dimensional form of the Lorentz transformation.

The three-dimensional form of the transformation is obtained by showing that  $x_r x_r - c^2 t^2$  is invariant under the transformation.

$$\left. \begin{aligned} x_r' &= \beta (a_{rs} x_s - v_r t / c), \\ t' &= \beta (-v_r x_r / c + t), \end{aligned} \right\} \quad (4.5)$$

where

$$a_{rs} = a_{sr}, \quad 1/\beta^2 = 1 - V^2/c^2, \quad V^2 = v_i v_i,$$

provided that

$$\alpha_{is}\alpha_{js} = \delta_{ij}/\beta^2 + v_i v_j/c^2, \quad v_s \alpha_{js} = v_j, \quad (4.6)$$

and the  $v_i$  are treated as constants. These conditions are satisfied by taking

$$\alpha_{ij} = (1 - \mu V^2) \delta_{ij} + \mu v_i v_j/c^2, \quad \mu V^2/c^2 = 1 - 1/\beta.$$

Now when  $dt=0$ , (4.5) gives

$$dx_r' = \beta \alpha_{rs} dx_s.$$

Consequently

$$\begin{aligned} dx_1'^2 + dx_2'^2 + dx_3'^2 &= \beta^2 \alpha_{mi} \alpha_{ni} dx_m dx_n \\ &= (\delta_{mn} + \beta^2 v_m v_n/c^2) dx_m dx_n \end{aligned} \quad (4.7)$$

on using (4.6).

For rotation about the  $x_3$ -axis, the components of velocity at the element considered are

$$v_1 = -\omega x_2, \quad v_2 = \omega x_1, \quad v_3 = 0.$$

With these values of  $v_i$ , (4.7) reduces to (3.4).

## 5. THE EQUATION OF EQUILIBRIUM OF A CONTINUOUS STATIC DISTRIBUTION OF MATTER

In the classical theory of fluids and elastic bodies the equations of equilibrium are

$$\frac{\partial p_{ms}}{\partial x_s} = -\rho X_m, \quad (5.1)$$

where  $p_{ms}$  are the components of stress,  $\rho$  the density, and  $X_m$  the gravitational attraction at the element considered. We shall now, in this final section, proceed to verify that, in the case of weak static fields for which the fourth power of the density may be neglected, the equation (5.1) is taken over unaltered in relativity mechanics if  $p_{ms}$  denotes covariant components of the stress and

$$\rho = (T_4^4 - p_i^i) \sqrt{(-g)}, \quad (5.2)$$

$$X_m = \{44, m\} g^{44} \sqrt{(-g)}, \quad (5.3)$$

where  $\{44, m\}$  is a Christoffel bracket and the components  $g_{4n}$  of the potentials  $g_{\mu\nu}$  are zero. The expressions (5.2), (5.3) are those occurring in Whittaker's (8) extension of Gauss' Theorem for a static system. This can be written in the form

$$4\pi \iiint \rho dx_1 dx_2 dx_3 = \iiint \frac{\partial X_s}{\partial x_s} dx_1 dx_2 dx_3,$$

or

$$4\pi \iiint (T_4^4 - p_i^i) \sqrt{(-g)} dx_1 dx_2 dx_3 = \iiint \frac{\partial}{\partial x_s} (\sqrt{(-g)} g^{44} \{44, s\}) dx_1 dx_2 dx_3.$$

From (5.2) and (5.3) we have

$$\rho X_m = (T_4^4 - p_i^i) (-g) \{44, m\} g^{44}. \quad (5.4)$$

If we denote the Newtonian potential by  $-\frac{1}{2}\phi$ , the non-vanishing components of  $g_{\mu\nu}$ ,  $g^{\mu\nu}$  are

$$g_{mn} = -\delta_{mn}(1 - \phi), \quad g^{mn} = -\delta_{mn}(1 + \phi), \quad g_{44} = 1 + \phi, \quad g^{44} = 1 - \phi,$$

and consequently

$$(-g) g^{44} \{44, m\} = \frac{1}{2}(1 - 2\phi) \frac{\partial g_{44}}{\partial x_m}, \quad (5.5)$$

correct to the second order in the density. From (5.4) and (5.5) we then obtain the relation

$$\rho X_m = \frac{1}{2}(T_4^4 - p_i^i)(1 - 2\phi) \frac{\partial g_{44}}{\partial x_m}, \quad (5.6)$$

correct to the third order in the density.

Now, in the case of strong static fields, the equations of equilibrium have the form

$$(T_m^s)_s = 0, \quad (5.7)$$

where ( ) denotes covariant differentiation. Written out in full, (5.7) becomes

$$\frac{\partial T_m^s}{\partial x_s} = -\{\alpha\beta, \beta\}T_m^\alpha + \{m\beta, \alpha\}T_{\alpha\beta}^{\beta} \quad (5.8)$$

where the Greek letters run over the values 1, 2, 3, 4. Writing  $T_m^s = p_m^s$ ,  $T_m^4 = 0$ , the equation (5.8) becomes

$$\frac{\partial p_m^s}{\partial x_s} = -\{rs, s\}p_m^r + \{ms, r\}p_r^s - \{r4, 4\}p_m^r + \{m4, 4\}T_4^4, \quad (5.9)$$

and, as we have already pointed out, the italic letters take the values 1, 2, 3 only.

Since the stresses are of the order of the square of the density in the case under discussion, we need only evaluate the first-order terms in the first three Christoffel brackets on the right-hand side of (5.9). Now,

$$p_m^s = g^{sr}p_{rm} = -(1+\phi)p_{ms}, \quad (5.10)$$

correct to the second order, and accordingly

$$\begin{aligned} -\{rs, s\}p_m^r &= \frac{1}{2}p_{mr}g^{st}\left(\frac{\partial g_{rt}}{\partial x_s} + \frac{\partial g_{st}}{\partial x_r} - \frac{\partial g_{rs}}{\partial x_t}\right) \\ &= \frac{1}{2}p_{mr}g^{st}\frac{\partial g_{st}}{\partial x_r} = -\frac{3}{2}p_{mr}\frac{\partial \phi}{\partial x_r}, \\ \{ms, r\}p_r^s &= -\frac{1}{2}p_{rs}g^{rt}\left(\frac{\partial g_{mt}}{\partial x_s} + \frac{\partial g_{st}}{\partial x_m} - \frac{\partial g_{ms}}{\partial x_t}\right) \\ &= \frac{1}{2}p_{rs}\left(\frac{\partial g_{mr}}{\partial x_s} + \frac{\partial g_{rs}}{\partial x_m} - \frac{\partial g_{ms}}{\partial x_r}\right) \\ &= \frac{1}{2}p_{rs}\frac{\partial \phi}{\partial x_m} = \frac{1}{2}p_{rs}\frac{\partial g_{44}}{\partial x_m}, \\ -\{r4, 4\}p_m^r &= \frac{1}{2}p_{mr}g^{44}\frac{\partial g_{44}}{\partial x_r} = \frac{1}{2}p_{mr}\frac{\partial \phi}{\partial x_r}. \end{aligned}$$

Also,

$$\begin{aligned} \{m4, 4\}T_4^4 &= \frac{1}{2}g^{44}\frac{\partial g_{44}}{\partial x_m}T_4^4 \\ &= \frac{1}{2}(1-\phi)\frac{\partial g_{44}}{\partial x_m}T_4^4. \end{aligned}$$

Inserting these values in the right-hand side of (5.9), we have, correct to the order stated,

$$\frac{\partial p_m^s}{\partial x_s} = -p_{mr}\frac{\partial \phi}{\partial x_r} + \frac{1}{2}(1-\phi)\frac{\partial g_{44}}{\partial x_m}(T_4^4 - p_i^i). \quad (5.11)$$

Again, from (5.10) we have

$$\frac{\partial p_m^s}{\partial x_s} = -(1+\phi)\frac{\partial p_{ms}}{\partial x_s} - \frac{\partial \phi}{\partial x_s}p_{ms}. \quad (5.12)$$

Combining (5.11) and (5.12), we then obtain the equation of equilibrium in the form

$$\begin{aligned} \frac{\partial p_{ms}}{\partial x_s} &= -\frac{1}{2}(1-2\phi)\frac{\partial g_{44}}{\partial x_m}(T_4^4 - p_i^i) \\ &= -\rho X_m \end{aligned}$$

on using (5.6).

We must emphasise that the theory only applies to weak fields, and that this form of the equation of equilibrium does not persist in the case of strong fields. This is most easily seen by considering a sphere composed of a perfect fluid, for which  $T_m^n = g_m^n p$ .

# SUMMARY

Little progress has been made in the development of a relativity theory of elasticity, although it has been realised that no disturbance can be propagated with a velocity greater than that of light. In 1917 Lorentz (1) gave a relativistic formulation of the laws of elasticity in the case of small strain and, applying the theory to the problem of a rotating, incompressible, homogeneous disc, he claimed that the radius as measured by an observer at rest on the disc undergoes a contraction. His result was accepted by Eddington (4) but was attacked by others. A great deal has been written on the subject, but it has never been pointed out that both Lorentz and Eddington were considering material in which the waves of dilatation travel with an infinite velocity. In this paper we define "incompressible" matter as that in which these waves are propagated with the velocity of light and Poisson's ratio tends to the value  $\frac{1}{2}$ . This gives an upper limit to the modulus of compression  $k$ , which in this case is the elastic constant  $\lambda$ , and as a result the expansion determined by the ordinary classical theory has to be taken into account. It is found that the "relativity contraction" is exactly cancelled by the "classical expansion". Throughout the discussion on the rotating disc the analysis is restricted to the case of small strain.

The equations of equilibrium of a continuous static distribution of matter are also investigated in the case of weak fields for which the fourth power of the density may be neglected.

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XLVII.—Non-Associative Arithmetics.\* By I. M. H. Etherington, University of Edinburgh. (With Three Text-figures)

(MS. received January 8, 1946. Revised MS. received July 11, 1947. Read July 1, 1946)

I. INTRODUCTION AND SUMMARY

THE systems of "partitive numbers" introduced in this paper differ from ordinary number systems in being subject to non-associative addition. They are intended primarily to serve as the indices of powers in algebraic systems having non-associative multiplication, or as the coefficients of multiples in systems with non-associative addition, but are defined more generally than is probably necessary for these purposes. They are essentially the same as root-trees (Setzbäume)† with non-branching knots other than terminal knots ignored, with operations of addition and multiplication defined.

Partitive numbers are of two kinds, *partitioned cardinals* and *partitioned serials*, defined respectively as the partition-types of repeatedly partitioned classes and series. For each kind, multiplication is binary (*i.e.* any ordered pair has a unique product) and associative. Addition is in general a *free* operation (*i.e.* the summands are not limited to two, and indeed, assuming the multiplicative axiom, may form an infinite class or series); but it is non-associative, which means that for example  $a + b + c$  (involving one operation of addition) is distinguished from  $(a + b) + c$  and  $a + (b + c)$  (involving two operations). A one-sided distributive law is obeyed:

$$a(\Sigma b_i) = \Sigma(ab_i); \quad \text{in general} \quad (\Sigma b_i)a \neq \Sigma(b_i a).$$

Partitioned cardinals are commutative in addition.

Closed subsystems of partitive numbers occur when we suitably restrict the meaning of *class*, *series* or *partitioned*, and correspondingly restrict the freedom of addition. Thus *partitioned ordinals* are got by considering only well-ordered series; *finite* partitive numbers arise from finite classes or series, forming systems closed under "finitely free" addition; and *n*-ary numbers may be considered; e.g. *binary* partitive numbers (corresponding to bifurcating trees), which form a system closed under binary addition, are obtained by taking *partition* to mean dichotomy.

*Altitude* is defined for partitive numbers as by Cayley for trees, and many difficulties are avoided by restricting attention throughout to numbers which may be infinite but are of finite altitude. Some theorems (*e.g.* unique factorization) are proved by "non-associative induction"—essentially an induction on the altitude.

The definitions are given first (§§ 4, 5, 7) in terms of the theory of classes, and are framed so as to emphasize analogies with cardinal, ordinal and serial numbers.‡ In §§ 8, 9 the arithmetics are redefined axiomatically. By adding fresh axioms further arithmetics are derived, appropriate for the indices of powers in algebras with special properties. The elements of all these arithmetics can be interpreted as classes of partitioned serials. (*Added in proof.*—A simpler axiomatic formulation for finite partitive numbers is given in a forthcoming paper by A. Robinson, 1949.)

2. LOGARITHMETIC

Using certain obvious conventions for denoting powers, the *logarithmic* of an algebra with non-associative binary multiplication was defined as the arithmetic of the indices of powers of the general element of the algebra.§ The logarithmic of an associative algebra is,

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† Cayley (1857, *etc.*). These are trees in which one knot is specified as the *root* of the tree.

‡ Also with the arithmetics of partially ordered systems (see Birkhoff, 1937, 1940). Generalization on the lines of Birkhoff (1942) would doubtless be possible.

§ Etherington (1939 a, 1941 b). We can consider also the logarithmic of a particular element or other subset, which will be a homomorphism of that of the general element.

generally, ordinary arithmetic; for a non-associative algebra the logarithmic has generally non-associative addition, and various non-associative arithmetics arise in this way from algebras of special types.

The conventions are:  $x^{a+b}$  means  $x^a x^b$ ,  $x^{ab}$  means  $(x^a)^b$ ,  $x^{a^n}$  means  $\dots (x^a)^a \dots$  with the power iterated  $n$  times. For example,

$$x^{(2+1)^2} = (x^2 x)^2, \quad x^{2(2+1)} = x^2 x^2 \cdot x^2, \quad x^{(2+1)^2} = (x^2 x)^2 (x^2 x).$$

Identities connecting indices  $a, b, c, \dots$  are used to express identities between powers. Thus we write

$$ab \cdot c = a \cdot bc, \quad a(b+c) = ab + ac$$

to indicate that in *any* algebra  $x^{ab \cdot c} = x^{a \cdot bc}$  since both mean  $((x^a)^b)^c$ , and  $x^{a(b+c)} = x^{ab+ac}$  since both mean  $(x^a)^b (x^a)^c$ . If multiplication in the algebra is associative, or merely associative for powers (i.e.  $x^a x^b \cdot x^c = x^a \cdot x^b x^c$ ),

$$(a+b)+c = a+(b+c);$$

if commutative,

$$a+b = b+a;$$

if the algebra is palintropic,\*

$$ab = ba, \quad \text{with consequences } (a+b)c = ac + bc, \quad ab \cdot cd = ac \cdot bd;$$

also for all the palintropic algebras which I have encountered,

$$(a+b)+(c+d) = (a+c)+(b+d).$$

Corresponding generalized non-associative arithmetics arise if we consider more general algebras in which multiplication is not necessarily binary. We shall speak of *n-ary, restricted, finitely free* or *free* algebras according to the nature of multiplication. The definitions to be given for partitive numbers are suggested by considering the logarithmic of a system in which the fundamental operation is free multiplication, with the convention for addition of indices generalized:  $x^{\sum a_i}$  means  $\prod x^{a_i}$ .

### 3. LAWS

We shall have to refer frequently to the following "laws", various sets of which are obeyed in the various number systems. Cardinals obey all except the (U) and cancellation laws; finite cardinals obey the latter and (U<sub>x</sub><sup>0</sup>); serials obey the associative and right distributive laws, and ordinals obey also (sub<sub>L</sub>); finite serials and ordinals are isomorphic with finite cardinals. These facts (assuming the multiplicative axiom where infinite sums or products are involved) are well known.

#### COMMUTATIVE LAWS

$$a+b = b+a, \quad (c_+)$$

$$ab = ba. \quad (c_x)$$

#### ASSOCIATIVE LAWS

$$a+(b+c) = (a+b)+c, \quad (a_+)$$

$$a \cdot bc = a \cdot bc \quad (a_x)$$

#### DISTRIBUTIVE LAWS

$$a(b+c) = ab+ac, \quad (d_R)$$

$$(b+c)a = ba+ca. \quad (d_L)$$

\* I.e.  $(x^a)^b = (x^b)^a$ ; Etherington (1945; cf. 1940, Theorem XIV, and 1941 b); also Murdoch (1939, Theorem 10, Corollary). The palintropic property is a consequence of  $x^{(a+b)+(c+d)} = x^{(a+c)+(b+d)}$  (Etherington, 1945, p. 120; Theorem 4 in the present paper).

To this familiar list we may add a further pair, obvious consequences of the (c) and (a) laws, to which the name *entropic* will be given:

## ENTROPIC LAWS

$$(a+b)+(c+d)=(a+c)+(b+d), \quad (e_+)$$

$$ab \cdot cd = ac \cdot bd. \quad (e_x)$$

The above laws refer to addition and multiplication as binary operations. They will require amplification when the operations are not restricted to being binary, and the corresponding statements will then be called *full laws*: these are to be interpreted as making certain assertions whenever these assertions have meaning, *i.e.*, whenever the sums and products mentioned exist.

## FULL COMMUTATIVE LAWS—FULL ASSOCIATIVE LAWS

*A sum  $\Sigma a_i$  or product  $\Pi a_i$  has the same value in whatever manner the  $a$ 's may be re-ordered—or brackets inserted.*

$$(C_+) (C_x) - (A_+) (A_x)$$

## FULL DISTRIBUTIVE LAWS

$$a(\Sigma b_i) = \Sigma(ab_i), \quad (\Sigma b_i)a = \Sigma(b_ia). \quad (D_R) (D_L)$$

Note that the *right* distributive law ( $d_R$ ) or ( $D_R$ ) asserts that a factor of a product can be distributed on to the summands of a factor on its immediate *right*.

## FULL ENTROPIC LAWS \*

$$\Sigma_i \Sigma_j a_{ij} = \Sigma_j \Sigma_i a_{ij}, \quad \Pi_i \Pi_j a_{ij} = \Pi_j \Pi_i a_{ij}. \quad (E_+) (E_x)$$

LAW OF INDUCTION (Definition of *finite* for cardinal numbers.)

*Any finite number other than 1 can be formed from 1 by repeated additions of 1.* (i)

The principle of "non-associative induction" (§§ 5, 8) may be regarded as the amplification of this.

If a number can be expressed as an unbracketed sum (product) of numbers, the latter are its *summands* (*factors*); a number other than 1 is *prime* if it has no factors other than itself and 1.

## UNIQUE SEPARATION

*Any number other than 1 can be resolved into summands which are uniquely determined in order.*

$$(U_+^S)$$

*... are uniquely determined except as regards order.*

$$(U_+^O)$$

\* These are consequences of the (C) and (A) laws. It may be remarked that in ordinary arithmetic the operations of subtraction and division are non-commutative and non-associative; also division is unlike multiplication as regards the distributive law ( $D_R$ ), though it satisfies a law corresponding to ( $D_L$ ). However, the four operations  $+$ ,  $-$ ,  $\times$ ,  $\div$  all satisfy full entropic laws; *e.g.* it is true that

$$(a \div b) \div (c \div d) = (a \div c) \div (b \div d),$$

$$(a - b - c) - (d - e - f) = (a - d) - (b - e) - (c - f).$$

( $e_x$ ) is important in the theory of quasi-groups (Murdoch, 1939).  $n$ -ary entropic systems have been studied by Toyoda (1941).

## UNIQUE FACTORIZATION

Any number, if not prime or 1, can be resolved into prime factors which are uniquely determined in order.  $(U_x^s)$

. . . are uniquely determined except as regards order.  $(U_x^0)$

## CANCELLATION LAWS

Each of the statements

$$a + b = a + c, \quad b + a = c + a, \quad ab = ac, \quad ba = ca$$

implies  $b = c$ .  $(\text{sub}_L) (\text{sub}_R) (\text{div}_L) (\text{div}_R)$

These can be amplified, e.g.

$\sum a_i = \sum b_i$ , where the two series have equal first terms, implies equality of these sums with the first terms omitted.  $(\text{SUB}_L)$

In all the arithmetics to be considered, multiplication will be a binary operation obeying  $(d_R)$  and  $(a_x)$ . Addition will be a free operation unless otherwise stated; but the results proved for free addition will be valid with suitable modifications when it is restricted in any way, and whenever full laws are referred to it is implied that they are interpreted according to the nature of the operation. Then an arithmetic will be called *commutative* if  $(C_+)$  holds, *associative* if  $(A_+)$ , *entropic* if  $(E_+)$ , *palintropic* if  $(c_x)$ .

## 4. PARTITIVE NUMBERS

The word *class* is used in its usual sense except that the null class will always be excluded, and *subclass* means "non-null subclass". A class is *simply ordered* and will be called a *series* if there is a transitive relation of order between any two distinct elements  $A, B$  of it (i.e. either  $A$  precedes  $B$  or  $B$  precedes  $A$ , not both; if  $A$  precedes  $B$  and  $B$  precedes  $C$ , then  $A$  precedes  $C$ ). It is *well ordered* if it is simply ordered and every subclass has a first element.

A *simple partition of a class* is a separation of the class into proper subclasses. A *simple partition of a series* means the following. The series considered as a class is simply partitioned, the ordering relation is preserved within each subclass, and the partitioning is such that the ordering relation applies also to the subclasses (i.e. is such that if  $P, Q$  are any two distinct subclasses, either all elements of  $P$  precede all elements of  $Q$  and we say  $P$  precedes  $Q$ , or *vice versa*).

Let any class of elements be partitioned in stages as follows. Let it be simply partitioned; let all subclasses which do not consist of single elements be again simply partitioned; and so on, until at a final stage all the subclasses consist of single elements. Such a "partitioned class" will be called a *clan*. If a series is dealt with in the same way, being partitioned perhaps repeatedly and ultimately into single elements, the resulting "partitioned series" or "simply ordered clan" will be called a *school*. The *degree*  $\delta = \delta(s)$  of a clan or school  $s$  is the cardinal number of its elements; the *altitude*  $\alpha = \alpha(s)$  is the ordinal number of partition stages. A clan or school is *finite* if its degree is finite.

The words "and so on" in the preceding paragraph could refer to an infinite series of stages preceding the final one if a rule determining the subclasses at any stage is given. Thus infinite altitudes could be considered; but for simplicity we shall assume throughout that the altitude is finite.

A class consisting of a single element, which is necessarily a series, cannot actually be partitioned, but will nevertheless be called a clan or school of zero altitude and denoted  $\mathbf{1}$ ; thus  $\delta(\mathbf{1}) = \mathbf{1}$ ,  $\alpha(\mathbf{1}) = \mathbf{0}$ .

The word *society* will be ambiguous: by giving it different meanings we shall arrive at different arithmetics. Interpreting it consistently as *class*, the numbers at which we shall arrive are the (assumed already familiar) cardinal numbers, and the interpretation *series* leads to serial numbers (or ordinals if the series is well ordered), with addition and multiplication

correctly defined in each case. The interpretations *clan* and *school* lead respectively to *partitioned cardinals* or *commutative partitive numbers*, and *partitioned serials* or *non-commutative partitive numbers* (*partitioned ordinals* if the school is well ordered). For any interpretation, similarity of societies must be suitably understood. We shall call two societies *similar* when there is a one-one correspondence between them element to element, and subclass to subclass at every stage if they are partitioned, in which all relations of inclusion of elements in subclasses and all relations of order in the one society hold for the corresponding elements and subclasses of the other.

By the word *number* in §§ 5-7 we shall understand a society, and we shall call two numbers *equal* ( $a=b$ ) and think of them as the same number \* if the two societies are similar.† A number is cardinal, ordinal, serial or partitive according to the meaning of *society* as already explained. With any interpretation of *society* it will be necessary to verify that the addition and multiplication which we are going to define are *consistent*, i.e. that if  $a_i=b_i$  for a series of values of the variable  $i$ , then  $\Sigma a_i=\Sigma b_i$  and  $\Pi a_i=\Pi b_i$ .

## 5. ADDITION

*Exclusive* societies are such as have no common element.

DEFINITION.—To form the sum of a given series of numbers  $s_i$  (where the different values of the variable  $i$  form a series). Taking it for granted that we can find exclusive societies with these numbers, combine them into a single society  $s$ , partitioned (if *society* means *clan* or *school*) into the already partitioned classes  $s_i$ , preserving (if *society* means *series* or *school*) the order in which they are given and the order within each society. Then we define  $\Sigma s_i=s$ ; the number  $s$  is the *sum* of the numbers  $s_i$ , which are the *summands* of  $s$ .

The exclusive societies required in the definition may be constructed by taking as the representative of each  $s_i$  the society consisting of all ordered pairs  $(i, S_i)$ , where  $S_i \in s_i$ , ordered and partitioned if necessary so that it is similar to  $s_i$ .

Evidently  $\delta(s)=\Sigma \delta(s_i)$ ; and, if *society* means *clan* or *school*,  $\alpha(s)=1+\max \alpha(s_i)$ .

Addition is a free operation, but can be restricted in any way by suitably restricting the meaning of the word *series* in the definition. Consistency for each interpretation of *society* may be verified: this requires the multiplicative axiom if the number of summands is infinite. Addition is fully associative if *society* means *class* or *series*, fully commutative if it means *class* or *clan*, fully entropic if it means *class*, fully commutative and entropic if *society* and *series* both mean *finite series*.

It follows from the definitions of addition and equality that the summands of a partitioned cardinal [partitioned serial]  $s \neq 1$  are uniquely determined [and their order is unique]; they are the numbers of the societies into which  $s$  is first separated in the process of partitioning, partitioned [and ordered] as in  $s$ ; their altitudes are all less than  $\alpha(s)$ , and at least one of them has altitude  $\alpha(s)-1$ .

Thus partitioned cardinals obey  $(U_+^0)$ , partitioned serials obey  $(U_+^S)$ , and in consequence both obey the full additive cancellation laws  $(SUB_L)$ ,  $(SUB_R)$ .

It also follows from the definitions that a partitive number  $s$  is equal to a bracketed sum of a series of  $\delta(s)$  1's. To obtain this representation, express  $s$  as the sum of its summands, express each summand which  $\neq 1$  as the sum of its summands, and so on. The process terminates after  $\alpha(s)$  such separations. We shall call this law

## NON-ASSOCIATIVE INDUCTION

*Any partitive number other than 1 can be formed entirely from 1's by repeated additions.* (I)

Note that (I) applies to *infinite* partitive numbers (of finite altitude).

Hence to prove a theorem  $\theta(a)$  involving an arbitrary partitive number  $a$ , it is enough to prove that a series of propositions  $\theta(s_i)$  together always imply  $\theta(\Sigma s_i)$ , and that  $\theta(1)$  is true.

\* E.g. when words such as "unique" are used.

† It is convenient to draw no distinction in their definitions between a number and the society to which it applies, but to distinguish them in use by calling two numbers equal where we should call the two societies similar. Alternatively, the number of a society could be defined as the class of all societies similar to it.

## 6. NOTATIONS

A finite clan or school, and the corresponding partitive number, can be represented in various ways:

(1) By inserting brackets in a row of symbols representing the elements. (2) By using dots instead of brackets, more dots between elements indicating prior partition. (3) By trees. The root of a tree stands for the complete class, the other knots for the subclasses, the free knots for the elements. The altitude is the number of knots above the root on the longest branch. (4) By the symbols of ordinary arithmetic, with the abbreviations  $* 2 = (1 + 1)$ ,  $3 = (1 + 1 + 1)$ ,  $4 = (1 + 1 + 1 + 1)$ , etc. for numbers of unit altitude. Until multiplication is introduced (§ 7) this notation scarcely differs from (1).

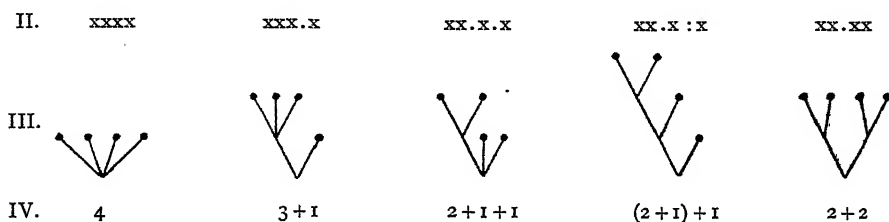


FIG. 1

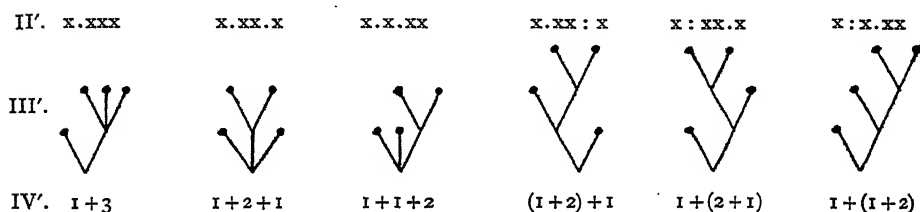


FIG. 2

From a class of four elements we can form five dissimilar clans, of altitudes 1, 2, 2, 3, 2, as shown above in the second, third and fourth notations, labelled II, III, IV. These therefore are the distinct partitioned cardinals of degree 4.

From a series of four elements we can form eleven schools, viz. the five societies already symbolized, now ordered, and the six further types shown in II', III', IV'. Thus there are eleven distinct partitioned serials of degree 4.†

More generally, any partitioned cardinal or serial of altitude 1 may be represented by the symbol used for the corresponding unpartitioned number (if an appropriate symbol exists), and the fourth mode of notation can then be used for infinite partitive numbers.

## 7. MULTIPLICATION

The definitions below give the correct result for cardinals and for serials; but for serials it should be pointed out that some writers denote by  $ba$  what we shall call  $ab$ . The notation used here is conditioned by the application to logarithmic.

DEFINITION.—To form the product of a given pair of societies or numbers  $a$ ,  $b$ , consider

\* For the logarithmic of a binary algebra it is more convenient to define  $3 = 2 + 1$ ,  $4 = (2 + 1) + 1$ ,  $5 = ((2 + 1) + 1) + 1$ , etc., as in my previous papers.

† The problem of enumerating the partitioned serials of given finite degree  $\delta$  is the same as that of enumerating the schools which can be formed from  $\delta$  elements, and was solved by Schröder, 1870, Problem 2; cf. Etherington, 1941 c, Case 5. But the enumeration of partitioned cardinals of given degree (Cayley, 1857, concluding paragraphs) is not the same as the corresponding problem on clans (Schröder, *ibid.*, Problem 4); e.g. from three elements  $A, B, C$  we can form four clans ( $ABC, A.BC, AB.C, AC.B$ ) but only two partitioned cardinals ( $3, 2 + 1$ ).

a society  $p$  similar to the second factor  $b$ , whose elements are societies similar to the first factor  $a$ ; and then regard  $p$  as consisting of the elements of the latter societies. Then we define

$$ab = p;$$

the number  $p$  is the *product* of  $a$  and  $b$ ;  $a, b$  are *left* and *right* factors of  $p$ . Evidently

$$\delta(p) = \delta(a)\delta(b), \quad \alpha(p) = \alpha(a) + \alpha(b).$$

This is illustrated in the following example, where the trees denote either clans or schools:—

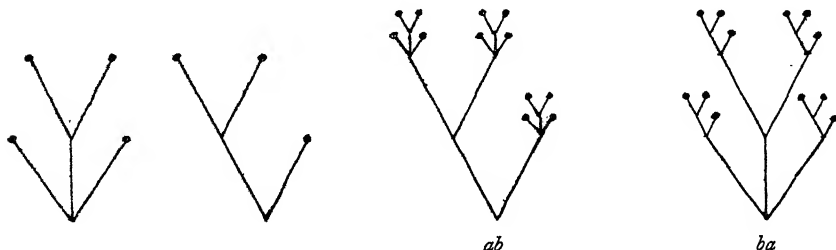


FIG. 3

The following is an equivalent constructive definition.

DEFINITION.—The product  $ba$  (not  $ab$ ) is the society consisting of all ordered pairs  $A_B$  where  $A \in a, B \in b$ ; ordered lexically if  $a$  and  $b$  are ordered; partitioned if  $a$  and  $b$  are partitioned, according to the following rule. Consider subclasses of the class of pairs, all pairs with the same  $A$  going into the same subclass; partition this class of subclasses similarly to the second factor  $a$ ; partition each subclass similarly to the first factor  $b$ .

To illustrate, using the same example as above in a more suitable notation:

$$a = A \cdot BC \cdot D, \quad b = EF \cdot G,$$

$$ab = E_A \cdot E_B E_C \cdot E_D : F_A \cdot F_B F_C \cdot F_D : G_A \cdot G_B G_C \cdot G_D,$$

$$ba = A_E A_F \cdot A_G : B_E B_F \cdot B_G : C_E C_F \cdot C_G : D_E D_F \cdot D_G.$$

The consistency of the definition, for each interpretation of society, is easily proved. Multiplication thus defined is a binary operation, in which 1 is always neutral. It is associative, but is non-commutative and non-entropic (unless society means *class* or *finite series*).

The right distributive law,  $a \sum b_i = \sum (ab_i)$ , may be proved from the definition, but is most easily seen to be true by considering trees. Place the trees representing the  $b_i$  in order; to add them, we have to join their roots to a single new root; to premultiply them by  $a$ , we have to erect the tree for  $a$  at each free knot of each of them. Then  $(D_R)$  merely asserts that the order of these two operations is immaterial. To illustrate the law, using the examples of multiplication already given, we have

$$(1 + 2 + 1)(2 + 1) = \{(1 + 2 + 1) + (1 + 2 + 1)\} + (1 + 2 + 1),$$

$$(2 + 1)(1 + 2 + 1) = (2 + 1) + \{(2 + 1) + (2 + 1)\} + (2 + 1).$$

We shall prove the law of unique factorization ( $U_{\times}^S$ ) for partitive numbers. As a lemma, we need the left cancellation law ( $ab = ac$  implies  $b = c$ ), and we shall prove this by non-associative induction (I). The right cancellation law will then follow from ( $U_{\times}^S$ ).

Considering partitioned cardinals, we shall say that a particular number  $a$  has the *left cancellation property* (l.c.p.) if, for all  $p, pa = pq$  implies  $a = q$ . Let  $s = \sum s_i$  be a number whose summands  $s_i$  all have the l.c.p. Then  $ps = pq$  implies  $a(q) = a(s) \neq 0$ , implies  $q \neq 1$ , and therefore implies  $p \sum s_i = p \sum q_j$ , where  $q_j$  are the summands of  $q$ . Applying in succession  $(D_R)$ ,  $(U_+^0)$  and the l.c.p. hypotheses, this implies that the series  $s_i$  is a derangement of the series  $q_j$  [for partitioned serials, obeying  $(U_+^S)$ , we should conclude further that  $s_i = q_i$ ], and hence implies  $s = q$ . Thus  $s$  has the l.c.p. if its summands all have this property. Now 1 has the l.c.p., and

hence by (I) all partitioned cardinals [and similarly all partitioned serials] have the l.c.p. This proves the l.c. law for partitive numbers.

It follows from  $(U_+)$  and  $(D_R)$  that a partitive number has a left factor  $a$  distinct from itself if and only if all its summands have the same left factor  $a$ . A partitive number other than  $1$  is *prime* if it has no factors other than itself and  $1$ : this is the case if and only if its summands have no common left factor other than  $1$ . If  $s = pqr \dots$  where  $p$  is prime,  $p$  is a *left prime factor* of  $s$ . Evidently every partitive number other than  $1$  has a left prime factor (equal to itself if it is prime). If a partitive number has two distinct left prime factors, then the same must be true of each of its summands. Now numbers of altitude  $1$ , being prime, have unique left prime factors, and hence by (I) it follows that the left prime factor of any partitive number other than  $1$  is unique. From this with  $(div_L)$ ,  $(U_x^s)$  follows, and  $(div_R)$ .

*Exponentiation.*—As multiplication is associative, the continued product of  $n$  equal numbers  $a$  may be denoted  $a^n$ . No definition of exponentiation of a partitive number with a partitive exponent is offered.

## 8. ABSTRACT NON-ASSOCIATIVE ARITHMETICS

In this and the next section we shall characterize various arithmetics abstractly. The four denoted  $P_s, P_c, A_s, A_c$  may be interpreted as consisting respectively of partitioned serials, partitioned cardinals, serials, cardinals. The rest are intermediate between these (homomorphisms of the whole or part of  $P_s$  which can be represented homomorphically on  $A_c$ ), with the exception of the *collapsed* arithmetics (lacking the second property).

An *arithmetic* will be taken to mean a system consisting of elements called *numbers* to which the following six postulates apply.

(+) For any series of numbers there is a unique number called their *sum*. We leave it to be stated in each case whether this is a free,  $n$ -ary or otherwise restricted operation, and interpret *series* accordingly, defining it as in § 4 in the free case.

( $\times$ ) For any ordered pair of numbers there is a unique number called their *product*.

(=) There is a relation symbolized as *equality* between elements, having the usual formal properties of an equivalence relation, and such that the operations  $(+)$ ,  $(\times)$  are consistent. Equality may mean absolute identity, or membership of the same equivalence class. For verbal convenience we assume the first meaning and use words such as "same", "unique" which will otherwise need paraphrasing.

$(D_R)$  The full right distributive law.

(1) There is a number  $1$  which is neutral in multiplication ( $1a = a1 = a$ ).

(I) Non-associative induction. All other numbers can be derived from  $1$  by finite repetition of the operation of addition (not necessarily in unique ways).

These postulates are logically consistent, for in partitive numbers, serials and cardinals we have four examples of systems to which they apply.

All six postulates are assumed and used in the following theorem.

THEOREM I.  $(a_x)$ .

*Proof.*—Consider a series of numbers  $s_i$  which are such that  $pq \cdot s_i = p \cdot qs_i$  for all numbers  $p, q$ . Then by  $(D_R)$

$$pq \cdot \Sigma s_i = \Sigma (pq \cdot s_i) = \Sigma (p \cdot qs_i) = p \cdot \Sigma qs_i = p \cdot q \Sigma s_i.$$

Thus  $s = \Sigma s_i$  has the property  $pq \cdot s = p \cdot qs$  for arbitrary  $p, q$  if the  $s_i$  separately have this property. Now  $s = 1$  has this property, by (1); and hence by (I) all numbers have it, *i.e.*  $(a_x)$ .

The postulate (I) implies that any number other than  $1$  has a series of summands; also that to every number there corresponds at least one partitioned serial, namely that which is derived from  $1$  by the same operations of addition; the correspondence in the opposite direction is unique. We shall call this correspondence *the p.s. notation* for the abstract numbers; using, *e.g.*, the bracketed numerical representation of § 6 (IV) (IV'), it enables us to denote in writing all numbers whose derivation from  $1$  is sufficiently simple, including all *finite* numbers (*i.e.* those representable by finite partitioned serials). In this notation we can add and, using  $(D_R)$ , multiply numbers correctly. Also we can speak of the *degree* and *altitude* of a number,



meaning those of a corresponding partitioned serial; but we cannot say that the summands, degree and altitude are uniquely determined (*i.e.* the same for all equal numbers), since we do not know without some further postulate what numbers in the p.s. notation can be equated. The existence of such a notation proves the following theorem.

**THEOREM 2.**—*Partitioned serials form the most general system satisfying the above six postulates; all other arithmetics are either isomorphisms or homomorphisms of it, or \* of subsystems of it.*

## 9. EXAMPLES OF ABSTRACT ARITHMETICS

We shall assume henceforward a further postulate concerning equality, namely:

[=] Two numbers, derived from 1 by stated operations of addition and multiplication, are equal if and only if their equality is a logical consequence of the other postulates of the arithmetic.

Thus we shall now always assume the seven postulates

$$(+)(\times)(=)[=](D_R)(I)(I),$$

together with certain identities specified in each case as *extra postulates*. Then [=] implies that two numbers given in the p.s. notation are equal if and only if they have the same notation or can be proved equal by application of the *extra postulates*.

Unless otherwise stated, addition is a free operation.

### *Partitive Arithmetics*

$P_S$  (*arithmetic of partitioned serials*): no extra postulates. This means that the p.s. notation is a one-one representation of the arithmetic, for there are no postulates by which two numbers with different p.s. notations could be proved equal. ( $U_+^S$ ) follows.

$P_C$  (*arithmetic of partitioned cardinals*): extra postulate ( $C_+$ ). ( $U_+^C$ ) follows.

### *Palintropic Arithmetics*

$\Pi_S$  (*palintropic serials*): extra postulate ( $c_x$ ).

$\Pi_C$  (*palintropic cardinals*): extra postulates ( $C_+$ ) ( $c_x$ ).

It will be shown in Theorem 3 that ( $c_x$ ) as an extra postulate can be replaced by ( $D_L$ ).

### *Entropic Arithmetics*

$E_S$  (*entropic serials*): extra postulate ( $E_+$ ).

$E_C$  (*entropic cardinals*): extra postulates ( $C_+$ ) ( $E_+$ ).

These arithmetics are also palintropic, for it will be shown in Theorem 4 that ( $E_+$ ) as an extra postulate includes ( $c_x$ ). Thus  $E_S, E_C$  are homomorphisms of  $\Pi_S, \Pi_C$ . The converse is not true; *e.g.*  $(1+2)+(3+4)=(1+3)+(2+4)$  in  $E_S, E_C$ , but not in  $\Pi_S, \Pi_C$ .

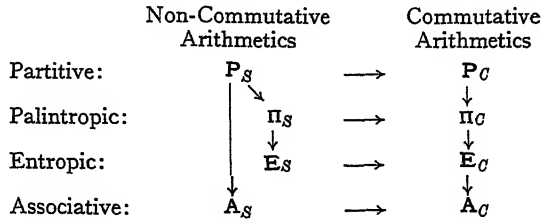
### *Associative Arithmetics*

$A_S$  (*serials*): extra postulate, either ( $A_+$ ) with addition free or ( $a_+$ ) with addition binary.

$A_C$  (*cardinals*): *ditto*, together with ( $C_+$ ) or ( $c_+$ ). We omit the proof that the arithmetics thus abstractly defined are isomorphic with those of serials and cardinals which we have already supposed defined class-theoretically. ( $A_+$ ) with ( $C_+$ ) of course implies ( $E_+$ ), so that  $A_C$  is a homomorphism of  $E_C$ , in fact of each of the seven preceding arithmetics. If finite numbers only are considered, ( $A_+$ ) as an extra postulate implies ( $C_+$ ), so that  $A_S$  and  $A_C$  are abstractly the same.

\* If addition is restricted.

The homomorphism relations of the above arithmetics are depicted by arrows in the following scheme:—



In the  $P$ ,  $\Pi$  and  $E$  arithmetics the degree and altitude of a given number are uniquely determined. For example, to prove this in  $E_S$  we have, from § 5,

$$\begin{aligned}\delta(\sum_{i,j} \sum a_{ij}) &= \sum_{i,j} \delta(a_{ij}) = \sum_{j,i} \delta(a_{ij}) = \delta(\sum_{j,i} \sum a_{ij}), \\ \alpha(\sum_{i,j} \sum a_{ij}) &= 1 + \max_{(all\ i)} \{1 + \max_{(fixed\ i)} \alpha(a_{ij})\} \\ &= 2 + \max_{j,i} \alpha(a_{ij}) = \alpha(\sum_{j,i} \sum a_{ij});\end{aligned}$$

thus no application of  $(E_+)$  can alter the degree or altitude of a number in the p.s. notation. Hence if two numbers of  $E_S$  are equal, they have the same degree and altitude. In  $A_S$  and  $A_G$  the degree of a number is of course uniquely determined, but not its altitude.

We recall that  $(U_x^S)$  and cancellation laws hold in  $P_S$  and  $P_G$ . (Proofs as in § 7.) It seems likely that  $(U_x^G)$  and cancellation laws can be proved for  $\Pi_S, \Pi_G, E_S, E_G$ . If not, further arithmetics may be obtainable by imposing them as extra postulates. Theorem 5 will show that if multiplicative cancellation laws are assumed, the postulate  $(c_x)$  of  $\Pi_S$  and  $\Pi_G$  can be replaced by  $(e_x)$ . Regarding factorization, it should be observed that the prime factors of a number will generally be different in the different arithmetics. For example,  $[(1+2)+(3+4)] + [(2+4)+(1+3)]$  is prime in  $P_S, P_G, \Pi_S, \Pi_G, E_S$ , but is equal to  $[(1+2)+(3+4)]2$  in  $E_G$ , and to  $2^2 5$  in  $A_S, A_G$ .

The proofs of the theorems referred to above will now be given. Their purpose is to show that certain selections of postulates do not give distinct arithmetics. The postulates  $(+)$   $(\times)$   $(=)$   $(1)$  are tacitly assumed.

**THEOREM 3.**— $(I) (D_R) (c_x)$  are together equivalent to  $(I) (D_R) (D_L)$ .

*Proof.*—It is obvious that  $(D_R)$  with  $(c_x)$  implies  $(D_L)$ , and we shall prove the converse, that  $(I) (D_R) (D_L)$  imply  $(c_x)$ .

Let  $s_i$  be a series of numbers such that for an arbitrary number  $p$ ,  $s_i p = p s_i$ . Then using both  $(D_R)$  and  $(D_L)$ , we have

$$(\sum s_i) p = \sum (s_i p) = \sum (p s_i) = p \sum s_i.$$

Thus  $s = \sum s_i$  has the property of commuting with an arbitrary number if the  $s_i$  all have this property. Now 1 has the property, hence all numbers have it, i.e.  $(c_x)$ .

**THEOREM 4.**— $(I) (D_R) (E_+)$  imply  $(c_x) (D_L)$ .

*Proof.*—Let  $s_i$  be a series of numbers such that, for an arbitrary series of numbers  $p_j$ ,

$$(\sum_j p_j) s_i = \sum_j (p_j s_i);$$

and let  $s = \sum s_i$ . Then

$$\begin{aligned}(\sum_j p_j) s &= \sum_i ((\sum_j p_j) s_i), \quad \text{by } (D_R), \\ &= \sum_{i,j} (p_j s_i), \quad \text{by hypothesis,} \\ &= \sum_{j,i} (p_j s_i), \quad \text{by } E_+, \\ &= \sum_j (p_j s), \quad \text{by } (D_R).\end{aligned}$$

Thus  $s$  has the left distributive property if its summands all have it. Now  $\mathbf{1}$  has this property, and hence by (I) all numbers have it; i.e.  $(D_L)$  holds and hence by Theorem 3  $(c_x)$ . (This generalizes a result previously proved for binary algebras; cf. § 2, second footnote.)

THEOREM 5.— $(a_x) (c_x) (div_L)$  are equivalent to  $(a_x) (e_x) (div_{L,R})$ .

*Proof.*—Obviously  $(a_x) (c_x)$  imply  $(e_x)$ ; and  $(c_x) (div_L)$  imply  $(div_R)$ . Conversely, if  $(a_x)$  holds,

$$\begin{aligned} ab \cdot cd &= ab \cdot c : d = a \cdot bc : d, \\ ac \cdot bd &= ac \cdot b : d = a \cdot cb : d; \end{aligned}$$

if  $(e_x)$  also holds, it follows that  $a \cdot bc : d = a \cdot cb : d$ ; and if  $(div_{L,R})$  also hold, it follows that  $bc = cb$ , i.e.  $(c_x)$ .

### Restricted Arithmetics

Binary, ternary, . . . ,  $n$ -ary, otherwise restricted, and finitely free arithmetics can be defined, forming closed subsystems of any of the previous arithmetics. With a few minor changes the whole of the above discussion applies; we are merely changing the meaning of the word *series* in the definition of addition.

Thus, suppose that in  $\mathbf{P}_S$  we consider only those numbers derived from  $\mathbf{1}$  by binary addition; they correspond to trees which bifurcate at every knot. (In Etherington, 1939 *a*, I called such trees *pedigrees*.) We are now taking *series* to mean *ordered pair*. We obtain an arithmetic, *binary*  $\mathbf{P}_S$ , which is the most general logarithmic applying to a non-commutative non-associative binary algebra. The corresponding *binary*  $\mathbf{P}_O$  applies similarly when the algebra is commutative; and the corresponding subsystems of  $\Pi_S$  and  $\Pi_O$ ,  $\mathbf{E}_S$  and  $\mathbf{E}_O$ , apply similarly when the algebra is palintropic, or is entropic as regards multiplication in its polynomial subalgebras.

### Collapsed Arithmetics

In all the above arithmetics the degree of a given number is uniquely determined and gives a homomorphic representation of the arithmetic on  $\mathbf{A}_O$ . Arithmetics which do not have this property will be called *collapsed*. For example, starting with the postulates of  $\mathbf{P}_O$ , let us as an extra postulate equate all numbers whose altitudes exceed some fixed integer  $m$ . The normalized elements of the gametic, zygotie and copular algebras for simple mendelian inheritance (Etherington, 1939 *b*, 1941 *a*) provide examples where the logarithmetics are collapsed and are of this type, with  $m=0, 1, 2$  respectively; so do Lie algebras ( $m=1$ ); and Boolean algebras show the extreme case ( $m=0$ ). The arithmetic of ordinary integers modulo an integer is an example of a finitely free collapsed arithmetic; the logarithmic of a cyclic group is of this type.

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XLVIII.—On Commuting Matrices and Commutative Algebras.\* By D. E. Rutherford, M.A., B.Sc., Dr.Math., United College, University of St Andrews

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INTRODUCTION

THE structure of commutative associative linear algebras is well known and is usually derived from more general results concerning non-commutative algebras (Cartan, Frobenius). The novelty of the present treatment is that while it avoids the complexities of the non-commutative case, it exhibits the essential relationship between the theory of commuting matrices and that of commutative algebras.

While theorems 1 and 2 of this paper are implicit in the writings of Voss (1889), Taber (1890), and Plemelj (1901), it has been considered worth while to recapitulate these results in the explicit form required for the discussion of commutative algebras. In doing so, some new facts emerge.

COMMUTING MATRICES

1. Let  $A$  and  $B$  be square matrices such that

$$AB = BA. \quad (1)$$

A non-singular matrix  $H$  exists, such that

$$HAH^{-1} = A_1 + A^*,$$

where every latent root of  $A_1$  is  $\alpha_1$ , and where no latent root of  $A^*$  is equal to  $\alpha_1$ . If  $HBH^{-1}$  is partitioned conformally with  $A_1 + A^*$ , say

$$HBH^{-1} = \begin{bmatrix} B_1 & B' \\ B'' & B^* \end{bmatrix},$$

then, from the fact that  $HAH^{-1}$  commutes with  $HBH^{-1}$ , we can deduce the following relations:—

$$A_1 B_1 = B_1 A_1, \quad A_1 B' = B' A^*, \quad B'' A_1 = A^* B'', \quad A^* B^* = B^* A^*.$$

From the second of these, we conclude that if the reduced characteristic function of  $A_1$  be  $(x - \alpha_1)^r$ , then

$$B' (A^* - \alpha_1 I^*)^r = (A_1 - \alpha_1 I_1)^r B' = 0,$$

where  $I^*$  and  $I_1$  denote unit matrices. Now  $A^* - \alpha_1 I^*$  is a non-singular matrix, for no latent root of  $A^*$  is equal to  $\alpha_1$ . Consequently  $B' = 0$ , and similarly  $B'' = 0$ . It has therefore been shown that a non-singular matrix  $H$  exists, for which

$$HAH^{-1} = A_1 + A^*, \quad HBH^{-1} = B_1 + B^*,$$

where

$$\begin{aligned} A_1 B_1 &= B_1 A_1, \\ A^* B^* &= B^* A^*. \end{aligned} \quad (2)$$

Treating equation (2) in the same way as equation (1), we eventually obtain after a number of steps

$$HAH^{-1} = A_1 + A_2 + \dots + A_t, \quad HBH^{-1} = B_1 + B_2 + \dots + B_t, \quad (3)$$

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to express the fact that  $KBK^{-1}$  is partitioned conformally with  $KAK^{-1}$ . Since  $KBK^{-1}$  commutes with  $KAK^{-1}$ , it also commutes with  $KAK^{-1} - \alpha I$ . Consequently, we have the matrix equation

$$\begin{bmatrix} I_{pq} B_{qp}, I_{pq} B_{qq}, I_{pq} B_{qr}, \dots \\ I_{qr} B_{rp}, I_{qr} B_{rq}, I_{qr} B_{rr}, \dots \\ \dots \\ \dots \\ 0 \quad 0 \quad 0 \quad \dots \end{bmatrix} = \begin{bmatrix} 0, B_{pp} I_{pq}, B_{pq} I_{qr}, \dots \\ 0, B_{qp} I_{pq}, B_{qq} I_{qr}, \dots \\ \dots \\ \dots \\ 0, B_{tp} I_{pq}, B_{tq} I_{qr}, \dots \end{bmatrix}.$$

Identification of the submatrices in respectively the (1,1)th, (2,2)th, (3,3)th, . . . , (2,1)th, (3,2)th, . . . , (3,1)th, . . . positions shows that every submatrix below the leading diagonal of  $KBK^{-1}$  vanishes. Comparison of submatrices in the (1,2)th, (2,3)th, . . . positions shows that

$$B_{pp} = \begin{bmatrix} B_{qq} & ? \\ 0 & ? \end{bmatrix}, \quad B_{qq} = \begin{bmatrix} B_{rr} & ? \\ 0 & ? \end{bmatrix}, \quad \dots \quad (6)$$

In the same way it may be shown that

$$B_{pq} = \begin{bmatrix} B_{qr} & ? \\ 0 & ? \end{bmatrix}, \quad B_{qr} = \begin{bmatrix} B_{rs} & ? \\ 0 & ? \end{bmatrix}, \quad \dots \quad (7)$$

and so on.

This result goes a little deeper than Taber's result, which states that the most general matrix which commutes with  $KAK^{-1}$  can be written in the following form

$$\begin{bmatrix} B_{pp} & B_{pq} & B_{pr} & \dots \\ 0 & B_{qq} & B_{qr} & \dots \\ 0 & 0 & B_{rr} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix},$$

in which

$$\begin{aligned} B_{pp} &= \begin{bmatrix} B_{qq} & ? \\ ? & ? \end{bmatrix}, & B_{qq} &= \begin{bmatrix} B_{rr} & ? \\ ? & ? \end{bmatrix}, & \dots, \\ B_{pq} &= \begin{bmatrix} B_{qr} & ? \\ ? & ? \end{bmatrix}, & B_{qr} &= \begin{bmatrix} B_{rs} & ? \\ ? & ? \end{bmatrix}, & \dots, \\ & \dots & \dots & \dots & \dots \end{aligned}$$

For although every matrix which commutes with  $KAK^{-1}$  is of the above form, not every matrix of this form commutes with  $KAK^{-1}$ . In general, certain matrix elements must vanish, as we have just seen. Thus the matrices

$$\begin{bmatrix} \alpha & 0 & 0 & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

do not commute, although the second is of Taber's form.

3. We now prove the principal result on commuting matrices.

**THEOREM 2.**—*A set of commuting matrices  $A, B, \dots, D$  can be reduced simultaneously by a similarity transformation to triangular form.*

By Theorem 1 it is sufficient to consider the case in which all the latent roots of  $A$  are  $\alpha$ , all those of  $B$  are  $\beta$ , and so on. Omitting the trivial case in which each matrix is scalar, we prove the theorem for matrices of order  $n+1$ , assuming that it is true for matrices of order not

greater than  $n$ . The basis of the induction is the fact that the most general matrix which commutes with  $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$  has the form  $\begin{bmatrix} y & z \\ 0 & y \end{bmatrix}$ .

It has been shown that a matrix  $K$  can be found such that  $KAK^{-1}$  has the form (5) and such that

$$KBK^{-1} = \begin{bmatrix} B_{pp} & B_{pq} & B_{pr} & \dots \\ 0 & B_{qq} & B_{qr} & \dots \\ 0 & 0 & B_{rr} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}, \dots, KDK^{-1} = \begin{bmatrix} D_{pp} & D_{pq} & D_{pr} & \dots \\ 0 & D_{qq} & D_{qr} & \dots \\ 0 & 0 & D_{rr} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

The fact that  $A, B, \dots, D$  commute, implies that  $A_{pp}, B_{pp}, \dots, D_{pp}$  commute. So do  $A_{qq}, B_{qq}, \dots, D_{qq}$ , and so on. By the induction hypothesis, we can find matrices  $L_p, L_q, \dots$  such that  $L_p A_{pp} L_p^{-1}, \dots, L_p D_{pp} L_p^{-1}$  are all triangular,  $L_q A_{qq} L_q^{-1}, \dots, L_q D_{qq} L_q^{-1}$  are all triangular, and so forth. It follows that if we write

$$L = L_p + L_q + \dots,$$

then  $(LK)A(LK)^{-1}, \dots, (LK)D(LK)^{-1}$  are all triangular. The theorem is therefore established.

The theorem that a set of commuting matrices can be reduced simultaneously to triangular form was known to Schur,\* but the proof given above affords further information concerning the nature of these triangular matrices. Let us express the formulæ (6) and (7) in the notation

$$B_{pp} = \begin{bmatrix} B_{qq} & B_{q, p-q} \\ 0 & B_{p-q, p-q} \end{bmatrix}, \quad B_{pq} = \begin{bmatrix} B_{qr} & B_{q, q-r} \\ 0 & B_{p-q, q-r} \end{bmatrix}.$$

It is clear from the first of these that we may choose  $L_p$  to be of the form

$$L_p = L_q + L_{p-q},$$

where  $L_{p-q}$  reduces  $B_{p-q, p-q}$  to triangular form. Similarly

$$L_q = L_r + L_{q-r},$$

$$\dots$$

Thus,

$$L_p = L_q + \dots + L_{q-r} + L_{p-q}.$$

In consequence, a typical submatrix of  $(LK)B(LK)^{-1}$  such as  $L_p B_{pq} L_q^{-1}$  has the form

$$\begin{bmatrix} L_q & 0 \\ 0 & L_{p-q} \end{bmatrix} \begin{bmatrix} B_{qr} & B_{q, q-r} \\ 0 & B_{p-q, q-r} \end{bmatrix} \begin{bmatrix} L_r^{-1} & 0 \\ 0 & L_{q-r}^{-1} \end{bmatrix}$$

or

$$\begin{bmatrix} L_q B_{qr} L_r^{-1} & L_q B_{q, q-r} L_{q-r}^{-1} \\ 0 & L_{p-q} B_{p-q, q-r} L_{q-r}^{-1} \end{bmatrix}.$$

We have therefore established the fact that the submatrices of the triangular matrix  $(LK)B(LK)^{-1}$  have all the properties established in § 2 for the submatrices of  $KBK^{-1}$ . In addition, since every latent root of  $B$  is  $\beta$  and since  $(LK)B(LK)^{-1}$  is triangular, this matrix must have  $\beta$  everywhere on the leading diagonal.

#### COMMUTATIVE ALGEBRAS

4. In this section we shall employ the summation convention that terms involving equal upper and lower Greek suffixes are to be summed from 1 to  $n$ , or from 1 to  $m$  as the case may be. Roman suffixes, however, do not imply summation. In the case of matrix elements the upper suffix denotes the row and the lower suffix the column.

\* Schur in his lectures deduced this theorem from certain results of Frobenius, but he attributed the theorem to Voss. The present author has, however, been unable to find an explicit statement of the theorem in the writings of Voss, although the theorem might be regarded as a corollary of the latter's paper of 1889.



Consider a commutative associative algebra whose basic elements  $u_1, \dots, u_n$  have the multiplication table (summation from 1 to  $n$ )

$$u_i u_j = \gamma_{ij}^a u_a.$$

The commutative law gives

$$\gamma_{ij}^k = \gamma_{ji}^k$$

and the associative law yields

$$\gamma_{ij}^a \gamma_{ak}^l = \gamma_{jk}^a \gamma_{ia}^l. \quad (8)$$

As is well known, it follows from (8) that the matrices

$$\Gamma_i \equiv [\gamma_{ij}^k] \quad (i = 1, \dots, n)$$

(where  $\gamma_{ij}^k$  is the  $(k, j)$ th element of  $\Gamma_i$ ) afford a representation of the basis. That is to say,

$$\Gamma_i \Gamma_j = \gamma_{ij}^a \Gamma_a = \Gamma_j \Gamma_i.$$

Consequently, if  $H$  is any non-singular matrix of order  $n$ , the matrices

$$\Lambda_i \equiv H \Gamma_i H^{-1} \quad (i = 1, \dots, n)$$

give another representation for which

$$\Lambda_i \Lambda_j = \gamma_{ij}^a \Lambda_a = \Lambda_j \Lambda_i.$$

Let  $H \equiv [h_j^k]$ ,  $H^{-1} \equiv [H_j^k]$ . A change to a new basis  $v_1, \dots, v_n$  by means of the transformation

$$u_i = h_i^a v_a \quad v_i = H_i^a u_a$$

gives

$$v_i v_j = H_i^a H_j^b \gamma_{ab}^c h_c^e v_e = \omega_{ij}^e v_e,$$

where

$$\omega_{ij}^k \equiv H_i^a H_j^b \gamma_{ab}^c h_c^k.$$

This new basis has a representation  $\Omega_1, \dots, \Omega_n$ , where

$$\Omega_i \equiv [\omega_{ij}^k] = H_i^a [h_c^k \gamma_{ab}^c H_j^b] = H_i^a (H \Gamma_a H^{-1}) = H_i^a \Lambda_a.$$

Now according to Theorem 1 (§ 1), a matrix  $H$  can be found for the commuting set of matrices  $\Gamma_i$ , such that the set  $H \Gamma_i H^{-1}$  (or  $\Lambda_i$ ) are simultaneously reducible to the direct sums of submatrices each of which has equal latent roots. Also by Theorem 2 (§ 3) the matrix  $H$  can be further specialised to ensure that each of these submatrices is triangular. Since  $\Omega_i = H_i^a \Lambda_a$ , the matrices  $\Omega_i$  will retain these properties. The given algebra is therefore a direct sum of subalgebras of which a typical one has a basis  $v_1, \dots, v_m$ , ( $m \leq n$ ) and a multiplication table (with summations from 1 to  $m$ )

$$v_i v_j = \omega_{ij}^a v_a$$

with the following properties:—

$$\omega_{ij}^k = \omega_{ji}^k \quad (9)$$

$$\omega_{i1}^1 = \omega_{i2}^2 = \dots = \omega_{im}^m \equiv \mu_i \quad \text{say,} \quad (10)$$

$$\omega_{ij}^k = 0 \quad \text{if } k > i, \text{ or if } k > j, \quad (11)$$

$$\omega_{ij}^a \omega_{ak}^l = \omega_{jk}^a \omega_{ia}^l. \quad (12)$$

(9) and (12) are true because the algebra is still commutative and associative; (10) and (11) arise from the special properties of the matrices  $\Omega_i$ .

A combination of (10) and (11) yields immediately

$$\mu_i = 0. \quad (i < m)$$

From this it follows that if  $\omega_i$  denotes the appropriate submatrix of order  $m \times m$  of the matrix  $\Omega_i$ , then each matrix  $\omega_i$ , with the possible exception of  $\omega_m$ , is singular. If  $\mu_m = 0$ , then  $\omega_m$  is

also singular and the subalgebra is nilpotent, for any element  $x = x^a v_a$  is represented by a matrix  $x^a \omega_a$  which is triangular and has zeros everywhere on the leading diagonal. The multiplication table of such a nilpotent subalgebra takes the form

$$v_i v_j = v_j v_i = \omega_{ij}^1 v_1 + \dots + \omega_{ij}^{i-1} v_{i-1}, \quad (i \leq j < m) \quad (13)$$

in which the coefficients  $\omega_{ij}^k$  must satisfy (9) and (12).

Alternatively, suppose  $\mu_m \neq 0$ . Then  $\omega_m$  is non-singular. Let  $(\omega_m)^{-1} \equiv [\sigma_j^k]$ . Then, using the Kronecker delta,

$$\omega_{ma}^k \sigma_j^a = \delta_j^k, \quad \sigma_a^k \omega_{mj}^a = \delta_j^k.$$

Accordingly, by (9) and (12),

$$\sigma_m^a \omega_{aj}^k = \sigma_m^a \omega_{a\beta}^k \delta_j^\beta = \sigma_m^a \omega_{a\beta}^k \omega_{m\gamma}^\beta \sigma_j^\gamma = \sigma_m^a \omega_{ma}^\beta \omega_{\beta\gamma}^k \sigma_j^\gamma = \delta_m^\beta \omega_{\beta\gamma}^k \sigma_j^\gamma = \omega_{m\gamma}^k \sigma_j^\gamma = \delta_j^k.$$

If we now write

$$V \equiv \sigma_m^a v_a$$

and choose  $v_1, \dots, v_{m-1}, V$  as a basis instead of  $v_1, \dots, v_m$ , then the formulæ for the products  $v_i v_j$  ( $i < m, j < m$ ) remain unchanged, but

$$V v_i = \sigma_m^a v_a v_i = \sigma_m^a \omega_{ai}^e v_e = \delta_i^e v_e = v_i,$$

and

$$V^2 = V \sigma_m^a v_a = \sigma_m^a v_a = V.$$

It is clear that  $V$  is the principal unit of the subalgebra. The multiplication table of this subalgebra now takes the form

$$\left. \begin{aligned} V^2 &= V, \\ V v_i &= v_i V = v_i, \\ v_i v_j &= v_j v_i = \omega_{ij}^1 v_1 + \dots + \omega_{ij}^{i-1} v_{i-1}, \quad (i \leq j \leq m-1) \end{aligned} \right\} \quad (14)$$

in which the coefficients  $\omega_{ij}^k$  satisfy (9) and (12).

To summarise these results, we observe that every commutative associative algebra is the direct sum of subalgebras of the types (13) and (14). This is the well-known result (cf. Dickson, 1930, p. 57). The distinction between types (13) and (14) is of course superficial, for we can always adjoin a principal unit to the given algebra, thereby ensuring that each subalgebra of the type considered has a principal unit.

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XLIX.—Generalizations of a Problem of Pillai. By L. Mirsky, Department of Mathematics, University of Sheffield. Communicated by Professor A. G. WALKER

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1. THROUGHOUT this paper  $k_1, \dots, k_s$  will denote  $s > 1$  fixed distinct positive integers. Some years ago Pillai (1936) found an asymptotic formula, with error term  $O(x/\log x)$ , for the number of positive integers  $n \leq x$  such that  $n+k_1, \dots, n+k_s$  are all square-free. I recently considered (Mirsky, 1947) the corresponding problem for  $r$ -free integers (*i.e.* integers not divisible by the  $r$ th power of any prime), and was able, in particular, to reduce the error term in Pillai's formula.

Our present object is to discuss various generalizations and extensions of Pillai's problem. In all investigations below we shall be concerned with a set  $A$  of integers. This is any given, finite or infinite, set of integers greater than 1 and subject to certain additional restrictions which will be stated later. The elements of  $A$  will be called *a-numbers*, and the letter  $a$  will be reserved for them. A number which is not divisible by any  $a$ -number will be called *A-free*, and our main concern will be with the study of *A-free* numbers. Their additive properties have recently been investigated elsewhere (Mirsky, 1948), and some estimates obtained in that investigation will be quoted in the present paper.

At various stages of our discussion one or more of the following assumptions will be made with regard to  $A$ :—

$$\sum_a a^{-1} \text{ converges.} \quad (1)$$

$$\sum_a a^{-1} \text{ diverges.} \quad (2)$$

$$\left. \begin{array}{l} \text{There exists a number } \vartheta \ (0 < \vartheta < 1) \text{ such that, for } x \rightarrow \infty, \\ A(x) = \sum_{a \leq x} 1 = O(x^\vartheta). \end{array} \right\} \quad (3)$$

$$\text{Any two elements of } A \text{ are coprime.} \quad (4)$$

We note that condition (1) is contained in (3).

Numbers of the form  $a_1^{t_1} \dots a_h^{t_h}$ , where  $h > 1$ ,  $t_1 > 0, \dots, t_h > 0$  will be called *c-numbers*, and the letter  $c$  will be reserved for them. In particular, 1 is a  $c$ -number, and so is every  $a$ -number.

Let  $C_1, \dots, C_s$  be  $s$  given, finite or infinite, sets of  $c$ -numbers. Denote by  $\gamma(n)$  the greatest  $c$ -number dividing  $n$ , and by

$$M = M(k_1, \dots, k_s; A; C_1, \dots, C_s)$$

the set of all integers  $n$  such that \*

$$\gamma(n+k_1) \in C_1, \dots, \gamma(n+k_s) \in C_s.$$

Furthermore, let  $M(x)$  be the number of positive integers  $n \leq x$  belonging to  $M$ . Our aim in §§ 3–6 will be to investigate the behaviour of  $M(x)$  for  $x \rightarrow \infty$  on the assumptions that  $A$  satisfies (3) and (4). We shall find an asymptotic formula for  $M(x)$ , and shall also prove that  $M$  is either empty or has a positive density.

In the remainder of the paper (§§ 7–11) we specialize our initial problem by taking each  $C_i$  to consist of the single number 1. The set  $M$  will in that case be denoted by  $N$  and will consist of the integers  $n$  such that  $n+k_1, \dots, n+k_s$  are all *A-free*. We shall denote by  $N(x)$

\* The symbol  $l \in L$  means that  $l$  is an element of the set  $L$ .

the number of positive integers  $n \leq x$  in  $\mathbf{N}$ , and we shall be concerned with the asymptotic behaviour of  $N(x)$  under various assumptions made about  $\mathbf{A}$ .

I wish to express my thanks to Dr R. Rado for his help in connection with § 8, and to the Referee for a number of useful comments which enabled me to make many textual improvements.

2. Our notation is as follows:—

If  $P_1(\eta)$ ,  $P_2(\eta)$  are two propositions concerning a variable  $\eta$ , then

$P_1(\eta) (P_2(\eta))$  means that  $P_1(\eta)$  holds for *every*  $\eta$  for which  $P_2(\eta)$  holds;

$P_1(\eta) [P_2(\eta)]$  means that  $P_1(\eta)$  holds for *some*  $\eta$  for which  $P_2(\eta)$  holds.

We shall employ an analogous notation when there is more than one variable.

The letters  $x, y, a, \beta, \xi$  will denote positive numbers and  $\epsilon$  an arbitrarily small positive number; all other small letters will denote positive integers unless otherwise stated. We shall write  $k = \text{Max}_{1 \leq i \leq s} k_i$ . The highest common factor and the lowest common multiple of  $n_1, \dots, n_s$  will be denoted by  $(n_1, \dots, n_s)$  and  $\{n_1, \dots, n_s\}$  respectively.

For typographical reasons it is frequently convenient to write  $l \equiv l' (\cdot m)$  in place of  $l \equiv l' \pmod{m}$ , and also

$$\Sigma_S [f(m, n, \dots)]$$

in place of

$$\sum_S f(m, n, \dots);$$

here  $S$  stands for the set of conditions which define the range of summation.

As usual  $\phi(n)$  denotes Euler's function,  $d(n)$  the number of divisors of  $n$ , and  $\pi(x)$  the number of primes not exceeding  $x$ .

$\Omega(n)$  is defined as 1 or 0 according as  $n$  is or is not  $\mathbf{A}$ -free; we also define  $\omega(n) = 0$  if  $a^2 \mid n [a]$ , and  $\omega(n) = (-1)^h$  if  $a^2 \nmid n [a]$ , and  $n$  is divisible by precisely  $h > 0$  distinct  $a$ -numbers.

$D(m)$  denotes the number of numbers among  $1, 2, \dots, m$  which are congruent  $(\text{mod } m)$  to at least one of  $k_1, \dots, k_s$ , i.e. the number of residue classes  $(\text{mod } m)$  represented by  $k_1, \dots, k_s$ .

We define  $E(n_1, \dots, n_s)$  as 1 or 0 according as the system of simultaneous congruences in  $n$ ,

$$n + k_i \equiv 0 \pmod{n_i}, \quad (1 \leq i \leq s) \quad (5)$$

is or is not soluble; moreover,  $T(x; n_1, \dots, n_s)$  denotes the number of positive integers  $n \leq x$  which satisfy the system (5).

We write

$$F(x; a) = F(x; a; k_1, \dots, k_s) = \sum_{n \leq x} \left| \begin{array}{l} n + k_i = c_i m_i \\ (1 \leq i \leq s); \quad c_1 \dots c_s > x^a \end{array} \right|_1,$$

$$N(x, y) = \sum_{n \leq x} \left| \begin{array}{l} a \nmid (n + k_i) \\ (1 \leq i \leq s; \quad a < y) \end{array} \right|_1.$$

A set of integers  $\mathbf{B}$  will be said to have *density*  $a$  if, for  $x \rightarrow \infty$ ,  $\sum_{n \leq x} n \in \mathbf{B} \sim ax$ . It will be said to have *logarithmic density*  $a'$  if  $\sum_{n \leq x} n \in \mathbf{B} \sim a' \log x$ .

The  $O$ -notation will be used in the following sense. If  $\Phi$  and  $\Psi$  are two functions of  $x$  and certain other parameters, say  $\lambda_1, \dots, \lambda_r$ , then  $\Phi = O(\Psi)$  means that there exist positive constants  $K, x_0$ , depending at most on  $k_1, \dots, k_s, \mathbf{A}, \mathbf{C}_1, \dots, \mathbf{C}_s, \vartheta, a, \beta, \epsilon, q$ , such that for  $x > x_0$  and all (or all specified) values of  $\lambda_1, \dots, \lambda_r$ , we have  $|\Phi| < K\Psi$ . If  $\Phi$  and  $\Psi$  do not depend on  $x$ , then the question of the existence of  $x_0$  does not, of course, arise.

3. We begin with some preliminary lemmas.

LEMMA 1.—*The system of congruences (5) is soluble if and only if  $(n_i, n_j) \mid (k_i - k_j)$  ( $1 \leq i < j \leq s$ ). In the case of solubility the solutions form precisely one residue class  $(\text{mod } \{n_1, \dots, n_s\})$ .*

For a proof of this well-known result see Scholz (1939, p. 49).

LEMMA 2. 
$$T(x; n_1, \dots, n_s) = x \frac{E(n_1, \dots, n_s)}{\{n_1, \dots, n_s\}} + O(1).$$

This is an immediate consequence of Lemma 1.

LEMMA 3. 
$$\frac{E(n_1, \dots, n_s)}{\{n_1, \dots, n_s\}} = O\left(\frac{1}{n_1 \dots n_s}\right).$$

For a proof see Mirsky (1947, Lemma 3).

LEMMA 4.—If  $A$  satisfies (1) and (4), then the multiple series

$$\sigma = \sum_{c_1, \dots, c_s} \left| \begin{matrix} c_1' \in C_1, \dots, c_s' \in C_s \\ c_1, \dots, c_s \end{matrix} \right| \omega(c_1) \dots \omega(c_s) \frac{E(c_1 c_1', \dots, c_s c_s')}{\{c_1 c_1', \dots, c_s c_s'\}}$$

converges absolutely.

Since  $A$  satisfies (1) the product  $\prod_a (1 - a^{-1})^{-1}$  converges, and therefore, by (4),  $\sum_c c^{-1}$  converges also.\* But, by Lemma 3,

$$\omega(c_1) \dots \omega(c_s) \frac{E(c_1 c_1', \dots, c_s c_s')}{\{c_1 c_1', \dots, c_s c_s'\}} = O\left(\frac{1}{c_1 c_1' \dots c_s c_s'}\right),$$

and the assertion therefore follows.

LEMMA 5.—If  $A$  satisfies (3) and (4), then

$$\sum_{c_1 \dots c_s > x} \frac{1}{c_1 \dots c_s} = O(x^{-1+\theta+\epsilon}),$$

$$\sum_{c_1 \dots c_s \leq x} \frac{1}{c_1 \dots c_s} = O(x^{\theta+\epsilon}).$$

For a proof of these two estimates see Mirsky (1948, Lemmas 9 and 10).

4. We next come to the crucial lemma required in the study of the function  $M(x)$ .

LEMMA 6.—If  $A$  satisfies (3) and (4), then

$$F(x; a) = O(x^{1-\alpha(1-\theta)+\epsilon}) + O(x^{2\theta/(1+\theta)+\epsilon}).$$

The proof is by induction. For  $s=1$  we have, by Lemma 5 and (3),

$$\begin{aligned} F(x; a) &= \sum_{n \leq x; n+k_1=cm; c > x^\alpha} 1 \\ &= \sum_{x^\alpha < c \leq x+k_1} \sum_{n \leq x; n+k_1 \equiv 0(c)} 1 \\ &= \sum_{x^\alpha < c \leq x+k_1} \left\{ O\left(\frac{x}{c}\right) + O(1) \right\} \\ &= O(x^{1-\alpha(1-\theta)+\epsilon}) + O(x^{\theta+\epsilon}). \end{aligned}$$

Thus the assertion holds for  $s=1$ . Assume next that it holds for  $s-1$  where  $s \geq 2$ . Denoting by  $\beta$  a number whose value will be fixed later, we have

$$\begin{aligned} F(x; a) &= F(x; a; k_1, \dots, k_s) = \sum_{n \leq x; \substack{n+k_i=c_i m_i \\ (1 \leq i \leq s)}} \frac{1}{c_1 \dots c_s} \Big|_{c_1 \dots c_s > x^\alpha} 1 \\ &< \sum_{n \leq x; \substack{n+k_i=c_i m_i \\ (1 \leq i \leq s)}} \frac{1}{c_1 \dots c_s} \Big|_{c_1 \dots c_s > x^\alpha} 1 \\ &\quad + \sum_{1 \leq j \leq s} \sum_{n \leq x; \substack{n+k_i=c_i m_i \\ (1 \leq i \leq s)}} \frac{1}{c_1 \dots c_s} \Big|_{c_1 \dots c_s/c_j > x^\beta} 1 \\ &= F_1 + \sum_{1 \leq j \leq s} F_2^{(j)}, \end{aligned} \tag{6}$$

\* We have, in fact,

$$\prod_a (1 - a^{-1})^{-1} = \sum_c c^{-1}.$$

say. Now, by Lemmas 2, 3, and 5,

$$\begin{aligned} F_1 &\leq \sum_{n \leq x} n \leq x; \quad \left. \begin{array}{l} n+k_i=c_i m_i \\ (1 \leq i \leq s) \end{array} \right| x^\alpha < c_1 \dots c_s \leq x^{\beta s/(s-1)} \Big|_1 \\ &= \sum x^\alpha < c_1 \dots c_s \leq x^{\beta s/(s-1)} \Big| T(x; c_1, \dots, c_s) \\ &= O(x^{1-\alpha(1-\vartheta)+\epsilon}) + O(x^{\vartheta \beta s/(s-1)+\epsilon}). \end{aligned} \quad (7)$$

Furthermore,

$$\begin{aligned} F_2^{(j)} &\leq \sum_{n \leq x} n \leq x; \quad \left. \begin{array}{l} n+k_i=c_i m_i \\ (1 \leq i \leq s) \end{array} \right| c_1 \dots c_s / c_j > x^\beta \Big|_1 \\ &= \sum_{n \leq x} n \leq x; \quad \left. \begin{array}{l} n+k_i=c_i m_i \\ (1 \leq i \leq s; i \neq j) \end{array} \right| c_1 \dots c_s / c_j > x^\beta \Big| \sum_{c_j m_j = n+k_j} \Big|_1 \\ &\leq \sum_{n \leq x} n \leq x; \quad \left. \begin{array}{l} n+k_i=c_i m_i \\ (1 \leq i \leq s; i \neq j) \end{array} \right| c_1 \dots c_s / c_j > x^\beta \Big| d(n+k_j) \\ &= O\left(x^\epsilon \sum_{n \leq x} n \leq x; \quad \left. \begin{array}{l} n+k_i=c_i m_i \\ (1 \leq i \leq s; i \neq j) \end{array} \right| c_1 \dots c_s / c_j > x^\beta \Big|_1\right) \\ &= O\{x^\epsilon F(x; \beta; k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_s)\}. \end{aligned}$$

Hence, by the induction hypothesis,

$$F_2^{(j)} = O(x^{1-\beta(1-\vartheta)+\epsilon}) + O(x^{2\vartheta/(1+\vartheta)+\epsilon}). \quad (8)$$

Putting  $\beta = (s-1)/(s-1+\vartheta)$  we obtain, by (6), (7), and (8),

$$F(x; \alpha) = O(x^{1-\alpha(1-\vartheta)+\epsilon}) + O(x^{\vartheta s/(s-1+\vartheta)+\epsilon}) + O(x^{2\vartheta/(1+\vartheta)+\epsilon}),$$

and the lemma now follows since  $\vartheta s/(s-1+\vartheta) \leq 2\vartheta/(1+\vartheta)$  for  $s \geq 2$ .

5. We are now in a position to obtain our main result concerning the asymptotic behaviour of  $M(x)$ .

THEOREM 1.—If **A** satisfies (3) and (4), then

$$M(x) = \sigma x + O(x^{2\vartheta/(1+\vartheta)+\epsilon}).$$

Throughout this proof it is assumed that  $c'_1 \in \mathbf{C}_1, \dots, c'_s \in \mathbf{C}_s$ . For given values of  $c'_1, \dots, c'_s$  we clearly have  $\gamma(n+k_1) = c'_1, \dots, \gamma(n+k_s) = c'_s$  if, and only if,

$$c'_1 \mid (n+k_1), \dots, c'_s \mid (n+k_s); \quad \Omega\left(\frac{n+k_1}{c'_1}\right) \dots \Omega\left(\frac{n+k_s}{c'_s}\right) = 1.$$

Now we know [Mirsky, 1948, equation (14)] that

$$\Omega(n) = \sum_{c \mid n} \omega(c),$$

and therefore

$$\begin{aligned} M(x) &= \sum_{c'_1, \dots, c'_s} \left| \sum_{n \leq x} n \leq x; \quad \left. \begin{array}{l} c'_i \mid (n+k_i) \\ (1 \leq i \leq s) \end{array} \right| \Omega\left(\frac{n+k_1}{c'_1}\right) \dots \Omega\left(\frac{n+k_s}{c'_s}\right) \right| \\ &= \sum_{n \leq x} n \leq x; \quad \left. \begin{array}{l} n+k_i=c_i c'_i m_i \\ (1 \leq i \leq s) \end{array} \right| \omega(c_1) \dots \omega(c_s) \\ &= \Sigma_1 + \Sigma_2, \end{aligned} \quad (9)$$

say, where  $c_1 c'_1 \dots c_s c'_s \leq x$  in  $\Sigma_1$  and  $c_1 c'_1 \dots c_s c'_s > x$  in  $\Sigma_2$ . Now, by Lemmas 2, 4, 3, and 5,

$$\begin{aligned} \Sigma_1 &= \sum_{c_1 c'_1 \dots c_s c'_s \leq x} \omega(c_1) \dots \omega(c_s) T(x; c_1 c'_1, \dots, c_s c'_s) \\ &= \sigma x + O(x^{\vartheta+\epsilon}). \end{aligned} \quad (10)$$

Furthermore,

$$\begin{aligned}
 |\Sigma_2| &< \sum_{n < x; \substack{n+k_i=c_i c_i' m_i \\ (1 \leq i \leq s)}}; c_1 c_1' \dots c_s c_s' > x \Big|_1 \\
 &= \sum_{n < x; \substack{n+k_i=c_i^* m_i \\ (1 \leq i \leq s)}}; c_1^* \dots c_s^* > x \Big| \sum_{(1 \leq i \leq s)} \frac{c_i c_i' = c_i^*}{c_i^*} \Big|_1 \\
 &< \sum_{n < x; \substack{n+k_i=c_i^* m_i \\ (1 \leq i \leq s)}}; c_1^* \dots c_s^* > x \Big| d(c_1^*) \dots d(c_s^*) \\
 &= O\left(x^\epsilon \sum_{n < x; \substack{n+k_i=c_i^* m_i \\ (1 \leq i \leq s)}}; c_1^* \dots c_s^* > x \Big|_1\right) \\
 &= O\{x^\epsilon F(x; 1)\}.
 \end{aligned} \tag{11}$$

Hence, by Lemma 6,

$$\Sigma_2 = O(x^{2\theta/(1+\theta)+\epsilon}), \tag{12}$$

and the theorem now follows by (9), (10), and (12).

It is, perhaps, worth noticing that Theorem 1 is only of interest when  $s > 2$ . When  $s = 1$  it can be sharpened and the expression for the constant  $\sigma$  simplified. Indeed, if  $\mathbf{C}_0$  is a prescribed class of  $c$ -numbers, it is easy to show that the number  $M_0(x)$  of positive integers  $n < x$  such that  $\gamma(n) \in \mathbf{C}_0$  is given by

$$M_0(x) = \left( \sum_{c \in \mathbf{C}_0} \frac{1}{c} \right) \prod_a \left( 1 - \frac{1}{a} \right) x + O(x^{\theta+\epsilon}).$$

We may also note that any improvement made in the critical Lemma 6 will result in a sharpening of the error term in Theorem 1. For suppose the relation

$$F(x; 1) = O(x^{\theta'+\epsilon})$$

is valid for some  $\theta' < 2\theta/(1+\theta)$ . Then, by (9), (10), and (11), we obtain

$$M(x) = \sigma x + O(x^{\theta+\epsilon}),$$

where  $\theta = \max(\theta, \theta') < 2\theta/(1+\theta)$ ; this would clearly constitute an advance on Theorem 1 provided that  $\theta > 0$ .

6. Theorem 1 shows, in particular, that the set  $\mathbf{M}$  has the density  $\sigma$ , but it gives us no information whether this density is positive or zero. This question will be settled now.

**THEOREM 2.**—*If  $\mathbf{A}$  satisfies (3) and (4), then  $\mathbf{M}$  is either empty or has a positive density.*

In view of Theorem 1 it is clearly sufficient to show that if  $\mathbf{M}$  is not empty, then

$$M(x) > \alpha x \tag{13}$$

for some  $\alpha > 0$  (independent of  $x$ ) and all sufficiently large values of  $x$ . As (13) will be needed again in § 11, we note that it will be established without making use of (4), and replacing (3) by the weaker condition (1).

Throughout the proof  $x$  is taken to be sufficiently large. If  $\mathbf{M}$  is not empty, then there exist numbers  $c_1, \dots, c_s, n_0$  such that  $c_1 \in \mathbf{C}_1, \dots, c_s \in \mathbf{C}_s$ , and

$$\gamma(n_0 + k_1) = c_1, \dots, \gamma(n_0 + k_s) = c_s. \tag{14}$$

Take  $\xi > 0$  such that

$$\sum_{a > \xi} \frac{1}{a} < \frac{1}{2s}. \tag{15}$$

Denote by  $a_1, \dots, a_r$  the  $a$ -numbers not exceeding  $\xi$ , and write  $c_0 = c_1 \dots c_s a_1 \dots a_r$ . Let  $\mathbf{D}$  be the set of all numbers  $n$  given by  $n = n_0 + c_0 t$  ( $t = 0, 1, 2, \dots$ ). Also let, for  $1 \leq i \leq s$  and any  $a$ ,

$$S_i(x, a) = \sum_{n < x; n \in \mathbf{D}; a c_i | (n + k_i)} 1.$$

By (14) it follows that the congruence in  $t$ ,

$$c_0 t \equiv -n_0 - k_i \pmod{ac_i},$$

is soluble if, and only if,  $(ac_i, c_0) \mid c_i$ . Hence it is certainly insoluble for  $a < \xi$ , while for  $a > \xi$  its solutions, if any exist, form precisely one residue class  $\pmod{a}$ . But

$$S_i(x, a) = \sum_{0 \leq t \leq (x-n_0)/c_0; c_0 t \equiv -n_0 - k_i \pmod{ac_i}} 1,$$

and therefore

$$S_i(x, a) = 0 \quad (a < \xi); \quad (16)$$

$$S_i(x, a) \leq \frac{x}{ac_0} + 1 \quad (a > \xi). \quad (17)$$

Also trivially

$$S_i(x, a) = 0 \quad (a > x + k). \quad (18)$$

Using (14) we have

$$\begin{aligned} M(x) &> \sum_{n \leq x; n \in D; \left. \begin{array}{l} \gamma(n+k_i) = c_i \\ (1 \leq i \leq s) \end{array} \right| 1} 1 \\ &> \sum_{n \leq x; n \in D} 1 - \sum_a \sum_{1 \leq i \leq s} S_i(x, a), \end{aligned}$$

and therefore, by (16), (17), (18), and (15),

$$\begin{aligned} M(x) &> \frac{x-n_0}{c_0} - \sum \xi < a \leq x+k \left| \sum_{1 \leq i \leq s} \left( \frac{x}{ac_0} + 1 \right) \right| \\ &> \frac{x-n_0}{c_0} - \frac{x}{2c_0} - sA(x+k) > \frac{x}{3c_0}, \end{aligned}$$

since (1) implies  $A(x) = o(x)$ .

7. From now on we shall be concerned with the function  $N(x)$  defined in § 1. We begin by observing a simple consequence of Theorem 1.

**THEOREM 3.**—If  $\mathbf{A}$  satisfies (3) and (4), then

$$N(x) = \tau x + O(x^{2\theta/(1+\theta)+\epsilon}), \quad (19)$$

where

$$\tau = \prod_a (1 - D(a)a^{-1}).$$

The convergence of the product expressing  $\tau$  follows, of course, trivially by (1). To prove Theorem 3 we take each  $C_i$  in Theorem 1 to consist of the single number 1. We then obtain

$$N(x) = \tau' x + O(x^{2\theta/(1+\theta)+\epsilon}), \quad (20)$$

where

$$\tau' = \sum c_1, \dots, c_s \left| \omega(c_1) \dots \omega(c_s) \frac{E(c_1, \dots, c_s)}{\{c_1, \dots, c_s\}} \right|;$$

the identity  $\tau' = \tau$  can now be established by means of a generalization of Euler's identity for multiplicative functions (Mirsky, 1948, Lemma A), the argument being straightforward though somewhat lengthy. This argument can, however, be avoided, and the identity  $\tau' = \tau$  will follow at once by (20) and Theorem 4.

A particularly interesting special case of Theorem 3 arises when  $\mathbf{A}$  is taken to be the class of  $r$ th power of all primes, where  $r$  is any integer greater than 1.  $N(x)$  then becomes the number of  $n \leq x$  such that  $n+k_1, \dots, n+k_s$  are all  $r$ -free, and we obtain the asymptotic formula referred to in § 1 (Mirsky, 1947), namely

$$N(x) = \prod_p (1 - D(p)p^{-r})x + O(x^{2/(r+1)+\epsilon}).$$

8. Our next problem is to investigate the asymptotic behaviour of  $N(x)$  when (3) is replaced by the weaker condition (1). We shall see that the formula (19) continues to be valid provided that the error term is suitably modified.



When (1) is satisfied we shall write

$$H(x) = \sum_{a > x} \frac{1}{a}.$$

THEOREM 4.—If **A** satisfies (1) and (4), then

$$N(x) = \tau x + O\{xH(\frac{1}{4} \log x)\} + O(x^{\frac{1}{2}}).$$

Let  $y$  be a function of  $x$ , to be chosen later, such that  $y \rightarrow \infty$  as  $x \rightarrow \infty$ , and take  $x$  to be sufficiently large throughout the proof.

Some of the results found below will be used again in later sections when the conditions imposed on **A** have been varied; we shall, therefore, note under what conditions each result we deduce is valid.

First let **A** satisfy (1). We then have

$$\begin{aligned} 0 &< N(x, y) - N(x) < \sum n < x; \quad a \mid (n + k_i) [1 \leq i \leq s; \quad a > y] \mid 1 \\ &< \sum_{a > y} \sum_{1 \leq i \leq s} \sum n < x; \quad a \mid (n + k_i) \mid 1 \\ &= \sum y < a \leq x + k \left| \sum_{1 \leq i \leq s} \left\{ O\left(\frac{x}{a}\right) + O(1) \right\} \right| 1 \\ &= O(xH(y)) + O(A(x)). \end{aligned} \quad (21)$$

Moreover,

$$\begin{aligned} A(x) &= \sum_{n < x} \{H(n-1) - H(n)\} n < \sum_{0 < n < x} H(n) \\ &= \sum_{0 < n < y} H(n) + \sum_{y < n < x} H(n) = O(y) + O(xH(y)), \end{aligned}$$

and we therefore see by (21) that, if **A** satisfies (1), then

$$N(x) = N(x, y) + O(xH(y)) + O(y). \quad (22)$$

Next, let **A** be subject to (4). For a given  $a$  the numbers  $n$  such that  $a \nmid (n + k_i)$  ( $1 \leq i \leq s$ ) form precisely  $a - D(a)$  residue classes (mod  $a$ ). Hence, by Lemma 1, the numbers  $n$  such that  $a \nmid (n + k_i)$  ( $1 \leq i \leq s$ ;  $a < y$ ) form precisely  $\prod_{a < y} (a - D(a))$  residue classes (mod  $\prod_{a < y} a$ ). Therefore

$$\begin{aligned} N(x, y) &= x \prod_{a < y} (1 - D(a)a^{-1}) + O\left(\prod_{a < y} a\right) \\ &= x \prod_{a < y} (1 - D(a)a^{-1}) + O(y^{A(y)}). \end{aligned} \quad (23)$$

But, since  $1 \leq D(a) \leq s(a)$ , we have, if **A** satisfies (1),

$$\begin{aligned} \prod_{a > y} (1 - D(a)a^{-1})^{-1} &= \exp \left\{ \sum_{a > y} -\log (1 - D(a)a^{-1}) \right\} \\ &= \exp \left\{ \sum_{a > y} O\left(\frac{1}{a}\right) \right\} = \exp \{O(H(y))\} \\ &= 1 + O(H(y)), \end{aligned}$$

and therefore, by (22) and (23), we see that if **A** satisfies (1) and (4), then

$$N(x) = \tau x + O(xH(y)) + O(y^{A(y)}). \quad (24)$$

But, if **A** satisfies (4), then

$$A(y) \leq \pi(y) \leq 2y/\log y. \quad (25)$$

Hence, by (24), if **A** satisfies (1) and (4), then

$$N(x) = \tau x + O(xH(y)) + O(e^{2y}).$$

The assertion now follows if we take  $y = (\log x)/4$ .

We conclude this section by mentioning some special cases of the problem just considered.

If the class  $\mathbf{A}$  is finite, then the integers  $n$  such that  $n + k_1, \dots, n + k_s$  are all  $\mathbf{A}$ -free form precisely  $\prod_a (a - D(a))$  residue classes  $(\text{mod } \prod_a a)$ . Hence  $N(x)$  is given by

$$N(x) = \tau x + O(1).$$

Again, if  $\mathbf{A}$  satisfies not only (1) but also the stronger condition (3), then we obtain by partial summation from (3)

$$H(x) = O(x^{-1+\theta}).$$

Hence, by Theorem 4,

$$N(x) = \tau x + O\{x (\log x)^{-1+\theta}\}.$$

A slightly sharper result can be deduced from (24). For we then have

$$N(x) = \tau x + O(xy^{-1+\theta}) + O(y^{\alpha\theta}),$$

where  $\alpha$  is a number independent of  $x$ . Putting  $y = \beta (\log x / \log \log x)^{1/\theta}$ , where  $\alpha\beta^\theta < \theta/2$ , we obtain

$$N(x) = \tau x + O\{x (\log \log x / \log x)^{(1-\theta)/\theta}\}.$$

However, this formula is still inferior to (19). We therefore recognize that the method of § 5 can deal with fewer cases than the method of the present paragraph, but that when it is applicable it yields much sharper results.

9. We already know by Theorem 2 that when  $\mathbf{A}$  satisfies (3) and (4)  $\mathbf{N}$  is either empty or has a positive density. This result will now be extended to the case when (3) is replaced by (1), and we shall, in fact, obtain a criterion for deciding whether  $\mathbf{N}$  is empty or not.

**THEOREM 5.**—Let  $\mathbf{A}$  satisfy (1) and (4). If, for every  $a$ , there exists some  $n$  such that

$$n + k_i \not\equiv 0 \pmod{a} \quad (1 \leq i \leq s), \quad (26)$$

then  $\mathbf{N}$  has a positive density. If, on the other hand, (26) cannot be satisfied for some  $a$ , then  $\mathbf{N}$  is empty.

It is clear that a necessary condition for  $\mathbf{N}$  to be non-empty is that (26) should be satisfied for every  $a$  and some  $n$ ; Theorem 5 shows that this obviously necessary condition is also sufficient.

The proof is now almost trivial. If (26) is satisfied for every  $a$  and some  $n$ , then clearly  $D(a) < a(a)$ , and therefore  $\tau > 0$ . Hence, by Theorem 4,  $\mathbf{N}$  has a positive density. On the other hand, if (26) cannot be satisfied for some  $a$ , then  $\mathbf{N}$  is obviously empty.

10. If condition (1) is replaced by (2), i.e. if it is assumed that  $\sum_a a^{-1}$  diverges, the problem of estimating  $N(x)$  naturally becomes much more difficult, and we are at present only able to obtain an upper estimate of  $N(x)$ .

We shall write  $V(x) = \sum_{a \leq x} \frac{1}{a}$ .

**THEOREM 6.**—If  $\mathbf{A}$  satisfies (2) and (4), then

$$N(x) = O\{x \exp(-sV(\log x))\}.$$

Denoting by  $y$  a function of  $x$  to be chosen later, we have, by (23) and (25),

$$N(x) \leq N(x, y) = x \prod_{a < y} (1 - D(a)a^{-1}) + O(e^{2y}).$$

But  $D(a) = s$  for  $a > k$ , and so

$$\begin{aligned} N(x) &= O\left\{x \prod_{k < a < y} \left(1 - \frac{s}{a}\right)\right\} + O(e^{2y}) \\ &= O\{x \exp(-sV(y))\} + O(e^{2y}). \end{aligned}$$

Furthermore,

$$\begin{aligned} V(4y) &= \sum_{a \leq y} \frac{1}{a} + \sum_{y < a \leq 4y} \frac{1}{a} < V(y) + \sum_{y < n \leq 4y} \frac{1}{n} \\ &= V(y) + O(1). \end{aligned}$$

Therefore

$$N(x) = O\{x \exp(-sV(4y))\} + O(e^{2y}),$$

and putting  $y = (\log x)/4$  we obtain

$$N(x) = O\{x \exp(-sV(\log x))\} + O(x^{\frac{1}{2}}).$$

But

$$V(x) < \sum_{n \leq x} \frac{1}{n} < 2 \log x,$$

and so,

$$x \exp(-sV(\log x)) > x \exp(-2s \log \log x) = x (\log x)^{-2s} > x^{\frac{1}{2}}.$$

This completes the proof.

We observe the following consequences of Theorem 6, valid for a set  $\mathbf{A}$  satisfying (4):—

$$\text{If } A(x) > ax/\log x \text{ for } x > \beta, \quad \text{then } N(x) = O\{x (\log \log x)^{-sa}\}. \quad (27)$$

$$\text{If } A(x) \sim ax/\log x, \quad \text{then } N(x) = O\{x (\log \log x)^{-sa+\epsilon}\}. \quad (28)$$

$$\text{If } A(x) = ax/\log x + O(x/\log^2 x), \quad \text{then } N(x) = O\{x (\log \log x)^{-sa}\}. \quad (29)$$

To prove these results we note that if  $A(x) > ax/\log x$  for  $x > \beta$ , then

$$\begin{aligned} V(x) &= \sum_{n \leq x} \frac{A(n) - A(n-1)}{n} = \sum_{n \leq x} \frac{A(n)}{n(n+1)} + \frac{A(x)}{[x]+1} \\ &> \sum_{\beta < n \leq x} \frac{a}{(n+1) \log n} = a \log \log x + O(1). \end{aligned} \quad (30)$$

Hence (27) follows by Theorem 6, and (28) is an immediate consequence of (27). Again, if  $A(x) = ax/\log x + O(x/\log^2 x)$ , then, by (30),  $V(x) = a \log \log x + O(1)$ , and (29) follows by Theorem 6.

As a final application of Theorem 6 we note that if  $\mathbf{A}$  is the set of all primes congruent to  $l \pmod{q}$ , where  $(l, q) = 1$ , then

$$N(x) = O\{x (\log \log x)^{-s/\phi(q)}\}.$$

This follows in view of the well-known relation

$$\sum_{\substack{p \leq x \\ p \equiv l \pmod{q}}} \frac{1}{p} = \frac{1}{\phi(q)} \log \log x + O(1).$$

11. So far we have always assumed that any two  $a$ -numbers are coprime. In this section we shall drop this restriction but shall suppose that  $\sum_a a^{-1}$  converges. What can then be said about  $\mathbf{N}$ ? This question is a generalization of the question considered in §§ 8–9, but since our present restrictions on  $\mathbf{A}$  are very much less severe only a weaker result can be expected. We shall, in fact, find an asymptotic formula for  $N(x)$ , but we shall not be able to obtain any estimation of the error term in this formula.

The corresponding problem when condition (1) is also dispensed with is even more difficult. Besicovitch (1934) has shown that in that case  $\mathbf{N}$  need not possess a density. In the opposite direction it was proved by Davenport and Erdős (1936) that for  $s = 1$   $\mathbf{N}$  must have a logarithmic density. It would be interesting to extend their result to any values of  $s$ .

**THEOREM 7.**—*If  $\mathbf{A}$  satisfies (1), then  $\mathbf{N}$  is either empty or has a positive density.*

Let  $y$  be a function of  $x$ , to be chosen later, such that  $y \rightarrow \infty$  as  $x \rightarrow \infty$ . We have, by (21) and (1),

$$N(x) = N(x, y) + o(x). \quad (31)$$

Let  $r=r(x)$  be the number of  $a$ -numbers not exceeding  $y$ , and let these numbers be denoted by  $a_1, \dots, a_r$ . Furthermore, let  $\tau_r$  be the density of the set of positive integers  $n$  such that

$$a_i \nmid (n+k_j) \quad (1 \leq i \leq r; 1 \leq j \leq s).$$

By Lemma 1 this set consists of certain residue classes (mod  $\{a_1, \dots, a_r\}$ ), and therefore

$$N(x, y) = \tau_r x + O(a_1 \dots a_r). \quad (32)$$

Choosing  $y$  such that  $a_1 \dots a_r = o(x)$  we have, by (31) and (32),

$$N(x) = \tau_r x + o(x). \quad (33)$$

Now clearly  $\tau_r \geq \tau_{r+1} > o(r > 1)$ . Therefore  $\lim_{r \rightarrow \infty} \tau_r$  exists. Denoting it by  $\tau^*$ , we have by (33), since  $r \rightarrow \infty$  as  $x \rightarrow \infty$ ,

$$N(x) \sim \tau^* x. \quad (34)$$

Finally, we make use of (13) in the case when each  $C_i$  consists of a single number 1. Hence, if  $A$  satisfies (1) and  $N$  is not empty, then  $N(x) > ax$  for some  $a > 0$  (independent of  $x$ ) and all sufficiently large values of  $x$ . It therefore follows that either  $N$  is empty or (34) holds with some  $\tau^* > 0$ .

It is worth observing that the inequality  $\tau^* > 0$  can be proved by a very simple and direct argument for  $s=1$ . In that case

$$\tau_r = \sum \delta_1, \dots, \delta_r = 0, 1 \left| \frac{(-1)^{\delta_1 + \dots + \delta_r}}{\{a_1^{\delta_1}, \dots, a_r^{\delta_r}\}} \right|$$

and therefore, by an inequality proved independently by Heilbronn (1937) and Rohrbach (1937), we have

$$\tau_r > \prod_{1 \leq i \leq r} \left( 1 - \frac{1}{a_i} \right).$$

Hence

$$\tau^* = \lim_{r \rightarrow \infty} \tau_r > \prod_a \left( 1 - \frac{1}{a} \right) > 0.$$

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**L.—Quantum Theory of Rest-Masses.\*** By **M. Born, F.R.S.,** and **H. S. Green.**  
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 (With Two Text-figures.)

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# 1. INTRODUCTION

It has been acknowledged for a long time that current quantum theory is incomplete. The difficulties and unanswered problems which have gradually become apparent during the development of the theory will not be discussed here, and only one aspect of the situation will be mentioned, that there seems to exist a large number of particles with different rest-masses, the numerical values of which demand a theoretical explanation. The experimental material has recently been greatly increased by the discovery of several kinds of mesons with different rest-masses.

It is well known that the mass  $\mu$  of a particle can be replaced by an equivalent length which is given by the formula  $l = \frac{\hbar}{\mu c}$ , where  $\hbar$  is Planck's constant and  $c$  the velocity of light.

Instead of speaking of different rest-masses, one can therefore say that each particle has a characteristic length which may be a numerical multiple of a certain absolute length  $a$ . The concept of such an absolute length has been suggested and developed by Fürth (1929), Born (1934, 1938 *a*), Heisenberg (1938, 1943), March (1938, 1947) and Snyder (1947) among others. However, the introduction of the absolute length has led to difficulties in connection with relativistic invariance, which is essential for any theory to be applied to fast particles. Some new principle is clearly required, which, to conform with the tendencies of modern physics, must be expressed by postulating the invariance of the fundamental laws under some kind of transformation.

Invariance under one such transformation is already apparent in the accepted laws of physics. The fundamental classical laws were expressed by the formulæ

$$\dot{x}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial x_k}, \quad (1.1)$$

giving the time variation of the co-ordinates  $x_k$  and associated momenta  $p_k$  in terms of derivatives of the Hamiltonian function  $H$ . These laws remain unchanged if  $x_k$  is replaced by  $p_k$  and  $p_k$  by  $-x_k$ . The same symmetry appears in the fundamental commutation laws

$$\left. \begin{aligned} p_k x_l - x_l p_k &= i\hbar \delta_k^l, & \delta_k^l &= \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases} \\ p_k &\equiv \left( \mathbf{p}, \frac{E}{c} \right), & x_l &\equiv (\mathbf{x}, ct) \end{aligned} \right\} \quad (1.2)$$

of relativistic quantum mechanics, in the Fourier transformation connecting wave functions and representatives of dynamical variables in the co-ordinate and momentum representations, and also in the formal expression

$$m_{kl} = x_k p_l - x_l p_k \quad (1.3)$$

for the important angular momentum tensor. If one introduces the fundamental length  $a$  and the corresponding momentum  $b$ , then Planck's constant is expressed as the symmetric product  $\hbar = ab$ . When  $a$  and  $b$  are used as units of distance and momentum, the equations (1.2) and (1.3) can be written in a simple dimensionless form.

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All these considerations strongly suggest that it should be possible to represent every fundamental physical law in this dimensionless symmetric way, but this has never been universally accepted as a general principle. It was, in fact, suggested some time ago by one of the authors (Born, 1938) that the symmetry between co-ordinates and momenta has a deeper significance than generally appreciated, but attempts to obtain new results from this "Principle of Reciprocity" (Landé, 1939; Born, 1939; Born and Fuchs, 1940, 1941; Fuchs, 1940, 1941; Sarginson, 1941) led to nothing of practical importance.

It has indeed become clear that a change of attitude towards quantum theory is required. The current quantum theory proceeds from a set of field equations for each kind of particle, which can be derived from the Hamiltonian  $H$  or, better, by a variational principle from the Lagrangian  $L$  for the corresponding field. The form of the Lagrangian is determined partly by application of the correspondence principle from classical mechanics, and partly by making use of quantum-mechanical considerations, such as spin, derived from observation. For every kind of particle one has therefore an essentially empirical Lagrangian function  $L$ , the structure of which indicates the spin, while the only numerical constant appearing is the rest-mass. The problem to be solved has hitherto been that of finding the wave function  $\psi$ , which determines the physical characteristics of the particle. As soon as the rest-masses themselves become the object of interest, however, this standpoint is inadequate, and the new problem consists in the finding of the function  $L$  itself from a general principle. The object of this paper is to show that the principle of reciprocity is a powerful tool for restricting the choice of  $L$  and producing those Lagrangians which correspond to observed particles.

This being accepted, the whole of quantum mechanics separates into two distinct fields: the first the determination of the different possible Lagrangians and the corresponding rest-masses for free particles (and the coupling between them, which will be given only slight consideration here); the second the derivation of the wave functions from the Lagrangians for a given experimental arrangement.

Before the first step could be taken a considerable amount of work has had to be done, of which the beginnings appear in another paper (Green, 1949), but which will be developed here in full. The Lagrangians which are obtained by the principle of reciprocity have a very general form, containing all derivatives of the wave function with respect to space and time. Such Lagrangians have been considered by Chang (1946, 1948) and de Wet (1948), whose calculations are, however, so complicated that it is impossible to apply them in practice. A definite meaning has to be given to the particle density, energy density, and the corresponding flux densities, and second quantisation applied so as to derive the particle aspect of the wave field.

In the first section following it is shown how the Lagrangians are to be derived from the Principle of Reciprocity, and this is in turn followed by the general field theory and second quantisation procedure which are independent of the actual Lagrangians used. The method is then applied in detail to the determination of the masses of particles with spin zero or one, and it is shown that among the most stable particles occur some which have masses in close agreement with experimental determinations of the meson masses. There exist also particles of spin one and zero rest-mass which may be interpreted as photons. Finally, in the first appendix, particles with spin half are considered, and it is found that they include mesons, as well as particles with vanishing rest-mass which may be interpreted as electrons or neutrinos, according as they are charged or not. The view is taken that the mass of the electron is wholly electromagnetic in origin, in conformity with the most recent expectations of quantum electrodynamics.

## 2. SELF-RECIPROCAL WAVE OPERATORS

Throughout this paper the notation of the theory of relativity is used, so that a relativistic affix  $k$  or  $l$  assumes the values 1 to 4, and is summed from 1 to 4 in any expression in which it occurs repeated. Also, for any four-vector

$$\begin{aligned} z_k &\equiv (z_1, z_2, z_3, z_4), \\ z^l &= g^{lk} z_k, \quad z_k = g_{kl} z^l, \\ g^{lk} &= g_{kl} = 0 \quad (k \neq l), \quad = 1 \quad (k = l = 4), \quad = -1 \quad (k = l = 1, 2, 3), \end{aligned} \quad (2.1)$$

so that  $g_{kl}$  is the metric tensor of Galilean space-time. It will usually be convenient to confine attention to a finite volume  $\Omega$  of real space, so that each component of the momentum of a particle has the discrete proper-values  $2\pi n\hbar\Omega^{-\frac{1}{3}}$ , where  $n$  is an integer, but the extension to infinite space when  $\Omega \rightarrow \infty$  is trivial.

Commencing with the consideration of particles, like the photon and certain kinds of meson, which have integral spin, the generally accepted wave equation for these particles has the form

$$\left(\frac{E^2}{c^2} - \mathbf{p}^2\right)\psi(x_k) = \mu^2 c^2 \psi(x_k), \quad \left(\mathbf{p}, \frac{E}{c}\right) \equiv p_k = i\hbar \frac{\partial}{\partial x^k}, \quad x_k \equiv (\mathbf{x}, ct), \quad (2.2)$$

where  $\psi(x_k)$  is the wave function, which may be a scalar or a vector, and  $\mu$  is the appropriate rest-mass. By expressing  $x_k$  in units of a fundamental length  $a$ , which may be taken to be the classical radius of the electron, and  $p_k$  in units of the corresponding fundamental momentum  $\delta = \hbar/a$ , the equations (2.2) are reduced to the dimensionless form

$$p_k p^k \psi(x_l) = \kappa^2 \psi(x_l), \quad p_k = i \frac{\partial}{\partial x^k}, \quad (2.3)$$

where

$$\kappa = \frac{\mu c}{\delta} = \frac{\mu c a}{\hbar} \quad (2.4)$$

in terms of the customary units. The most general solution of (2.3) is known to be

$$\psi(x_l) = \Omega^{-\frac{1}{2}} \sum_{p_i'} \psi(p_i') e^{-ip_i' x^k}, \quad (2.5)$$

where the  $p_i'$  are constants connected by the relation  $p_i' p^{i'} = \kappa^2$ , and the  $\psi(p_i')$  are also (scalar or vector) constants. The same wave function (2.5), however, also satisfies the equation

$$F(p_i p^i) \psi(x_k) = 0, \quad (2.6)$$

where  $F(p_i p^i)$  is any function of the form

$$F(p_i p^i) = F_1(p_i p^i) (p_k p^k - \kappa^2), \quad (2.7)$$

and if  $F_1(z) = 0$  has no root, (2.5) is the only solution. If, however,  $F_1(p_i p^i)$  is itself of the form  $F_2(p_i p^i) (p_i p^i - \kappa_1^2)$ , then the equation (2.6) will have solutions corresponding to particles of rest-mass  $\mu_1 = \frac{\hbar \kappa_1}{ca}$  also; and it can be seen quite generally that a wave equation of the type (2.6) may have solutions corresponding to particles with any number of different rest-masses.

The wave equation

$$a_k p^k \psi(x_l) = \kappa \psi(x_l), \quad a_k \alpha_l + a_l \alpha_k = 2g_{kl}, \quad (2.8)$$

of the electron, or any other particle with spin half, can in a similar way be shown to have solutions satisfying the equation

$$F(a_k p^k) \psi(x_l) = 0, \quad (2.9)$$

where  $F(a_k p^k)$  is any function with a factor  $a_k p^k - \kappa$ ; and, though this is not necessary, (2.9) may have solutions corresponding to particles with different rest-masses from those originally considered. In the same way, the equation

$$F(p_k) \psi(x_l) = 0 \quad (2.10)$$

for any spin may characterise particles either all with the same rest-mass or with many different rest-masses.

It is obvious that the choice of the function  $F(p_k)$  is *a priori* very arbitrary. For any given spin, the requirement that it should be relativistically invariant imposes a considerable limitation on the form of  $F$ , but some further restriction is still required in order that it should provide a description of the particles which are actually observed. Such a restriction is provided by the Principle of Reciprocity, which imposes on  $F$  the condition that it should be

reciprocally invariant as well as relativistically invariant. A scalar reciprocal invariant is defined as a function which satisfies the equation

$$S(x_k, p_k)F(x_i) = sF(x_i), \quad (2.11)$$

where

$$S(x_k, p_k) = S(p_k, -x_k). \quad (2.12)$$

From this symmetrical property  $S\left(x_k, i\frac{\partial}{\partial x^k}\right) = S\left(p_k, i\frac{\partial}{\partial p^k}\right)$  of what will be called the *metric operator*  $S$ , it follows that the equation (2.11) will still be satisfied when  $F(x_i)$  is replaced by  $F(p_i)$ .

To demonstrate the connection of this definition of reciprocal invariance with Born's original definition (1938) in terms of the Fourier transformation, it is sufficient to consider the equation

$$F(x) = (2\pi)^{-\frac{1}{2}} \int F(p) e^{-ipx} dp \quad (2.13)$$

defining a reciprocally invariant function of one variable  $x$ . If  $F_m(x)$  is any solution, one has

$$\begin{aligned} p F_m(x) &= i \frac{\partial}{\partial x} F_m(x) = (2\pi)^{-\frac{1}{2}} \int \{p F_m(p)\} e^{-ipx} dp; \\ (2\pi)^{-\frac{1}{2}} \int \{x F_m(p)\} e^{-ipx} dp &= - (2\pi)^{-\frac{1}{2}} \int \left\{ i \frac{\partial}{\partial p} F_m(p) \right\} e^{-ipx} dp = -x F_m(x), \end{aligned} \quad (2.14)$$

by integration by parts. Hence

$$\begin{aligned} S(x, p) F_m(x) &= (2\pi)^{-\frac{1}{2}} \int \{S(x, p) F_m(p)\} e^{-ipx} dp \\ &= (2\pi)^{-\frac{1}{2}} \int \{S(p, -x) F_m(p)\} e^{-ipx} dp, \end{aligned} \quad (2.15)$$

on the assumption that  $S$  satisfies (2.12), *i.e.*

$$S\left(x, i\frac{\partial}{\partial x}\right) F_m(x) = (2\pi)^{-\frac{1}{2}} \int \left\{ S\left(p, i\frac{\partial}{\partial p}\right) F_m(p) \right\} e^{-ipx} dp.$$

From this it is evident that  $S\left(x, i\frac{\partial}{\partial x}\right) F_m(x)$  is a linear combination of the solutions  $F_n(x)$  of (2.13),

$$S(x, p) F_m = \sum_n S_{mn} F_n, \quad (2.16)$$

or, making the matrix  $S_{mn}$  diagonal,

$$S(x, p) F = s F.$$

The theorem just proved shows that for the determination of  $F$  it is sufficient to proceed from the simplest relativistically invariant function satisfying (2.12), which is

$$S = x_k x^k + p_k p^k. \quad (2.17)$$

This may be used to determine the scalar, vector and tensor reciprocal invariants. As shown by Dr K. C. Cheng in Appendix I to this paper, for the determination of spinor and other more complex reciprocal invariants, the appropriate metric operator  $S$  may be defined to satisfy

$$S(x_k, p_k) = T^{-1} S(p_k, -x_k) T, \quad (2.18)$$

where  $T$  is a unitary operator not involving  $x_k$  or  $p_k$ . Multiplying (2.11) by  $T$  before and  $T^{-1}$  afterwards, one then obtains

$$S\left(i\frac{\partial}{\partial x^k}, -x_k\right) T F(x_i) T^{-1} = T F(x_i) T^{-1} S, \quad (2.19)$$



or, replacing  $x_k$  by  $-\dot{p}_k$  throughout,

$$S(x_k, \dot{p}_k)TF(\dot{p}_l)T^{-1} = TF(\dot{p}_l)T^{-1}S. \quad (2.20)$$

The reciprocal of  $F(x_k)$  is then  $TF(\dot{p}_k)T^{-1}$ .

The reciprocally invariant wave operators  $F$  will be obtained by the detailed solution of the equation (2.11) in the later sections of this paper. Before this is done, it is necessary to inquire more closely into the observational properties of systems described by wave equations of the type (2.10). In particular, proof must be given that the modification of the original wave equation (2.2) has not invalidated the interpretation of the wave field in terms of an assembly of particles with different rest-masses given by the factors of the wave operator in the way suggested by the early considerations of this section. Now the properties of wave fields are most conveniently derived from the Lagrangian function, and it is important, therefore, to be able to construct this function from a knowledge of the form of  $F(\dot{p}_k)$ . For linear fields of the type here considered, the Lagrangian is a linear combination of the *statistical operator*

$$\rho(x_k, x_k') = \psi(x_k)\psi^*(x_k'), \quad (2.21)$$

and its derivatives of all orders with respect to  $x_k$  and  $x_k'$ . Writing for brevity  $\dot{p}^m = \dot{p}_1^{m_1}\dot{p}_2^{m_2}\dot{p}_3^{m_3}\dot{p}_4^{m_4}$ , etc., the Lagrangian may therefore be represented in the form

$$\left. \begin{aligned} L(x_k, x_k') &= \sum_{mn} c_{mn} \dot{p}^m \dot{p}'^n \rho(x_k, x_k') = F(\dot{p}_l, \dot{p}_l') \rho(x_k, x_k'), \\ \dot{p}_k &= i \frac{\partial}{\partial x_k}, \quad \dot{p}_k' = -i \frac{\partial}{\partial x_k'} \end{aligned} \right\} \quad (2.22)$$

The field equations result from the variational principle

$$\delta \iint L(x_k, x_k') d\Omega dt = 0,$$

and by the well-known procedure one obtains easily from (2.22),

$$\sum c_{mn} \dot{p}^{m+n} \psi(x_k) = F(\dot{p}_l, \dot{p}_l) \psi(x_k) = 0. \quad (2.23)$$

It is therefore clear that the function  $F(\dot{p}_k)$  of (2.10) must be identical with  $F(\dot{p}_k, \dot{p}_k)$ , where  $F(\dot{p}_k, \dot{p}_k')$  is defined by (2.22).

The construction of the Lagrangian function from the wave operator  $F(\dot{p}_k)$  is now obviously not a unique procedure; however, it will appear in the next section that the observable properties of the field do not depend on the way in which it is done. For  $F = F(\dot{p}_k, \dot{p}_k)$  and  $F = F(a_k \dot{p}_k)$ , the most convenient choice is to take  $L = F(\dot{p}_k, \dot{p}_k) \rho$  and  $L = F\{\frac{1}{2} a_k (\dot{p}_k + \dot{p}_k')\} \rho$  respectively.

### 3. THE FIELD THEORY

In order to ascertain the observational properties of fields described by equations of the type  $F(\dot{p}_k) \rho = 0$ , which result from multiplying (2.23) by  $\psi^*(x_k')$ , it is necessary to investigate the field theory and second quantisation of fields whose Lagrangian operators have the general form

$$L(x_k, x_k') = F(\dot{p}_k, \dot{p}_k') \rho(x_k, x_k') \quad (3.1)$$

suggested by (2.22). If the operation of setting  $x_k' = x_k$  is denoted by the brackets  $\langle \rangle$ , then  $\langle L \rangle$  is the ordinary Lagrangian density of the field. Although the statistical operator has been defined by (2.21), there are grounds, suggested by the work of Dirac (1942), for supposing that  $\rho$  in general is not positive definite, so that  $\rho(x_k, x_k')$  is henceforth regarded as an arbitrary function of  $x_k$  and  $x_k'$ , subject only to the Hermitian property  $\rho^*(x_k, x_k') = \rho(x_k', x_k)$ .

The method of Heisenberg and Pauli (1929) fails to quantise fields for which  $F$  is other than a linear function of  $\dot{p}_k$  and  $\dot{p}_k'$ , and that of Chang (1946, 1948) or de Wet (1948), though applicable in principle to any polynomial form of  $F$ , is very cumbersome if at all applicable to the transcendental functions at present under consideration. The method here adopted

is a development of the procedure proposed by one of the authors (Green, 1949) in another paper.

If  $G^k$  and  $G^{k'}$  are defined by

$$\left. \begin{aligned} F(\dot{p}_k', \dot{p}_k') - F(\dot{p}_k, \dot{p}_k') &= \pi_k G^k(\dot{p}_i, \dot{p}_i'), & \pi_k &= \dot{p}_k' - \dot{p}_k \\ F(\dot{p}_k, \dot{p}_k) - F(\dot{p}_k, \dot{p}_k') &= \pi_k' G^{k'}(\dot{p}_i, \dot{p}_i'), & \pi_k' &= \dot{p}_k - \dot{p}_k' \end{aligned} \right\} \quad (3.2)$$

then the field equations  $F(\dot{p}_k', \dot{p}_k')\rho = 0$ ,  $F(\dot{p}_k, \dot{p}_k)\rho = 0$ , become

$$(F + \pi_k G^k)\rho = 0, \quad (F + \pi_k' G^{k'})\rho = 0. \quad (3.3)$$

Subtracting the second of the equations (3.3) from the first, one obtains, since  $\pi_k + \pi_k' = 0$ , the equation of continuity

$$\pi_k (G^k + G^{k'})\rho = 0 \quad (3.4)$$

in operational form, or  $\frac{\partial}{\partial x^k} \langle (G^k + G^{k'})\rho \rangle = 0$  in the more customary notation, by setting  $x_k' = x_k$ . It will be observed that if  $A(x_k, x_k')$  is any operator, then

$$\langle \pi_k A \rangle = -i \left\langle \left( \frac{\partial}{\partial x^{k'}} + \frac{\partial}{\partial x^k} \right) A(x_i, x_i') \right\rangle = -i \frac{\partial}{\partial x^k} \langle A \rangle.$$

According to (3.4), the four-vector

$$R^k = \frac{1}{2} (G^k + G^{k'})\rho \quad (3.5)$$

is to be interpreted as the density-density flux vector. If the flux across a surface at infinity vanishes, it follows also from (3.4) that

$$\frac{dN}{dt} = 0, \quad N = \int \langle R^4 \rangle d\Omega, \quad (3.6)$$

where the integration extends over all real space  $\Omega$ . This shows that the total amount of matter  $N$  is conserved.

Next, by multiplying the first of the equations (3.3) by  $\dot{p}_i$  and the second by  $\dot{p}_i'$  before subtraction, one has

$$\pi_k (-F\delta_i^k + G^k \dot{p}_i + G^{k'} \dot{p}_i')\rho = 0, \quad (3.7)$$

which may be interpreted as the equation of conservation of energy in operational form, the canonical energy-energy flux tensor being defined by

$$L_i^k = \frac{1}{2} (-F\delta_i^k + G^k \dot{p}_i + G^{k'} \dot{p}_i')\rho. \quad (3.8)$$

The Hamiltonian energy-momentum vector is defined as

$$P_k = \int \langle L_k^4 \rangle d\Omega. \quad (3.9)$$

It may be noticed that the momentum density  $L_k^4$  so defined does not coincide with the energy flux  $g_{ki} L_4^i$  unless  $L_{ki}$  is symmetrical; it will be found, however, that this condition is satisfied in the application to meson fields envisaged. It will now be shown that

$$P_k = \int \langle \frac{1}{2} (\dot{p}_k + \dot{p}_k') R^4 \rangle d\Omega. \quad (3.10)$$

For  $k=4$  this follows from the fact, seen from the field equations (3.3), that

$$F\rho = \frac{1}{2} (\dot{p}_k - \dot{p}_k') (G^k - G^{k'})\rho, \quad (3.11)$$

so  $\langle L_4^4 \rangle$  differs from  $\langle \frac{1}{2} (\dot{p}_4 + \dot{p}_4') R^4 \rangle$  only by a term

$$\frac{1}{4} \sum_{k=1}^3 \langle \pi_k (G^k - G^{k'})\rho \rangle = -\frac{1}{4} i \sum_{k=1}^3 \frac{\partial}{\partial x^k} \langle (G^k - G^{k'})\rho \rangle,$$

which vanishes on integration over  $\Omega$ . For  $k \neq 4$ ,  $\langle L_k^4 \rangle$  differs from  $\langle \frac{1}{2} (\dot{p}_k + \dot{p}_k') R^4 \rangle$  by a term  $\frac{1}{4} \langle \pi_k (G_4 - G_4')\rho \rangle$ , which equally vanishes on integration over  $\Omega$ .

To solve the field equations (3.3), one may substitute for  $\rho$  the form

$$\Omega^{-1} \rho_{ij}(\mathbf{p}, \mathbf{p}') \exp \{i(\mathbf{p} \cdot \mathbf{x} - E_i t - \mathbf{p}' \cdot \mathbf{x}' + E_j t')\}, \quad (3.12)$$

where  $\rho_{ij}(\mathbf{p}, \mathbf{p}')$ ,  $\mathbf{p}$ ,  $\mathbf{p}'$ ,  $E_i$ ,  $E_j'$  are constants; then  $E_i$  and  $E_j'$  are solutions of the equations

$$\left. \begin{aligned} F(p_k^{(i)}, p_k^{(i)}) &= 0, & F(p_k^{(j)'}, p_k^{(j)'}) &= 0, \\ p_k^{(i)} &\equiv (\mathbf{p}, E_i), & p_k^{(j)'} &= (\mathbf{p}', E_j'). \end{aligned} \right\} \quad (3.13)$$

There will in general be several roots  $E_i$  and  $E_j'$  of these equations, and the most general solution of the field equations is then obtained by summing over all possible values of  $i, j, \mathbf{p}$  and  $\mathbf{p}'$ . If this solution is substituted in (3.5) and (3.6), one obtains

$$\left. \begin{aligned} N &= \sum_i N_i, & N_i &= \sum_{\mathbf{p}} n_i(\mathbf{p}), \\ n_i(\mathbf{p}) &= \Gamma_i(\mathbf{p}) \rho_{ii}(\mathbf{p}, \mathbf{p}), \\ \Gamma_i(\mathbf{p}) &= \frac{1}{2} \{ G^4(p_k^{(i)}, p_k^{(i)}) + G^{4'}(p_k^{(i)}, p_k^{(i)}) \}. \end{aligned} \right\} \quad (3.14)$$

Terms involving  $\rho_{ij}(\mathbf{p}, \mathbf{p}')$  with different values of  $\mathbf{p}$  and  $\mathbf{p}'$  obviously vanish on integration over  $\Omega$ , and those corresponding to different values of  $i$  and  $j$  must also disappear because according to (3.6)  $N$  does not depend on time. From (3.10) one obtains similarly

$$P_k = \sum_i P_k^{(i)}, \quad P_k^{(i)} = \sum_{\mathbf{p}} n_i(\mathbf{p}) p_k^{(i)}. \quad (3.15)$$

Turning now to the special example of the meson wave field, where  $F$  may be taken to be a function  $F(p_k, p^k)$  of  $p_k, p^k$  only, one sees from (3.13) that  $E_i = (\kappa_i^2 + \mathbf{p}^2)^{\frac{1}{2}}$ ,  $E_j' = (\kappa_j^2 + \mathbf{p}'^2)^{\frac{1}{2}}$ , where the  $\kappa_i$  are the roots of the equation  $F(\kappa_i^2) = 0$ . To calculate  $G^k$ , define a function  $G(z, z')$  by means of

$$\left. \begin{aligned} G(z, z') &= \frac{F(z) - F(z')}{z - z'} = G(z', z), \\ G(z, z) &= F'(z) = \frac{dF(z)}{dz}; \end{aligned} \right\} \quad (3.16)$$

then it follows from (3.2) that

$$\left. \begin{aligned} \pi_k G^k &= (p_k' p^{k'} - p_k p^k) G(p_k' p^{k'}, p_k p^k), \\ G^k &= p^{k'} G(p_k' p^{k'}, p_k p^k); \end{aligned} \right\} \quad (3.17)$$

so, according to (3.14),

$$\Gamma_i(\mathbf{p}) = E_i F'(\kappa_i^2). \quad (3.18)$$

When the function  $F(p_k, p^k)$  has been determined, by substituting into (3.18), one obtains immediately the total amount of matter, momentum and energy present in the field from (3.14) and (3.15). This latter equation justifies the interpretation of  $n_i(\mathbf{p})\mathbf{p}$  and  $n_i(\mathbf{p})E_i$  as the momentum and energy respectively associated with the amount of matter  $n_i(\mathbf{p})$  in the field, but the fact that  $n_i(\mathbf{p})$  is an integer cannot be inferred until second quantisation has been effected in the next section.

The example of the spinor field may be treated similarly.  $F$  is then regarded as a function of  $\frac{1}{2} \alpha_k (p^k + p^{k'})$  alone. Defining  $G(z, z')$  again by (3.16), one obtains instead of (3.17),

$$\left. \begin{aligned} \pi_k G^k &= \frac{1}{2} \alpha_k (p_k' - p^k) G\{\alpha_k p^{k'}, \frac{1}{2} \alpha_k (p_k + p^{k'})\}, \\ G^k &= \frac{1}{2} \alpha^k G\{\alpha_k p^{k'}, \frac{1}{2} \alpha_k (p^k + p^{k'})\}. \end{aligned} \right\} \quad (3.19)$$

The  $\rho_{ij}$  in (3.12) have now to be regarded as Dirac operators of the form

$$\rho_{ij} = \rho'_{ij}(\mathbf{p}, \mathbf{p}') \frac{\kappa_i + \alpha^k p_k^{(i)}}{2\kappa_i} \cdot \frac{\kappa_j + \alpha^k p_k^{(j)'}}{2\kappa_j}, \quad (3.20)$$

where the  $\rho'_{ij}$  are numerical factors; and the equations (3.13) show that the  $\kappa_i = (E_i^2 - \mathbf{p}^2)$  are the roots of the equation  $F(\kappa_i) = 0$ . From (3.14) one infers that

$$\Gamma_i(\mathbf{p}) = \frac{1}{2} \alpha^4 F'(\kappa_i), \quad (3.21)$$

which, coupled with (3.14) and (3.15), again supplies explicit expressions for the amount of matter, energy and momentum when the form of  $F$  is known.

#### 4. SECOND QUANTISATION

To show that the wave field may be regarded as consisting of particles with definite energies and momenta, it is necessary to proceed to the second quantisation of the field, as became apparent in the previous section. The correct commutation rules have been given by one of the authors (Green, 1949), but it is necessary to remember that  $R^4$  as defined by (3.5), and not  $\rho$ , is the density operator for a system of many particles like that now considered. The commutation rule is then

$$\zeta_i R^4 \pm R^4 \zeta_i = \zeta_i, \quad \zeta_i^* \zeta_i = 1, \quad (i' = i), \quad (4.1)$$

where  $\zeta_i$  is a unitary "creation" operator corresponding to the solution  $E_i$  of the equation (3.13), and the positive or negative sign is employed according as Fermi or Bose statistics are appropriate. Defining the operator  $\sigma$  in such a way that

$$\left. \begin{aligned} \rho_{ij}(\mathbf{p}, \mathbf{p}') &= \sigma_i(\mathbf{p}) \zeta_j \sigma_j(\mathbf{p}') \zeta_j = \sigma_i(\mathbf{p}) \sigma_j^+(\mathbf{p}'), \\ \sigma_i(x_k) &= \Omega^{-\frac{1}{2}} \sum_{\mathbf{p}} \sigma_i(\mathbf{p}) \exp \{i(\mathbf{p} \cdot \mathbf{x} - E_i t)\}, \\ \sigma_j^+(x_k) &= \Omega^{-\frac{1}{2}} \sum_{\mathbf{p}'} \sigma_j^+(\mathbf{p}') \exp \{-i(\mathbf{p}' \cdot \mathbf{x} - E_j' t)\}, \\ \rho(x_k, x_k') &= \sum_{ij} \sigma_i(x_k) \sigma_j^+(x_k'), \quad (t' = t), \end{aligned} \right\} \quad (4.2)$$

it is assumed that  $\zeta_i$  commutes with  $\sigma_j$  (or anticommutes for Fermi statistics) if  $j \neq i$ , but not with  $\sigma_i$ . Transforming the commutation rule (4.1) to the momentum representation by expanding both sides in Fourier series, and remembering that  $\delta(\mathbf{x} - \mathbf{x}')$  is the unit "matrix" implicit on the right-hand side, one has

$$\left. \begin{aligned} \Gamma_i(\mathbf{p}) \{ \zeta_i \rho_{ii}(\mathbf{p}, \mathbf{p}) \pm \rho_{ii}(\mathbf{p}, \mathbf{p}) \zeta_i \} &= \zeta_i, \\ \zeta_i \rho_{jk}(\mathbf{p}, \mathbf{p}') \pm \rho_{jk}(\mathbf{p}, \mathbf{p}') \zeta_i &= 0, \quad j \text{ or } k \neq i \text{ or } \mathbf{p} \neq \mathbf{p}'. \end{aligned} \right\} \quad (4.3)$$

By post-multiplication with  $\zeta_i^*$ , this commutation rule reduces to the more usual form

$$\Gamma_i(\mathbf{p}) \{ \sigma_i^+(\mathbf{p}) \sigma_j(\mathbf{p}') \pm \sigma_j(\mathbf{p}') \sigma_i^+(\mathbf{p}) \} = \delta_{ij} \delta_{\mathbf{p}\mathbf{p}'}, \quad (4.4)$$

which may equally well be adopted as the fundamental commutation rule. It may be remarked that  $\sigma_i^+$  is not necessarily the complex conjugate  $\sigma_i^*$  of  $\sigma_i$ , but may also be taken to be  $-\sigma_i^*$ ; it is convenient to have this sign at one's disposal, as one can then make  $n_i(\mathbf{p}) = \Gamma_i(\mathbf{p}) \sigma_i(\mathbf{p}) \sigma_i^+(\mathbf{p})$  positive definite independently of the sign of  $\Gamma_i(\mathbf{p})$ . With the help of the commutation rule it is then easily shown that  $n_i$  has integral eigenvalues.

For Bose statistics one sees in succession that

$$\left. \begin{aligned} n_i &= \Gamma_i \sigma_i \sigma_i^+, \\ \Gamma_i \sigma_i n_i \sigma_i^+ &= \Gamma_i \sigma_i \sigma_i^+ (n_i - 1) = n_i (n_i - 1), \\ \Gamma_i \sigma_i n_i (n_i - 1) \sigma_i^+ &= \Gamma_i \sigma_i \sigma_i^+ (n_i - 1) (n_i - 2) = n_i (n_i - 1) (n_i - 2), \text{ etc.} \end{aligned} \right\} \quad (4.5)$$

are all positive definite, and this can only be if  $n_i$  is an integer in a diagonal representation. From (3.14) it then follows that  $n_i(\mathbf{p})$  may be interpreted as the number of particles with momentum  $\mathbf{p}$  and energy  $E_i$ , i.e. with rest-mass  $\kappa_i = (E_i^2 - \mathbf{p}^2)^{\frac{1}{2}}$ . It is then clear that the reciprocally invariant Lagrangian operators derived by the method indicated in § 2 will describe assemblies of particles which will later be shown to have various rest-masses in agreement with the experimentally determined meson masses.

For Fermi statistics one has

$$\left. \begin{aligned} n_i (1 - n_i) &= \Gamma_i \sigma_i \sigma_i^+ (1 - \Gamma_i \sigma_i \sigma_i^+) \\ &= \Gamma_i^2 \sigma_i^2 \sigma_i^{+2}, \end{aligned} \right\} \quad (4.6)$$

from which it follows that  $0 < n_i \leq 1$ ; and as  $n_i = 0$  is certainly a permitted value, the only other eigenvalue is 1. The obvious interpretation is that one and only one particle may occupy each momentum state with assigned energy and spin.

It remains to complete the second quantisation of the field. By multiplying (4.4) by the factor  $\exp \{i(\mathbf{p} \cdot \mathbf{x} - E_i t - \mathbf{p}' \cdot \mathbf{x}' + E_j t')\}$  and summing over all values of  $\mathbf{p}$  and  $\mathbf{p}'$ , one obtains

$$\frac{1}{2}(G^4 + G^4)[\sigma_i^+(x_k), \sigma_j(x_k')]_{\pm} = \delta(\mathbf{x} - \mathbf{x}')\delta_{ij} \quad (i' = i). \quad (4.7)$$

Hence with the help of (3.6) and (3.14),

$$\left. \begin{aligned} [N, \sigma_i(x_k'')]_{-} &= \sum_j \int \left\langle \frac{1}{2}(G^4 + G^4)\sigma_j(x_k)[\sigma_j^+(x_k'), \sigma_i(x_k'')]_{\pm} \right\rangle d\Omega \\ &= \sum_j \int \sigma_j(x_k)\delta(\mathbf{x} - \mathbf{x}'')\delta_{ij}d\Omega = \sigma_i(x_k''). \end{aligned} \right\} \quad (4.8)$$

To show similarly that

$$[P_k, \sigma_i(x_k'')]_{-} = i \frac{\partial \sigma_i(x_k'')}{\partial x^k}, \quad (4.9)$$

it is convenient to use the unsymmetrical expression

$$P_k = \int \langle p_k R^4 \rangle d\Omega \quad (4.10)$$

for  $P_k$  instead of (3.10). The two expressions are equivalent, since they differ by  $\frac{1}{2}i \int \frac{\partial}{\partial x^k} \langle R^4 \rangle d\Omega$ , which vanishes by transformation to a surface integral when  $k = 1, 2, 3$ ,

and by the relation  $\frac{dN}{dt} = 0$  when  $k = 4$ . Using (4.10), one has immediately

$$[P_k, \sigma_i(x_k'')]_{-} = i \sum_j \int \left\langle \frac{1}{2}(G^4 + G^4) \frac{\partial \sigma_j(x_k')}{\partial x^k} [\sigma_j(x_k'), \sigma_i(x_k'')]_{\pm} \right\rangle d\Omega,$$

which reduces to (4.9). It is easily verified now that the different components of  $P_k$  commute with one another. The total momentum and energy  $P_k$  have therefore all the properties of the differential operators  $i \frac{\partial}{\partial x^k}$ , and starting from the field equations (3.3), the entire theory of § 3 may be transcribed into a fully quantised form.

## 5. PARTICLES WITH SPIN ZERO OR ONE

As was shown in § 2, the most general scalar, vector, or tensor reciprocal invariant is an eigensolution of the equation

$$\left( -\frac{\partial^2}{\partial p_k \partial p^k} + p_k p^k \right) F(p^i) = s F(p^i). \quad (5.1)$$

The relativistically covariant solutions of this equation are obtained most readily by transformation to four-dimensional polar co-ordinates. It has been shown by Born and Fuchs (1940) that the eigenfunctions of the square of the angular momentum tensor, which satisfy the equation

$$M^2 Y_k = \frac{1}{2} m_j m^{ij} Y_k = k(k+2) Y_k, \quad (5.2)$$

are four-dimensional spherical functions, and that  $k$  is a positive integer ( $k = 0, 1, 2, \dots$ ). The operator  $x_k x^k$  is transformed to

$$-\frac{\partial^2}{\partial p_k \partial p^k} = -\left( \frac{d^2}{dp^2} + \frac{3}{p} \frac{d}{dp} \right) + \frac{M^2}{p^2}, \quad p^2 = p_i p^i. \quad (5.3)$$

Putting  $p^2 = P$ , so that

$$\frac{d^2 F}{dP^2} + \frac{3}{P} \frac{dF}{dP} = 4P \frac{d^2 F}{dP^2} + 8 \frac{dF}{dP},$$

and factorising  $F$  into a radial part  $F_k(P)$  and a spherical function  $Y_k$ , so

$$F = F_k(P) Y_k(\theta, \phi, \omega),$$

one has

$$\frac{d^2 F_k}{dP^2} + \frac{2}{P} \frac{dF_k}{dP} - \frac{1}{4} \left\{ 1 - \frac{s}{P} + \frac{k(k+2)}{P^2} \right\} F_k = 0. \quad (5.4)$$

This is solved by making the substitution

$$F_k = P^{k/2} e^{-P/2} f_k, \quad (5.5)$$

which leads to

$$\frac{d^2 f_k}{dP^2} + \left( \frac{k+2}{P} - 1 \right) \frac{df_k}{dP} + \left( \frac{s-2k-4}{4P} \right) f_k = 0, \quad (5.6)$$

of which the known solution is

$$f_k = L_n^{k+1}(P) = \frac{d^{k+1} L_n(P)}{dP^{k+1}}, \quad (5.7)$$

where  $L_n(P)$  is the Laguerre polynomial of order  $n$ . The eigenvalue  $s$  is given by the relation

$$s = 2(2n - k). \quad (5.8)$$

Clearly only values of  $n$  and  $k$  for which  $k \geq 0$  and  $n \geq k+1$  correspond to admissible solutions.

It has been seen that the rest-masses of the particles represented by the wave equation  $F(p_k)\psi(x_i)$  are given as the roots of the equation  $F(\kappa^2) = 0$ , so that one has now

$$\kappa^k L_n^{k+1}(\kappa^2) = 0. \quad (5.9)$$

The distribution of the roots of these equations (5.9) is investigated by Dr A. E. Rodriguez in Appendix II. The "ground state" in the spectrum of masses so determined corresponds to the values  $k=0$ ,  $n=2$ , and is characterised by  $\kappa^2=2$ ,  $\kappa=\sqrt{2}$ . Substituting the accepted value  $a=e^2/mc^2$  for the electronic radius into the equation (2.4), one obtains the rest-mass

$$\mu = \frac{\hbar c}{e^2} \kappa m = 137\sqrt{2} m = 194m, \quad (5.10)$$

where  $m$  is the electronic mass, regarded as electromagnetic in origin. This is in good agreement with the observed values of the rest-mass of the stable  $\mu$ -meson. The first excited state corresponds to the values  $k=0$ ,  $n=3$ , and requires that

$$\kappa^4 - 6\kappa^2 + 6 = 0, \quad \kappa^2 = 3 + \sqrt{3};$$

so

$$\mu = \frac{\hbar c}{e^2} (3 + \sqrt{3})^{\frac{1}{2}} m = 298m. \quad (5.11)$$

This agrees well with the observed mass of the less stable  $\pi$ -meson. There exists also a second root  $\kappa^2 = 3 - \sqrt{3}$  which is associated with a particle of mass  $154m$  which has not yet been observed.

For  $k=2$ ,  $n=3$ , and generally for  $n=k+1$ ,  $k \geq 2$ , the equation (5.9) reduces to  $\kappa^2=0$ , which is the characteristic equation for the photon. There exist, however, many other excited states in the mass-spectrum with masses corresponding to particles which have presumably not yet been observed, though it is possible that the supposed  $\rho$ -meson can be accounted for in this way, and also, since some of the masses are not far from  $194m$  and  $298m$ , that they have been confused experimentally with the  $\mu$ - and  $\pi$ -mesons. As these other particles are in excited states of rest-energy, it is, however, likely that they will have a short lifetime, decaying into the  $\mu$ - and  $\pi$ -mesons. This question can be discussed properly only by considering the interaction between the elementary particles. There are good grounds for supposing that this also can be treated quite generally by the methods of this paper, but for the present it is proposed to confine attention to certain aspects of the theory already developed. It has been seen in § 3 that the total energy of the field derived from a single  $F$ -operator is the sum of the

energies of the constituent particles. Therefore there can be no direct interaction between particles with masses derived from the same equation. Particles with masses derived from different equations may, however, be expected to interact with one another, leading to the decay processes which are experimentally observed.

## 6. IDENTIFICATION OF THE ABSOLUTE LENGTH

There will now be given a preliminary treatment of the problem of the interaction between the electron and photon fields, which is the subject-matter of quantum electrodynamics. The difficulties associated with this subject are so well known that it is unnecessary to discuss them here; nor is it the object of this section to suggest a final solution. It will appear that there are indeed certain features of the present theory which suggest that the divergences appearing in current quantum electrodynamics may be eliminated, but for the present the authors are content to set aside this aspect. The primary question to be considered is the identification of the fundamental length  $\alpha$  with a quantity almost equal to the classical radius  $e^2/mc^2$  of the electron, on which the mass determinations of the previous sections were based.

In order to make apparent the full significance of the calculation, ordinary units will be employed. Maxwell's equations for the electromagnetic field may be written in the form

$$FA = \beta\rho, \quad \beta = 4\pi e\hbar^2, \quad (6.1)$$

where  $F$  is the self-reciprocal wave operator  $P e^{-P/2\delta}$  appropriate to the photon field,  $\rho$  is the statistical matrix for the electron, and

$$A = A^k \alpha_k + A', \quad (6.2)$$

where  $A^k$  is the four-vector potential of the photon field, and  $A'$  is some operator such that the spur of  $\alpha_k A'$  vanishes. By multiplying (6.1) by  $\alpha_k$  and taking the quarter-spur of both sides, one then obtains what is effectively the ordinary form of Maxwell's equations, since  $\delta$  is large. Writing

$$F\sigma = FA - \beta\rho, \quad (6.3)$$

the field equations (6.1) can be derived in the usual way from the Lagrangian

$$\begin{aligned} L_1 &= \frac{1}{2} \sigma F \sigma \\ &= \frac{1}{2} A F A - \frac{1}{2} \beta (A \rho + \rho A) + \frac{1}{2} \beta^2 \rho F^{-1} \rho. \end{aligned} \quad (6.4)$$

As this Lagrangian is somewhat different from that usually adopted, it is necessary to consider briefly its distinctive features. The statistical matrix  $\rho$  of the electron, regarded as a particle with spin half and zero rest-mass, has the form

$$\rho = \Omega^{-1} \frac{\alpha_k \dot{p}^k}{\dot{p}^4} e^{i p_k (x^k - x^k)/\hbar}, \quad (6.5)$$

so  $A'$  may be assumed to vanish. Then the first term on the right-hand side of (6.4) reduces to

$$\left. \begin{aligned} \frac{1}{2} (A \eta) (\eta A) &= \frac{\hbar^2}{2} \left( \frac{\partial A^k}{\partial x^k} - \frac{1}{2} \alpha_k \alpha_l F^{kl} \right) \left( \frac{\partial A^k}{\partial x^k} + \frac{1}{2} \alpha_k \alpha_l F^{kl} \right) \\ &= \frac{\hbar^2}{2} \left\{ \left( \frac{\partial A^k}{\partial x^k} \right)^2 + \frac{1}{2} F^{kl} F_{kl} + 2\gamma \epsilon^{ijkl} F_{ij} F_{kl} \right\}, \\ \eta &= \alpha_k \dot{p}^k, \quad F_{kl} = \frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l}, \quad \gamma = \alpha_1 \alpha_2 \alpha_3 \alpha_4. \end{aligned} \right\} \quad (6.6)$$

On account of the auxiliary condition  $\frac{\partial A^k}{\partial x^k} = 0$ , which is necessary to the conservation of charge, (6.6) differs from the usual Lagrangian of a pure radiation field only by the term  $\epsilon^{ijkl} F_{ij} F_{kl}$ , which, being the scalar product of the electric and magnetic field vectors, vanishes on integration over all space. The second term in (6.4) is obviously the usual one representing the interaction between the pure radiation field and the electron. The last term is new, but can be

seen without much difficulty to be substantially equivalent to the energy of the longitudinal part of the electromagnetic field, reducing to the Coulomb energy for a charge distribution at rest, but having the property of relativistic invariance.

To the Lagrangian (6.4) for the electromagnetic field must be added the term

$$L_2 = F\rho = 4\pi c\hbar^2\eta\rho \quad (6.7)$$

for the electrons. The usual variational procedure applied to the total Lagrangian  $L = L_1 + L_2$  leads to the field equations (6.1) for the Maxwell field, and

$$(4\pi c\hbar^2\eta - \beta A)\rho + \beta^2(\rho F^{-1})\rho = 0 \quad (6.8)$$

for the electron. This is a non-linear equation not easily solved exactly, but evidently it is practically identical with the Dirac equation provided the mass  $m$  of the electron is taken to be of the order of the expectation value of the operator  $\frac{\beta^2\rho F^{-1}}{4\pi c^2\hbar^2}$ , *i.e.* provided

$$m \doteq \frac{\beta^2\Omega^{-1}}{4\pi c^2\hbar^2} \sum_k \frac{e^{-k^2/2b^2}}{k^2} = \frac{\beta^2(2\pi\hbar)^{-3}}{4\pi c^2\hbar^2} \int_0^\infty \frac{e^{-k^2/2b^2}}{k^2} 4\pi dk \left. \vphantom{\sum_k} \right\} \quad (6.9)$$

$$= \sqrt{\frac{2}{\pi}} \frac{e^2 b}{\hbar c^2} = 0.8 \frac{e^2}{ac^2}.$$

This is the first approximation to the self-energy of the electron, and converges. It is likely that higher approximations will increase the value slightly, so that, as nearly as can be estimated, the value  $a = e^2/mc^2$  assumed in the last section was exact: though further calculations may show the need for a small correction, the meson masses thereby determined were probably correct within the present limits of experimental error.

## 7. APPENDIX I

### THE RECIPROCAL INVARIANTS FOR SPIN HALF. By K. C. CHENG

The problem of the determination of the reciprocally invariant function  $F(\alpha_i p^j)$  appropriate to particles of spin half will here be considered. It is first necessary to formulate the metric operator  $S(x_j, p_j)$  which, to comply with the requirements of the principle of reciprocity, must satisfy the equation

$$S(x_j, p_j) = TS(p_j, -x_j)T^{-1}, \quad (7.1)$$

where  $T$  is a unitary operator depending only on the Dirac matrices  $\alpha_j$ 's. To (7.1) must be added the conditions (i) that  $S$  should be relativistically invariant and linear in both  $x_i$  and  $p_j$ ; (ii) that the solutions  $F$  obtained from the metric operator  $S$  should be also the solutions of the equation

$$(R + P + C)F = Fs^2, \quad (7.2)$$

where  $C, s^2$  are constants; and (iii) that for integral spin the functions  $F$  should be identical with those obtained in § 5, *i.e.* in this case  $F$  must commute with  $C$ .

$S$  is therefore some combination of the relativistic invariants:

$$\left. \begin{aligned} \xi &= \alpha_j x^j, & \eta &= \alpha_j p^j, \\ \gamma &= \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \frac{1}{4!} \epsilon^{jlmn} \alpha_j \alpha_l \alpha_m \alpha_n. \end{aligned} \right\} \quad (7.3)$$

It will be shown that it is sufficient to choose the simple form:

$$S = \eta + i\gamma\xi. \quad (7.4)$$



The condition (7.1) is thereby satisfied provided  $T = \frac{1+i\gamma}{1-i}$ , so that  $T^{-1} = \frac{1-i\gamma}{1-i}$ . For, since  $\gamma$  anti-commutes with  $a_j$ , one has

$$\begin{aligned} TS(x_j, p_j)T^{-1} &= \frac{1+i\gamma}{1-i}(\eta+i\gamma\xi)\frac{1-i\gamma}{1-i} \\ &= \frac{(1+i\gamma)^2}{(1+i)(1-i)}(\eta+i\gamma\xi) \\ &= (i\gamma\eta - \xi) \\ &= S(p_j, -x_j). \end{aligned}$$

The wave operator  $F(x^j, p^j)$  from which the Lagrangian is to be derived will now be determined as the solution of the equation

$$F^{-1}SF = s,$$

or

$$SF = Fs, \quad (7.5)$$

where  $S$  is an eigensymbol depending only on  $\gamma$ . For convenience in calculation define the following quantities:—

$$\left. \begin{aligned} L &= -ia_j\alpha_j m^{jl}, \\ m^{jl} &= x^j p^l - x^l p^j. \end{aligned} \right\} \quad (7.6)$$

By direct substitution it is easily verified that

$$L^2 + 4L = 2m_{jl}m^{jl}. \quad (7.7)$$

Assume that  $F(p^j)$  is expressible in the form

$$F = (S + s)F_k(\eta^2)g(\theta, \phi, \omega), \quad (7.8)$$

where  $F_k(\eta^2)$  is a function of  $\eta^2 = p^2 = p_j p^j$  only, and  $g$  is an eigensymbol depending on the polar angles  $\theta, \phi, \omega$  of the four-vector  $p_j$ , and satisfies the equation

$$Lg = 2kg, \quad (7.9)$$

so that  $2k$  is the corresponding eigenvalue of  $L$ . These eigenvalues and eigensymbols can be determined in the following way. Writing

$$g = (L + 2k + 4)Y_k, \quad (7.10)$$

and using the relation (7.7), one obtains

$$(L - 2k)(L + 2k + 4)Y_k = 2(m_{jl}m^{jl} - 2k^2 - 4k)Y_k. \quad (7.11)$$

Hence (7.9) is satisfied if  $Y_k$  is an eigenfunction and  $k(k+2)$  is the corresponding eigenvalue of  $\frac{1}{2}m_{jl}m^{jl}$ .

It is well known that the solutions of the equation

$$\frac{\partial^2 U_k}{\partial p^j \partial p_j} = 0 \quad (7.12)$$

are

$$\frac{1}{p^{2j}} \quad \frac{\partial}{\partial p_j} \frac{1}{p^{2j}} \quad \dots \quad U_k = \frac{\partial^k}{\partial p_1^{k_1} \partial p_2^{k_2} \partial p_3^{k_3} \partial p_4^{k_4}} \frac{1}{p^{2j}} \quad (7.13)$$

where  $k = k_1 + k_2 + k_3 + k_4 = 0, 1, 2, \dots$ , and  $p^2 = p^j p_j$ . Putting  $Y_k = p^{k+2} U_k$ , then  $Y_k$  is independent of  $p$ , and

$$0 = \frac{\partial^2 U_k}{\partial p^j \partial p^j} = \left( \frac{\partial^2}{\partial p^2} + \frac{3}{p} \frac{\partial}{\partial p} \right) (Y_k p^{-k-2}) - p^{-k-4} \frac{1}{2} m_{jl} m^{jl} Y_k,$$

or

$$Y_k \left( \frac{\partial^2}{\partial p^2} + \frac{3}{p} \frac{\partial}{\partial p} \right) p^{-k-2} - p^{-k-4} \frac{1}{2} m_{ji} m^{ji} Y_k = 0.$$

Hence

$$\frac{1}{2} m_{ji} m^{ji} Y_k = k(k+2) Y_k. \quad (7.14)$$

The four-dimensional spherical function  $Y_k$  is therefore an eigenfunction of  $\frac{1}{2} m_{ji} m^{ji}$ . The proper values of  $k$  are 0, 1, 2, . . . , as was found otherwise by Born and Fuchs (1940).

Now the components of the angular momentum tensor  $m_{ji} = (x_i p_j - x_j p_i)$  are well known to be expressible in terms of angular operators only. The same holds for  $L$ , according to (7.6). Therefore

$$g = (L + 4 + 2k) Y_k \equiv Z_k \quad (7.15)$$

is a function of the angles  $\theta, \phi, \omega$  alone, and may therefore be regarded as a generalisation of the spherical harmonics for spin half. Notice that  $g$  commutes with  $\gamma$ , therefore with  $s$ . Hence equation (7.5) becomes

$$S^2 F_k g = F_k g S^2. \quad (7.16)$$

Since  $\gamma$  anticommutes with  $a_j$ 's, one has from (7.4)

$$S^2 = \eta^2 + \xi^2 + i\gamma(\xi\eta - \eta\xi) = \eta^2 + \xi^2 - \gamma(L + 4). \quad (7.17)$$

Using (7.9), one obtains

$$\gamma(L + 4) F_k g = F_k g (2k + 4) \gamma. \quad (7.18)$$

Substituting (7.17) and (7.18) into (7.16), one has

$$\left\{ \frac{\partial^2}{\partial p^2} + \frac{3}{p} \frac{\partial}{\partial p} - \frac{m_{ji} m^{ji}}{2p^2} - p^2 + s^2 + (2k + 4) \gamma \right\} F_k g = 0. \quad (7.19)$$

On account of the relation (7.9) and (7.7), one has

$$2m^{ji} m_{ji} g = (L^2 + 4L) g = 4k(k+2) g,$$

and thus the equation (7.19) becomes

$$\left\{ \frac{\partial^2}{\partial p^2} + \frac{3}{p} \frac{\partial}{\partial p} - \frac{k(k+2)}{p^2} - p^2 + s^2 + (4 + 2k) \gamma \right\} F_k = 0. \quad (7.20)$$

Putting  $P = p_i p^i$ , one obtains

$$\left\{ \frac{d^2}{dP^2} + \frac{2}{P} \frac{d}{dP} - \frac{1}{4} \left[ 1 - \frac{1}{P} (s^2 + 4\gamma + 2k\gamma) + \frac{k(k+2)}{P^2} \right] \right\} F_k = 0. \quad (7.21)$$

Now (7.21) is identical with (5.4), so that the eigenfunction  $F_k(P)$  and the eigenvalues  $(s^2 + 4\gamma + 2\gamma k)$  are the same as those given by (5.5), (5.7) and (5.8), namely

$$F_k = P^{k/2} e^{-P/2} L_n^{k+1}(P), \quad s^2 + 4\gamma + 2\gamma k = 4n - 2k, \quad (7.22)$$

or

$$s_{\pm} = \pm \sqrt{4n - 2(1 - \gamma)k - 4\gamma(k+1)}. \quad (7.23)$$

The complete wave function for particles having spin half is

$$F(\eta) = \{ \eta + i\gamma\xi + s_{\pm} \} P^{k/2} e^{-P/2} L_n^{k+1}(P) Z_k, \quad (7.24)$$

$$s_{\pm} = \pm \sqrt{4n - 2(1 - \gamma)k - 4\gamma(k+1)}. \quad (7.25)$$

Lastly, we shall show that the requirements in (7.2) are fulfilled. By multiplying (7.5) from the left by  $S$ , one has

$$S^2 F = F S^2, \quad (7.26)$$

or, in view of (7.17) and (7.18),

$$\{R + P - (4 + 2k)\gamma\}F = Fs^2. \quad (7.27)$$

This shows that all the three conditions are satisfied: (i) The relativistic invariance is obvious; (ii) the equation (7.27) has the form of (7.2) with  $C = -(4 + 2k)\gamma$ ; (iii) for particles of integral spin,  $\gamma$  commutes with the function  $F$  determined in § 5.

In the following we give an explicit formula of the wave operator  $F(\eta)$  suitable for determining the masses, which are the roots of the equation  $F(\eta) = 0$ . We note that if we multiply  $F(\eta)$  in (7.22) from the right by  $(1 \pm \gamma)$ , this function still satisfies (7.5), since  $\gamma$  commutes with  $\pm s$ . This process is equivalent to putting  $\gamma$ , wherever it appears on the right side of  $\eta$  in  $F(\eta)$ , equal to  $\pm 1$ . With this procedure, we obtain the following functions:—

$$F(\eta) = \{\eta[(1 - \gamma)L_n^{k+1} + 2\gamma L_n^{k+2}(\eta^2)] \pm 2\sqrt{\{n - \frac{1}{2}(1 - \gamma)k - \gamma(k + 1)\}}L_n^{k+1}(\eta^2)\} \cdot \eta^k Z_k(1 \pm \gamma). \quad (7.28)$$

The equation  $F(\eta) = 0$  gives the masses in the following cases:—

$$(i) \quad \gamma = 1, \\ \eta L_n^{k+2}(\eta^2) \pm \sqrt{(n - k - 1)}L_n^{k+1}(\eta^2) = 0; \quad (7.29)$$

$$(ii) \quad \gamma = -1, \\ \eta(L_n^{k+1}(\eta^2) - L_n^{k+2}(\eta^2)) \pm \sqrt{(n + 1)}L_n^{k+1}(\eta^2) = 0; \quad (7.30)$$

and  $\eta^k = 0$  for all  $k > 0$ .

## 8. APPENDIX II

### THE CALCULATION OF THE REST-MASSSES. By A. E. RODRIGUEZ

(i) It has been seen in § 5 that there are an infinite number of particles of *integral spin* with rest-masses

$$\mu = \frac{\hbar c}{e^2 \kappa m}, \quad (8.1)$$

$\kappa^2$  being given by the roots of the equation (5.9).

The Laguerre polynomials are easily calculated with the help of the recurrence formula

$$\left. \begin{aligned} L_{n+1}(x) - (2n + 1 - x)L_n(x) + n^2 L_{n-1}(x) &= 0, \\ L_0(x) &= 1, \quad L_1(x) = 1 - x, \\ L_n^{(j)}(x) &= \frac{d^j}{dx^j} L_n(x). \end{aligned} \right\} \quad (8.2)$$

The equation (5.9) reduces, after putting  $\kappa^2 = x$ , to

$$\left. \begin{aligned} \text{For } k=0, \quad n=2: \quad x - 2 &= 0, \\ n=3: \quad x^2 - 6x + 6 &= 0, \\ n=4: \quad x^3 - 12x^2 + 36x - 24 &= 0, \\ n=5: \quad x^4 - 20x^3 + 120x^2 - 240x + 120 &= 0, \\ n=6: \quad x^5 - 30x^4 + 300x^3 - 1200x^2 + 1800x - 720 &= 0. \end{aligned} \right\} \quad (8.3)$$

$$\left. \begin{aligned} \text{For } k=1, \quad n=4: \quad x - 4 &= 0, \\ n=5: \quad x^2 - 10x + 20 &= 0, \\ n=6: \quad x^3 - 18x^2 + 90x - 120 &= 0. \end{aligned} \right\} \quad (8.4)$$

$$\text{For } k=2, \quad n=6: \quad x - 6 = 0. \quad (8.5)$$

The solutions of these equations are given in the following table.

TABLE

$n \backslash k$	0	1	2	3	4
2	1.41	..	..	..	..
3	1.13 2.17	1.73	..	..	..
4	0.97 1.82 2.78	1.41 2.45	2.00	..	..
5	0.86 1.60 2.39 3.31	1.20 2.07 3.06	1.66 2.70	2.24	..
6	0.73 1.45 2.14 2.92 3.78	1.10 1.85 2.62 3.52	1.46 2.30 3.24	1.88 2.91	2.45

The associated rest-masses, calculated assuming the validity of (8.1), are represented in fig. 1. The asymptotic distribution of these roots for large values of  $n$  has been studied by Tricomi (1940), who has given the formula

$$x = 4n' \sin^2 \theta_{n',j} \quad (8.6)$$

for the  $j$ th root of the Laguerre polynomial  $L_{n'}^j(x)$ , where  $\theta_{n',j}$  satisfies the transcendental equation

$$2\theta_{n',j} + \sin 2\theta_{n',j} + \frac{1+j}{n'} \theta_{n',j} = \left(k - \frac{1-2j}{4}\right) \frac{\pi}{n'}, \quad (8.7)$$

$n'$  being related to  $n$  by  $n' = n - 1$ .

For a given value  $n'$  it follows from (8.6) and (8.7),

$$\left. \begin{aligned} dx &= 8n' \sin \theta_{n',j} \cos \theta_{n',j} d\theta_{n',j} \\ \frac{\pi}{n'} dk &= \left(2 + 2 \cos 2\theta_{n',j} + \frac{1+j}{n'}\right) d\theta_{n',j} \end{aligned} \right\} \quad (8.8)$$

For  $j=1$ , (8.8) can be reduced to

$$\frac{dk}{dx} = \frac{2 + \frac{1}{n'} - \frac{x}{2n'}}{2\pi \sqrt{\frac{x}{n'} - \frac{x^2}{4n'^2}}}, \quad x < 4n'. \quad (8.9)$$

This formula gives the asymptotic density of the masses for large values of  $n'$ , corresponding to the highly excited states of the mass spectrum.

(ii) For *half integral* spin there are two cases to consider, corresponding to the values  $\pm \frac{1}{2}$  of  $\gamma$  in the equation (7.22). For  $\gamma = \frac{1}{2}$ , (7.8) with (7.22) becomes

$$F_1(\eta) = \{2\eta L_n^{k+2}(\eta^2) \pm 2\sqrt{n - (k+1)} L_n^{k+1}(\eta^2)\} \eta^k e^{-\eta^2/2} g, \quad (8.10)$$

which vanished identically for  $k = n - 1$ ; for  $k = n - 2$  one obtains

$$F_1(\eta) = 2\{\eta L_n^n(\eta^2) \pm L_n^{n-1}(\eta^2)\} \eta^{n-2} e^{-\eta^2/2} g, \quad (8.11)$$

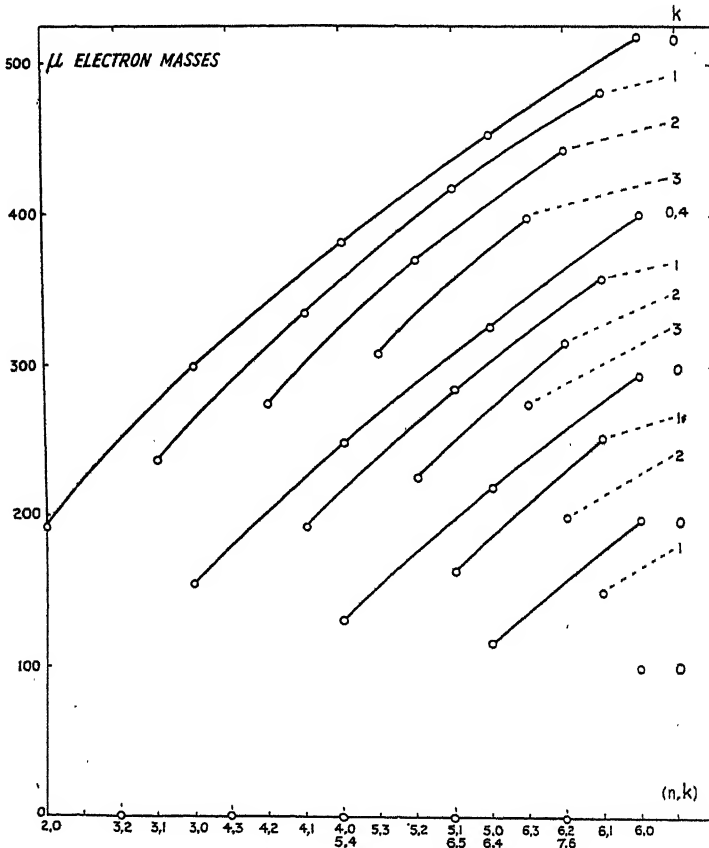


FIG. 1.—The rest-masses for integral spin.

which becomes

$$F_1 = 2L_n^n(\eta^2)\{\eta \pm (\eta^2 - n)\} \eta^{n-2} e^{-\eta^2/2} g \quad (8.12)$$

on using the relation

$$L_n^{n-1}(\eta^2) = L_n^n(\eta^2)\{\eta^2 - n\}. \quad (8.13)$$

For  $n=2$  there are only two masses, namely

$$\sqrt{\eta^2} = \begin{cases} 1 \\ 2 \end{cases}.$$

This can be interpreted as indicating a state for which only spin mesons of masses 1 and 2 are present.

For  $n=3, 4, 5, \dots$ , one obtains

$$\sqrt{\eta^2} = \begin{cases} 0 \\ \frac{\pm 1 + \sqrt{1 + 4n}}{2} \end{cases}.$$

The root which gives  $\eta = 0$  may be regarded as representing states for which there are neutrinos present or electrons for charged particles by coupling with an electromagnetic field. The other solutions represent states for which there are spin mesons of masses

$$\frac{\pm 1 + \sqrt{1 + 4n}}{2}.$$

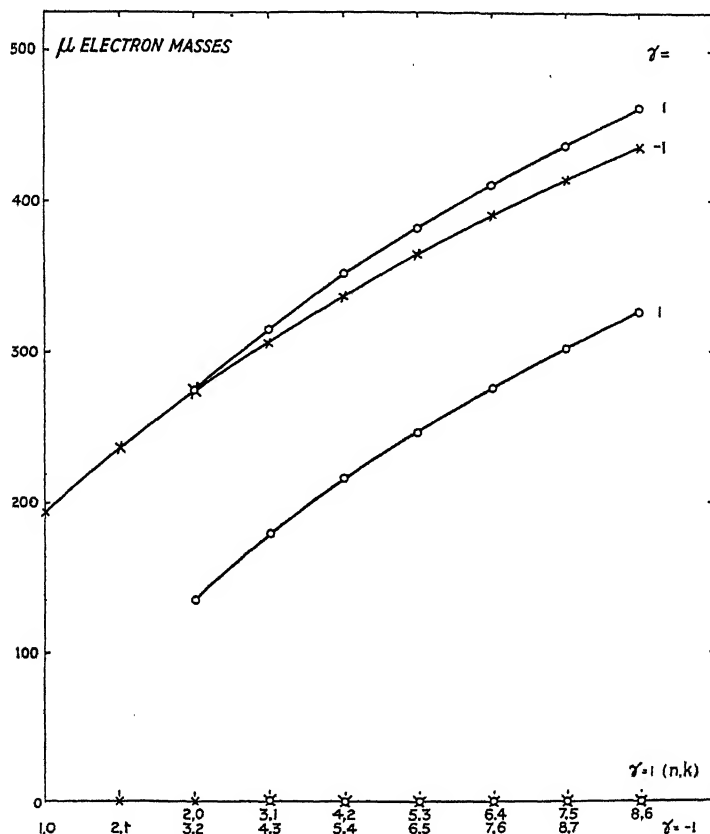


FIG. 2.—The rest-mass for half spin.

For obtaining rest-masses for  $k = n - i$ , others than those already given, the same process may be applied by repeated use of the formula

$$L_n^{n-i}(\eta^2) = L_n^n(\eta^2) \left\{ \frac{\eta^{2i}}{i!} - \frac{\binom{n}{1}}{(i-1)!} \eta^{2i-2} + \frac{\binom{n}{2}}{(i-2)!} \eta^{2i-4} - \dots \pm \binom{n}{n-i} \right\}. \quad (8.14)$$

For  $\gamma = -1$ , (7.8) with (7.22) becomes

$$F_1(\eta) = 2\{\eta L_n^{k+1}(\eta^2) - \eta L_n^{k+2}(\eta^2) \pm \sqrt{n+1} L_n^{k+1}(\eta^2)\} \eta^{.k} e^{.-\eta^{1/2}g}, \quad (8.15)$$

which for  $k = n - 1$  reduces to

$$F_r(\eta) = 2L_n^n(\eta^2) \{\eta \pm \sqrt{n+1}\} \eta^{.n-1} e^{.-\eta^{1/2}g}. \quad (8.16)$$

For  $n=1$  there is only one mass  $\eta = \sqrt{2}$ . This indicates a state for which there are spin mesons of mass  $\sqrt{2}$  present.

For  $n=2, 3, 4, \dots$ , one obtains

$$\sqrt{\eta^2} = \begin{cases} 0 \\ \sqrt{n+1} \end{cases}$$

Again this result may be regarded as representing states for which there exist neutrinos (or electrons for charged particles by coupling with an electromagnetic field) and spin mesons of masses  $\sqrt{n+1}$ .

For  $k=n-2$ , (8.15) becomes, by using (8.13),

$$F_{-1}(\eta) = 2L_n^n(\eta^2) \{ \eta^3 \pm \sqrt{n+1} \eta^2 - (n+1) \eta \mp n \sqrt{n+1} \} \eta^{n-2} e^{-\eta^2/2} g. \quad (8.17)$$

A similar consideration to the one given above shows that for  $n=2$  there are only spin mesons present, and again neutrinos (or electrons) and spin mesons for  $n=3, 4, 5, \dots$

The associated rest-masses for the simple cases discussed in this section, calculated assuming the validity of (8.1), are represented in fig. 2.

[Remark added in proof.] Meanwhile it has been found that there is another procedure for forming the Lagrangian operator in a reciprocally invariant manner, which is in some ways preferable. It consists in assuming, instead of (2.22),

$$L(x_k, x_k') = F(\phi) F(\phi') \rho(x_k, x_k'),$$

which leads to the same values for the rest-masses.

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